Can. J. Math., Vol. XXVI, No. 5, 1974, pp. 1069–1085

LOCALIZATION OF RIGHT NOETHERIAN RINGS AT SEMIPRIME IDEALS

J. LAMBEK AND G. MICHLER

In [**11**] and [**12**] we investigated the process of localization of right Noetherian rings R at prime ideals. We shall now extend these investigations to semiprime ideals N of R.

In Section 2 we show that localizing at the injective right R-module $E(R/N)$ is the same as localizing with respect to the multiplicative set

$$\mathscr{C}(N) = \{c \in R \mid \forall_{r\in R}(cr \in N \Rightarrow r \in N)\}.$$

We say we are *localizing at* N and call the localization $h : R \rightarrow R_N$ the *ring of right quotients* of R at N.

Extending a theorem of Heinicke [**7**] we show that the localization functor Q_N is right exact if and only if the N-closure $\tilde{N}$ of $h(N)$ in R_N is such that $R_N/\tilde{N}$ is a finite direct sum of simple R_N-modules containing at least one representative of each isomorphism class of simple R_N-modules. In Theorem 2.6 we prove that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$ if and only if $\tilde{N}$ is the Jacobson radical of R_N and $R_N/\tilde{N}$ is semisimple Artinian. Another equivalent statement asserts that $\tilde{N}$ is a two-sided ideal and Q_N is right exact.

In Section 3 we consider Small's characterization of those right Noetherian rings which are right orders in right Artinian rings. Our Theorem 3.3 asserts that, if N is any semiprime ideal of the right Noetherian ring R, then R_N is a right Artinian classical ring of right fractions of R with respect to $\mathscr{C}(N)$ if and only if some power of N is N-torsion and

$$\forall_{r\in R}(\exists_{c\in\mathscr{C}(N)} cr = 0 \Rightarrow \exists_{c'\in\mathscr{C}(N)} rc' = 0).$$

These two conditions are trivially satisfied when $\mathscr{C}(N)$ consists of regular elements, hence one obtains Small's Theorem as a corollary. We also show that, when N is the prime radical of the right Noetherian ring R and Q_N is right exact, then R_N is right Artinian.

In Sections 4 and 5 we generalize the results of [**12**] to semiprime ideals. In Proposition 4.3 we consider a right Noetherian ring R with a semiprime ideal N such that R/N is semisimple Artinian. If I is the injective hull $E(R/N)$, we show that on any finitely generated right R-module the N-adic and I-adic topologies coincide if and only if N has the *Artin-Rees property*: for every right ideal E of R there is a natural number n such that $E \cap N^n \subseteq EN$. It then also

Received January 31, 1973 and in revised form, July 3, 1973.

https://doi.org/10.4153/CJM-1974-099-4 Published online by Cambridge University Press

follows if N is the Jacobson radical then every right ideal of R is closed in the N-adic topology, as in the commutative case. Such a ring is called a *classical semilocal ring*.

Results of earlier sections are applied to establish Theorem 5.3: if N is a semiprime ideal of the right Noetherian ring R then the ring R_N of right quotients of R at N is a classical semilocal ring if and only if N has the *right symbolic Artin-Rees property*. This property was introduced by Goldie [**4**] for prime ideals of right and left Noetherian rings and is here extended to semiprime ideals of right Noetherian rings. It follows from [**10**] that the $\tilde{N}$-adic completion $\hat{R}_N$ of R_N is the bicommutator of $E(R/N)$.

Throughout this paper, R will be an associative ring with 1. $E(A)$ denotes the injective hull of the right R-module A.

The authors are endebted to Bill Schelter for his critical reading of the manuscript and for several helpful suggestions, to Allan Heinicke for spotting a false example and to one referee for his constructive criticism.

1. Localization at an injective. In this section we recall some definitions, notations and results that depend on a given injective right R-module I. (For details see [**9**].)

A right R-module A is called *I-torsion* if $\mathrm{Hom}_R(A, I) = 0$. We note that A is $E(B)$-torsion if and only if

$$(1) \qquad \forall_{a \in A} \forall_{0 \neq b \in B} \exists_{r \in R} (ar = 0 \;\&\; br \neq 0).$$

(See [**9**, Proposition 0.2].) A is called *I-torsionfree* if it is isomorphic to a submodule of some power of I. A is called *I-divisible* if $E(A)/A$ is *I-torsionfree.*

The *I-torsion submodule* of A is given by

$$T_I(A) = \{a \in A | \mathrm{Hom}_R(aR, I) = 0\},$$

and the *I-divisible hull* $D_I(A)$ is given by

$$D_I(A)/A = T_I(E(A)/A).$$

By the *localization* of A at I we mean the R-module homomorphism $h : A \to A_I = Q_I(A) = D_I(A/T_I(A))$. One also calls A_I the *module of quotients* of A at I and Q_I the *quotient functor.* Q_I is a left exact functor of Mod R into itself.

It is well-known that R_I is a ring, the *ring of quotients* of R at I, and that $h : R \to R_I$ is a ring homomorphism. Moreover, every I-torsionfree and I-divisible R-module is an R_I-module, and every R-homomorphism between such is an R_I-homomorphism.

A right ideal D of R is called *I-dense* if R/D is I-torsion. The I-dense right ideals of R form an idempotent filter $\mathscr{D}_I$ in the sense of Gabriel [**2**]. Conversely every idempotent filter $\mathscr{D}$ of right ideals may be obtained from an injective R-module I such that $\mathscr{D} = \mathscr{D}_I$. (See [**9**].)

Let $h : A \to A_I$ be the localization of A at I. Assume that N is an *I-closed* submodule of A, that is, that A/N is I-torsionfree. Let

$$\tilde{N} = \{q \in A_I | q^{-1}h(N) \in \mathscr{D}_I\}.$$

Then $\tilde{N}$ is called the *I-closure* of $h(N)$ in A_I. We note that

$$\tilde{N}/h(N) = T_I(A_I/h(N)), \quad \tilde{N} \cap h(A) = h(N), \quad N = h^{-1}(\tilde{N}).$$

PROPOSITION 1.1. *Let I be any injective right R-module, and N an I-closed submodule of the right R-module A. Then there exist canonical monomorphisms*

$$A/N \to A_I/\tilde{N} \to (A/N)_I.$$

Moreover, $A_I/\tilde{N} \to (A/N)_I$ is an isomorphism if and only if A_I/N is I-divisible.

Proof. Let p be the canonical projection $A_I \to A_I/\tilde{N}$ and consider the composite homomorphism

$$A \xrightarrow{h} A_I \xrightarrow{p} A_I/\tilde{N}.$$

Its kernel is $h^{-1}(\tilde{N}) = N$, and therefore the induced mapping $h' : A/N \to A_I/\tilde{N}$ is a monomorphism.

Next, consider the following diagram with two exact rows and two commutative squares:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & A & \longrightarrow & A/N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} & & \\ 0 & \longrightarrow & \tilde{N} & \longrightarrow & A_I & \underset{p}{\longrightarrow} & A_I/\tilde{N} & \longrightarrow & 0\,. \end{array}$$

Suppose $f : A_I/\tilde{N} \to I$ and $fh' = 0$. Then $fp : A_I \to I$ and $fph = 0$. Since $A_I/h(A)$ is I-torsion, it follows that $fp = 0$. Since p is an epimorphism, we have $f = 0$. Thus

$$\operatorname{cok} h' = (A_I/\tilde{N})/\operatorname{im} h'$$

is I-torsion. If K is any submodule of $A_I/\tilde{N}$ such that $K \cap \operatorname{im} h' = 0$, then K is both I-torsion and I-torsionfree, hence zero. Thus $A_I/\tilde{N}$ is an essential extension of A/N, and we may regard $A_I/\tilde{N}$ as an R-submodule of $(A/N)_I$.

Finally, if $A_I/\tilde{N}$ is I-divisible, the monomorphism $A_I/\tilde{N} \to (A/N)_I$ is clearly an isomorphism.

Remark 1.2. The hypothesis that $A_I/\tilde{N}$ is I-divisible is fulfilled whenever Q_I is exact. (See [**10**] for examples.) When R is right Noetherian, Q_I is exact if and only if it preserves all colimits, or equivalently, all R_I-modules are I-torsionfree. Walker and Walker [**20**] have shown that Q_I preserves all colimits if and only if $DR_I = R_I$ for all $D \in \mathscr{D}_I$. When this is the case, it also follows that R_I is flat as a left R-module and that R_I is right Noetherian.

From this remark one easily deduces the following:

LEMMA 1.3. *If N is any I-closed submodule of A, then $NR_I \subseteq \tilde{N}$, with equality holding when Q_I preserves all colimits.*

2. Localization at a semiprime ideal. Following Goldie [**5**], we associate with any two-sided ideal N of R the multiplicatively closed set

$$\mathscr{C}(N) = \{c \in R | \forall_{r \notin N} cr \notin N\}.$$

This set (called $\mathscr{C}'(N)$ in [**5**]) determines the idempotent filter $\mathscr{D}_N$ of right ideals D such that

$$\forall_{r \in R} r^{-1}D \cap \mathscr{C}(N) \neq \emptyset.$$

For convenience, we collect here some results by Goldie which will be referred to frequently.

LEMMA 2.1. *Let N be a semiprime ideal of the right Noetherian ring R. Then*
(1) *$(\mathscr{C}(N) + N)/N$ is the set of regular elements of R/N;*
(2) *a right ideal D of R containing N meets $\mathscr{C}(N)$ if and only if D/N is an essential right ideal of R/N;*
(3) *R/N satisfies the right Ore condition with respect to $(\mathscr{C}(N) + N)/N$;*
(4) *R/N has a classical ring of right quotients $Q_{cl}(R/N)$;*
(5) *for each $c \in \mathscr{C}(N)$, $cR + N \in \mathscr{D}_N$.*

Proof. (1) follows from the definition of $\mathscr{C}(N)$ and [**3**, Lemma 3.8]. (2) is [**3**, Theorem 3.9]. (3) and (4) are [**3**, Theorem 4.1]. (5) is an immediate consequence of (3) and appears in [**4**, Lemma 3.1].

PROPOSITION 2.2. *If N is a semiprime ideal of the right Noetherian ring R, then*

$$\mathscr{D}_{E(R/N)} = \mathscr{D}_N.$$

Proof. Assume that $D \in \mathscr{D}_N$. To show that $D \in \mathscr{D}_{E(R/N)}$, we require that R/D be $E(R/N)$-torsion, that is, by §1 (1), that

$$\forall_{r \in R} \forall_{s \notin N} \exists_{c \in R} (rc \in D \;\&\; sc \notin N).$$

By assumption, we may pick $c \in \mathscr{C}(N)$ such that $rc \in D$, then $sc \notin N$, by definition of $\mathscr{C}(N)$.

Conversely, assume that $D \in \mathscr{D}_{E(R/N)}$. Let $r, r' \in R$ and $r \notin N$. Again, by §1 (1),

$$\exists_{t \in R} (rr't \in D \;\&\; rt \notin N).$$

This means that $(r^{-1}D + N)/N$ is an essential right ideal of R/N. By Lemma 2.1, $r^{-1}D + N$ meets $\mathscr{C}(N)$, say $c = d + n$, for $c \in \mathscr{C}(N)$, $d \in r^{-1}D$, $n \in N$. Then $d \in r^{-1}D \cap \mathscr{C}(N)$, hence $D \in \mathscr{D}_N$.

In view of Proposition 2.2, we write *N-torsion*, *N-torsionfree*, *N-divisible*,

N-dense and *N-closed* instead of $E(R/N)$-torsion, etc. We also write $T_N(A)$, $D_N(A)$, A_N, R_N and Q_N in place of $T_{E(R/N)}(A)$ etc.

The following could be deduced from Proposition 1.1, but it seems more instructive to prove it directly.

LEMMA 2.3. *Let N be a semiprime ideal of the right Noetherian ring R, and assume that its N-closure $\tilde{N}$ in R_N is an ideal in R_N. Then there exists a ring monomorphism $\tau : R_N/\tilde{N} \to Q_{cl}(R/N)$, where $Q_{cl}(R/N)$ is the classical ring of quotients of R/N.*

We may write

$$R/N \subseteq R_N/\tilde{N} \subseteq Q_{cl}(R/N) \subseteq Q_N(R/N).$$

Proof. We may assume that R is N-torsionfree. Hence for every $q \in R_N$ there exists $c \in \mathscr{C}(N)$ such that $qc \in R$. Define $\tau : R_N/\tilde{N} \to Q_{cl}(R/N)$ by $\tau([q]) = [qc][c]^{-1}$. To check that τ is a mapping, suppose $q \in \tilde{N}$, then $qc \in \tilde{N} \cap R = N$. To see that τ is one-to-one, suppose $qc \in N$; then $q(cR + N) \subseteq \tilde{N}$, hence q is in the N-closure of $\tilde{N}$, by Lemma 2.1, and therefore $q \in \tilde{N}$. If q_1 and $q_2 \in R_N$, we may pick a single $c \in \mathscr{C}(N)$ such that $q_1c \in R$ and $q_2c \in R$, and from this it follows that τ is additive.

Finally, we will show that τ preserves multiplication. Let $q_1, q_2 \in R_N$ be given, then there exist $c_1, c_2 \in \mathscr{C}(N)$ such that q_1c_1 and $q_2c_2 \in R$. Now R/N satisfies the right Ore condition with respect to $\mathscr{C}(N)$ modulo N (see Lemma 2.1), hence we can find $c \in \mathscr{C}(N)$ and $r \in R$ so that $c_1r - q_2c_2c \in N$. Then $q_1c_1r - q_1q_2c_2c \in \tilde{N}$, since $\tilde{N}$ is an ideal. Pick $c' \in \mathscr{C}(N)$ such that $q_1c_1rc' - q_1q_2c_2cc' \in N$. Then

$$\begin{aligned}\tau([q_1q_2]) &= [q_1q_2c_2cc'][c_2cc']^{-1}\\ &= [q_1c_1rc'][c_2cc']^{-1}\\ &= [q_1c_1][r][c]^{-1}[c_2]^{-1}\\ &= [q_1c_1][c_1]^{-1}[q_2c_2][c_2]^{-1}\\ &= \tau([q_1])\tau([q_2]).\end{aligned}$$

PROPOSITION 2.4. *Let N be a semiprime ideal of the right Noetherian ring R. Then a right ideal A of R containing N is N-closed if and only if A/N is a right complement. Moreover a right ideal A is maximal among right ideals not meeting $\mathscr{C}(N)$ if and only if A/N is a maximal right complement.*

Proof. Suppose A contains N, B is the N-closure of A and $r \in B$. Then $rD \subseteq A$ for some N-dense right ideal D. Now D meets $\mathscr{C}(N)$, and so $rc \in A$ for some $c \in \mathscr{C}(N)$. If $r \notin N$ then $rc \notin N$, hence B/N is an essential extension of A/N. If A/N is a right complement, $B/N = A/N$, hence $B = A$.

Conversely, suppose B/N is an essential extension of A/N. Then, for any $r \in B$, $r^{-1}A$ is an essential right ideal of R, hence meets $\mathscr{C}(N)$, by Lemma 2.1. Therefore B/A is N-torsion. If A is N-closed, it follows that $B = A$, hence A/N is a right complement.

Suppose A is maximal among right ideals not meeting $\mathscr{C}(N)$. Then $A + N$ does not meet $\mathscr{C}(N)$, hence $N \subseteq A$. Suppose $rD \subseteq A$, where D is N-dense and $r \notin A$. Then $A + rR$ meets $\mathscr{C}(N)$, say $c = a + rs$, where $a \in A$, $s \in R$. Now $s^{-1}D$ meets $\mathscr{C}(N)$, say $sc' \in D$ for some $c' \in \mathscr{C}(N)$. Therefore $cc' = ac' + rsc' \in A + rD \subseteq A$, a contradiction. Thus A is N-closed. If B properly contains A, B meets $\mathscr{C}(N)$, hence B is N-dense, by Lemma 2.1. Thus A is maximal among proper N-closed right ideals containing N.

Conversely, let A be maximal among proper N-closed right ideals containing N. Then A is not N-dense, hence A does not meet $\mathscr{C}(N)$. Suppose B contains A, then the N-closure of B is R, hence B is N-dense, and so B meets $\mathscr{C}(N)$. Thus A is maximal among right ideals not meeting $\mathscr{C}(N)$.

The following generalizes a result by Heinicke [**7**, Theorem 4.3].

PROPOSITION 2.5. *Let N be a semiprime ideal of the right Noetherian ring R. Then the localization functor Q_N is right exact if and only if the N-closure $\tilde{N}$ of N in R_N is such that*

(1) *$R_N/\tilde{N}$ is a direct sum of a finite number of simple R_N-modules, and*

(2) *every simple R_N-module is isomorphic to one of these.*

If furthermore A is an R/N-module, then $Q_N(A)$ is a direct sum of simple R_N-modules.

Proof. We may assume that R is N-torsionfree.

First, suppose Q_N is right exact. Then it follows from Proposition 1.1 and Remark 1.2 that

$$R/N \subseteq R_N/\tilde{N} = (R/N)_N.$$

Let $U_1/N \oplus U_2/N \oplus \ldots \oplus U_d/N$ be a maximal direct sum of uniform submodules of R/N, hence an essential right ideal of R/N. By Lemma 2.1, $U_1 + \ldots + U_d$ contains an element of $\mathscr{C}(N)$, hence is N-dense in R, as it contains N. Therefore

$$\begin{aligned} R_N/\tilde{N} &= Q_N(R/N) \\ &= Q_N(U_1/N \oplus \ldots \oplus U_d/N) \\ &\cong Q_N(U_1/N) \oplus \ldots \oplus Q_N(U_d/N). \end{aligned}$$

We shall prove that the direct summands are simple R_N-modules.

Let U/N be a uniform submodule of R/N, we may as well assume it to be N-closed, and B any nonzero R_N-submodule of $Q_N(U/N)$. Since Q_N is exact, by Remark 1.2, B is torsionfree and divisible as an R-module, hence N-closed, and therefore $B \cap (U/N) = V/N$ is a nonzero N-closed submodule of U/N. By Proposition 2.4, V/N is a complement right ideal, hence $V = U$. Therefore, B contains U/N, and so $B = Q_N(B) = Q_N(U/N)$.

Now suppose A is any simple R_N-module. Since Q_N is exact, A is N-torsionfree as an R-module, hence $\mathrm{Hom}_R(A, E(R/N)) \neq 0$. But both A and $E(R/N)$ are N-torsionfree and N-divisible, hence we may write this $\mathrm{Hom}_{R_N}(A, E(R/N))$

$\neq 0$. Since

$$E(R/N) \cong E(U_1/N) \times \ldots \times E(U_d/N),$$

there exists $i \in \{1, \ldots, d\}$ such that $\mathrm{Hom}_{R_N}(A, E(U_i/N)) \neq 0$. Since A is simple, we may write $A \subseteq E(U_i/N)$, hence A meets $Q_N(U_i/N) \subseteq E(U_i/N)$. Since $Q_N(U_i/N)$ is simple, $A = Q_N(U_i/N)$.

We have thus shown that right exactness of Q_N implies (1) and (2). Conversely, assume (1) and (2). By (2), the injective hull of the R_N-module $R_N/\tilde{N}$ is a cogenerator of Mod R_N. If we can show it is N-torsionfree, it will follow that every R_N-module is N-torsionfree, hence that Q_N is exact.

In view of (1), it suffices to show that the injective hull of every simple R_N-module A is N-torsionfree as an R-module. Let T be its N-torsion submodule, we shall show that $T = 0$, using an argument due to Heinicke [**7**, p. 710].

Suppose $T \neq 0$, then $A \subseteq TR_N$. Therefore any element a of A could be written as $a = t_1q_1 + \ldots + t_nq_n$, where $t_i \in T$ and $q_i \in R_N$, and so we could find $D \in \mathscr{D}_N$ such that $aD \subseteq T$. But then the element a would be N-torsion, whereas we know from (2) that A is N-torsionfree.

Finally, to prove the last assertion of the proposition, let A be an R/N-module. Since $Q_N(A) = Q_N(A/T_N(A))$, we may assume that A is N-torsionfree. Then every nonzero submodule of A contains a uniform R/N-module U which is isomorphic to a uniform right ideal of R/N. (For, if $0 \neq a \in A$, aR being N-torsionfree, it easily follows from Lemma 2.1 that $a^{-1}0/N$ is not an essential right ideal of R/N. Pick a uniform right ideal V/N of R/N such that $a^{-1}0/N \cap V/N = 0$, then $U = a(V/N)$ is isomorphic to V/N.) By Zorn's Lemma, there is a maximal direct sum S of such uniform R/N-submodules U of A. Thus A is an essential extension of S, hence A/S is N-torsion. Since Q_N is exact, $Q_N(A)/Q_N(S) \cong Q_N(A/S) = 0$, hence $Q_N(A) = Q_N(S)$. Since Q_N commutes with direct sums, $Q_N(S)$ is the direct sum of the $Q_N(U)$, and these are simple R_N-modules, as above.

According to Gabriel [**2**, p. 415], a ring homomorphism $h : R \to R_\Sigma$ is called a (*classical*) *ring of right fractions* with respect to the multiplicative set Σ if and only if

(a) for all $r \in R$, $h(r) = 0 \Rightarrow$ there exists $\sigma \in \Sigma$ such that $r\sigma = 0$,

(b) for all $\sigma \in \Sigma$, $h(\sigma)$ is invertible,

(c) for all $q \in R_\Sigma$ there exists $\sigma \in \Sigma$ such that $qh(\sigma) \in h(R)$.

He showed that such an R_Σ exists (and is unique up to isomorphism) if and only if

(*) $\quad \forall_{r\in R}\forall_{\sigma\in\Sigma}\exists_{r'\in R}\exists_{\sigma'\in\Sigma}\, r\sigma' = \sigma r'$,

the so-called *right Ore condition* with respect to Σ, and

(**) $\quad \forall_{r\in R}(\exists_{\sigma\in\Sigma}\,\sigma r = 0 \Rightarrow \exists_{\sigma'\in\Sigma}\, r\sigma' = 0)$.

The following is well-known.

Remark 2.6. When R is right Noetherian, (*) implies (**).

https://doi.org/10.4153/CJM-1974-099-4 Published online by Cambridge University Press

Proof. Suppose $\sigma r = 0$. Pick the natural number n such that the right annihilator of σ^n is maximal. By (*), there exist $r' \in R$ and $\sigma' \in \Sigma$ such that $\sigma^n r' = r\sigma'$, hence $\sigma^{n+1} r' = \sigma r \sigma' = 0$. But the right annihilator of σ^{n+1} contains that of σ^n, hence coincides with it, and therefore $\sigma^n r' = 0$, whence $r\sigma' = 0$.

With the help of Proposition 2.5, we can now prove the main result of this section, which generalizes [**11**, Theorem 5.6] and Heinicke's result [**7**, Theorem 4.6].

THEOREM 2.7. *Let N be a semiprime ideal of the right Noetherian ring R, and let $\tilde{N}$ be the N-closure of N in R_N. Then the following statements are equivalent:*

(1) *R satisfies the right Ore condition with respect to $\mathscr{C}(N)$.*

(2) *Q_N is right exact and $\tilde{N}$ is a two-sided ideal of R_N.*

(3) *$\tilde{N}$ is the Jacobson radical of R_N and $R_N/\tilde{N}$ is a semisimple Artinian ring.*

(4) *N/cN is N-torsion for every $c \in \mathscr{C}(N)$.*

Furthermore, if these equivalent conditions hold, there is a commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & R_N \\ \downarrow & & \downarrow \\ R/N & \longrightarrow & R_N/\tilde{N} \cong Q_{cl}(R/N). \end{array}$$

Proof. We show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

Assume (1). By Remark 2.6, $R \to R_{\mathscr{C}(N)}$ is a classical ring of fractions of R with respect to $\mathscr{C}(N)$. It is easily seen that $R_{\mathscr{C}(N)} = R_N$. Let $D \in \mathscr{D}_N$, then D meets $\mathscr{C}(N)$, say $c \in D \cap \mathscr{C}(N)$. By condition (b) above, $h(c)$ is invertible in R_N, hence $R_N \subseteq ch(c)^{-1} R_N \subseteq DR_N$. By Remark 1.2, Q_N is exact. By Lemma 1.3, $NR_N = \tilde{N}$. An easy computation using the right Ore condition then shows that $\tilde{N}$ is a two-sided ideal.

Assume (2). By Proposition 2.5, $R_N/\tilde{N}$ is a finite direct sum of simple right R_N-submodules, hence it is a semisimple Artinian ring. Let M be any maximal right ideal of R_N, then, again by Proposition 2.5, R_N/M is isomorphic to a submodule of $R_N/\tilde{N}$. It follows that M is the kernel of some R_N-homomorphism $R_N \to R_N/\tilde{N}$. Suppose this homomorphism sends 1 onto $[q]$; then $M = \{q' \in R_N | [qq'] = 0\} = q^{-1}\tilde{N} \supseteq \tilde{N}$. Hence $\tilde{N}$ is contained in the Jacobson radical of R_N. Since $R_N/\tilde{N}$ is semiprimitive, $\tilde{N}$ is the Jacobson radical.

Assume (3). Then $R_N/\tilde{N}$ is the direct sum of a finite number of minimal right ideals, hence of simple R_N-modules. Let A be any simple R_N-module. By Nakayama's lemma, $A\tilde{N} \neq A$, hence $A\tilde{N} = 0$, and so A is also a simple $R_N/\tilde{N}$-module, hence is isomorphic to one of the direct summands of $R_N/\tilde{N}$. By Proposition 2.5, Q_N is right exact. Let $c \in \mathscr{C}(N)$; then $cR + N \in \mathscr{D}_N$, by Lemma 2.1, hence $cR_N + \tilde{N} = (cR + N)R_N = R_N$, by Lemma 1.3 and Remark 1.2. Since $\tilde{N}$ is the Jacobson radical, $cR_N = R_N$. Therefore $cR \in \mathscr{D}_N$, and so R/cR is N-torsion. But then so is $N/cN = N/(cR \cap N) \cong (cR + N)/cR$.

Assume (4), and take $c \in \mathscr{C}(N)$. Then $(cR + N)/cR \cong N/cN$ is N-torsion, and so is $R/(cR + N)$, by Lemma 2.1. Therefore R/cR is N-torsion, that is, for all $r \in R$, $r^{-1}(cR)$ meets $\mathscr{C}(N)$. This is just another way of stating the right Ore condition.

Finally, we invoke Proposition 1.1 and Lemma 2.3 to show that (2) implies that $R_N/\tilde{N} \cong Q_{cl}(R/N)$. The proof of the theorem is now complete.

As a first application of Theorem 2.7 we give a variation of the characterization of right orders in semilocal rings by Faith [**1**]. The ring R is called *semilocal* if it is semisimple Artinian modulo its Jacobson radical.

COROLLARY 2.8. *The right Noetherian ring R is a right order in a semilocal ring S if and only if there exists a semiprime ideal N of R such that $\mathscr{C}(N)$ is the set of all regular elements of R and R satisfies the right Ore condition with respect to $\mathscr{C}(N)$. Moreover, N is then the intersection of R with the Jacobson radical of S.*

Proof. In view of Theorem 2.7, the two conditions are clearly sufficient. Conversely, let R be a right order in a semilocal ring S with Jacobson radical J. Put $N = J \cap R$, then N is a semiprime ideal of R, since every prime ideal of S intersects R in a prime ideal of R [**5**, (2.18), p. 247].

Suppose $c \in \mathscr{C}(N)$ and $cq \in J$, where $qd \in R$ for some regular element d of R. Then $cqd \in J \cap R = N$, hence $qd \in N$ and therefore $q \in J$. Since S/J is semisimple Artinian, it follows that $[c]$ is a unit in S/J, and therefore c is a unit in S, hence regular in R. Thus all elements of $\mathscr{C}(N)$ are regular.

Conversely, if c is regular in R, then $[c]$ is a unit in S/J. If $cr \in N \subseteq J$, then $r \in J \cap R = N$. Thus $c \in \mathscr{C}(N)$. Therefore, all regular elements are in $\mathscr{C}(N)$.

3. Artinian localization. Given a semiprime ideal N of the right Noetherian ring R, we shall investigate when the ring R_N of right quotients at N is right Artinian. As a corollary, we will obtain Small's theorem when R is a right order in a right Artinian ring.

First, we require a well-known lemma.

LEMMA 3.1. *Let R be a right Noetherian ring, A a two-sided ideal of R, and $c \in \mathscr{C}(0)$ a right regular element of R. Then cA is essential in A.*

Proof. Otherwise A contains $cA + B$, where $cA \cap B = 0$ and $B \neq 0$. But cA is isomorphic to A and contains $c^2A + cB$, where $c^2A \cap cB = 0$ and $cB \neq 0$. We continue in this manner and obtain an "infinite" direct sum $B + cB + c^2B + \ldots$, which would violate the maximal condition.

PROPOSITION 3.2. *Let R be a right Noetherian ring, N a semiprime ideal, and assume that there exists an ideal $K \subseteq N$ such that $\mathscr{C}(N) \subseteq \mathscr{C}(K)$ and a natural number m such that $N^m \subseteq K$. Then R satisfies the right Ore condition with respect to $\mathscr{C}(N)$ if and only if K/cK is N-torsion for each $c \in \mathscr{C}(N)$.*

Proof. Let us write

$$N_k = \{n \in N | nN^{m-k} \subseteq K\}$$

for $k = 1, \ldots, m$. Then clearly $N_1 = N$ and $N_m = K$.

Take any $c \in \mathscr{C}(N)$ and suppose $cr \in N_k$. Then $cr \in N$ (hence $r \in N$) and $cr \in N^{m-k} \subseteq K$, so that $rN^{m-k} \subseteq K$, that is, $r \in N_k$. Thus $\mathscr{C}(N) \subseteq \mathscr{C}(N_k)$.

Now let us look at condition (4) of Theorem 2.7. N/cN is N-torsion if and only if each of $N/(N_2 + cN)$ and $(N_2 + cN)/cN \cong N_2/(cN \cap N_2) = N_2/cN_2$ are N-torsion.

Similarly, N_2/cN_2 is N-torsion if and only if each of $N_2/(N_3 + cN_2)$ and N_3/cN_3 are N-torsion. Iterating this argument, we reduce the problem to showing that all $N_k/(N_{k+1} + cN_k)$ and $N_m/cN_m = K/cK$ are N-torsion.

Let $D_k = N_{k+1} + cN_k$. We claim that N_k/D_k is N-torsion for each $k = 1, \ldots, m - 1$. Take any $n \in N_k$, we wish to show that $n^{-1}D_k$ meets $\mathscr{C}(N)$. In view of Lemma 2.1, it suffices to show that $n^{-1}D_k/N$ is essential in R/N. Note that

$$nN \subseteq N_kN \subseteq N_{k+1} \subseteq D_k \subseteq N_k \subseteq N \subseteq n^{-1}D_k \subseteq R.$$

Given $r \notin N$, we seek $s \in R$ such that $rs \notin N$ and $nrs \in D_k$.

Since c is a right regular element modulo N_{k+1}, Lemma 3.1 tells us that cN_k/N_{k+1} is essential in N_k/N_{k+1}. Hence D_k/N_{k+1} is essential in N_k/N_{k+1}. Thus, if $nr \notin N_{k+1}$, we can find $s \in R$ such that $nrs \notin N_{k+1}$, hence $rs \notin N$ and $nrs \in D_k$. If however $nr \in N_{k+1}$, we just take $s = 1$.

The proof is now complete.

THEOREM 3.3. *Let R be right Noetherian, N a semiprime ideal of R. Then R_N is a right Artinian classical ring of right fractions of R with respect to $\mathscr{C}(N)$, if and only if*

(a) *some power of N is N-torsion, and*

(b) *$\mathscr{C}(N)$ satisfies*

$$(**) \quad \forall_{r \in R}(\exists_{c \in \mathscr{C}(N)} cr = 0 \Rightarrow \exists_{c' \in \mathscr{C}(N)} rc' = 0).$$

Proof. First, we show the sufficiency of the conditions. Suppose $c \in \mathscr{C}(N)$, $r \in R$, and $cr \in T_N(R)$. Then, for each $s \in R$, there exists $c' \in \mathscr{C}(N)$ such that $crsc' = 0$. By (**), there exists $c'' \in \mathscr{C}(N)$ such that $rsc'c'' = 0$, hence $r \in T_N(R)$. Thus $\mathscr{C}(N) \subseteq \mathscr{C}(T_N(R))$.

Take $K = T_N(R)$. Then $N^m \subseteq K$ for some natural number m. Moreover, K is N-torsion. Therefore, R satisfies the right Ore condition with respect to $\mathscr{C}(N)$, by Proposition 3.2. Thus $R \to R_N$ is a classical ring of right fractions with respect to $\mathscr{C}(N)$, in view of Remark 2.6.

Furthermore, Theorem 2.7 asserts that $\tilde{N}$ is the Jacobson radical of R_N and $R_N/\tilde{N}$ is semisimple Artinian. Since Q_N is right exact, R_N is right Noetherian and $\tilde{N} = NR_N$. Thus $\tilde{N}$ is nilpotent, and so R_N is right Artinian, by Hopkins' Theorem.

Conversely, assume that $h : R \to R_N$ is a classical ring of right fractions with respect to $\mathscr{C}(N)$ and that R_N is right Artinian.

Then R satisfies Gabriel's condition (**) as in § 2. Furthermore, $h(N) = h(R) \cap \tilde{N}$ is nilpotent. Thus $N^m \subseteq T_N(R)$ for some natural number m.

Theorem 3.3 contains the crux of Small's Theorem [**17**; **18**] and could have been proved with the help of the latter, which may be stated as follows:

COROLLARY 3.4. *Let R be a right Noetherian ring with prime radical N. Then R is a right order in a right Artinian ring if and only if every element of $\mathscr{C}(N)$ is regular in R.*

Proof. The necessity of the condition is an immediate consequence of Corollary 2.8. Conversely, suppose every element of $\mathscr{C}(N)$ is regular in R. Then clearly R is N-torsionfree. By Levitski's Theorem, $N^k = 0$ for some positive integer k. Therefore, Theorem 3.3 applies, $h : R \to R_N$ is a classical ring of right fractions of R with respect to $\mathscr{C}(N)$, h is injective and R_N is right Artinian. It will follow that $R_N = Q_{cl}(R)$ if we show that all regular elements of R are in $\mathscr{C}(N)$.

Suppose c is a regular element of R. Then c is right regular in R_N. As R_N is right Artinian, c is a right unit in R_N, that is, $cq = 1$ for some $q \in R_N$. Since c is right regular in R_N, also $qc = 1$. Suppose $cr \in N$, then $r = qcr \in R_N N \subseteq \tilde{N}$, hence $r \in \tilde{N} \cap R = N$. Thus $c \in \mathscr{C}(N)$.

An immediate consequence of Corollary 3.4 is the following:

COROLLARY 3.5. *Let R be a right Noetherian ring with prime radical N. Suppose R is N-torsionfree and satisfies the right Ore condition with respect to $\mathscr{C}(N)$. Then R satisfies the right Ore condition with respect to the set of all regular elements of R.*

COROLLARY 3.6. *Let N be a semiprime ideal of the right Noetherian ring R and suppose that some power of N is N-torsion. If Q_N is exact then R_N is right Artinian.*

Proof. We may assume that R is N-torsionfree. Let σ_N denote N-closure in R_N, then

$$R_N \supseteq \sigma_N(N) = \tilde{N} \supseteq \sigma_N(N^2) \supseteq \ldots \supseteq \sigma_N(N^m) = \sigma_N(0) = 0$$

for some natural number m. Consider

$$A_k = (N^k + \sigma_N(N^{k+1}))/\sigma_N(N^{k+1}) \cong N^k/(N^k \cap \sigma_N(N^{k+1})).$$

This is a finitely generated R/N-module, it is N-torsionfree, and its N-closure is

$$Q_N(A_k) = \sigma_N(A_k) = \sigma_N(N^k)/\sigma_N(N^{k+1}).$$

By Proposition 2.5, this is a direct sum of simple R_N-modules, hence we obtain a composition series for R_N.

4. Classical semilocal rings. If N is an ideal of R and $I = E(R/N)$, we shall be comparing two topologies on an R-module G:

(a) the N-adic topology, which has a fundamental system of open neighborhoods of zero consisting of submodules of the form GN^n, n any natural number,

(b) the I-adic topology, which has a fundamental system of open neighborhoods of zero consisting of kernels of homomorphisms $f : G \to I^n$, n finite.

Before stating the main result of this section, we require two lemmas:

LEMMA 4.1. *Let R be a right Noetherian ring, N an ideal such that R/N is semisimple Artinian, and $I = E(R/N)$. Then, on any finitely generated R-module G, the N-adic topology is contained in the I-adic topology.*

Proof. We claim that $GN^n \in \mathscr{F}$, the class of all R-modules isomorphic to submodules of finite powers of I. Since $\mathscr{F}$ is closed under module extensions, it suffices to show that $GN^k/GN^{k+1} \in \mathscr{F}$, for $k = 0, \ldots, n-1$. Put $H = GN^k$; then H/HN is an R/N-module, hence a finite direct sum of minimal right ideals of R/N. Since $R/N \subseteq I$, $H/HN \subseteq I^r$.

The following is the same as [**12**, Lemma 4], but we give the proof for completeness.

LEMMA 4.2. *Suppose N is an ideal of R and every finitely generated right ideal of R is closed in the N-adic topology. Then N is small.*

Proof. Suppose E is any right ideal of R such that $N + E = R$. Without loss in generality, we may take E to be finitely generated. Now $N = RN = N^2 + EN$, hence $N^2 + E = N^2 + EN + E = N + E = R$. Similarly $N^3 + E = R$, and so on. Hence the N-adic closure $\bigcap_{n=1}^{\infty}(E + N^n)$ of E is also R. Since E is closed, $E = R$.

PROPOSITION 4.3. *Let R be right Noetherian, N an ideal of R such that R/N is semisimple Artinian, $I = E(R/N)$. Then the following statements are equivalent:*

(a) *For any right ideal E of R there exists a natural number n such that $E \cap N^n \subseteq EN$.*

(b) *For every element $i \in I$ there exists a natural number n such that $iN^n = 0$.*

(c) *On every finitely generated right R-module the N-adic and I-adic topologies coincide.*

Moreover, these equivalent conditions together with the assertion that N is the Jacobson radical of R are equivalent to the following:

(d) *Every right ideal of R is closed in the N-adic topology.*

Definition. A semilocal ring R with Jacobson radical N satisfying the equivalent conditions (a) to (d) above will be called a *classical right semilocal ring.*

Proof. Assume (a). Let $i \in I$, put $E = \{r \in R | irN = 0\}$, and pick n such that $E \cap N^n \subseteq EN$. Suppose $iN^n \neq 0$, then iN^n meets R/N, hence there exists $r \in N^n$ such that $0 \neq ir \in R/N$. But then $irN = 0$, hence $r \in E \cap N^n \subseteq EN$, and so $ir = 0$, a contradiction. Thus (a) $\Rightarrow$ (b).

Assume (b). Let G be a finitely generated right R-module, $f : G \to I^n$ any R-homomorphism, $p_k : I^n \to I$ the canonical projections for $k = 1, \ldots, n$.

Then $p_k f(G)N^{m(k)} = 0$ for some $m(k)$. Let $m = \max\{m(1), \ldots, m(n)\}$; then $f(G)N^m = 0$, hence $\operatorname{Ker} f$ contains GN^m. Therefore the I-adic topology on G is contained in the N-adic one. The converse is true by Lemma 4.1. Thus (b) $\Rightarrow$ (c).

Assume (c). Let E be any right ideal of R. Since R is right Noetherian, E is finitely generated. Now EN is an open subset of E in the N-adic topology, hence in the I-adic topology. Now the I-adic topology of any submodule of R is induced by that of R, hence $EN = E \cap V$, where V is an open subset of R in the I-adic topology, hence in the N-adic topology. Therefore $N^n \subseteq V$ for some n, and so $E \cap N^n \subseteq E \cap V \subseteq EN$. Thus (c) $\Rightarrow$ (a).

Assume (d). Since R/N is right Artinian, the finitely generated R/N-module E/EN is also Artinian. Pick n such that

$$((E \cap N^n) + EN)/EN = (E \cap (N^n + EN))/EN$$

is minimal. Now EN is closed, hence

$$EN = \bigcap_{k=1}^{\infty} (N^k + EN).$$

Therefore

$$((E \cap N^n) + EN)/EN = (E \cap EN)/EN = 0,$$

hence

$$E \cap N^n \subseteq E \cap (N^n + EN) \subseteq EN.$$

Thus (d) $\Rightarrow$ (a).

Clearly, N contains the Jacobson radical of R. Therefore, by Lemma 4.2, it is the Jacobson radical, when (d) holds.

Assume (a) and suppose that N is the Jacobson radical. Then (d) follows as in [**12**, Theorem (5), (*) $\Rightarrow$ (**)]. Indeed, let F be any right ideal and E its N-adic closure. Pick n as in (a); then

$$E \subseteq (F + N^n) \cap E = F + (N^n \cap E) \subseteq F + EN,$$

hence $E/F \subseteq (E/F)N$, and so $E = F$, by Nakayama's Lemma.

Note that all the assumptions of Proposition 4.3 are used only in the implication (b) $\Rightarrow$ (c). (c) $\Rightarrow$ (d) depends only on R being right Noetherian, (d) $\Rightarrow$ (a) also on R/N being right Artinian, and (a) $\Rightarrow$ (b) holds without any assumptions whatsoever.

5. Semiprime ideals with the Artin-Rees property. Before turning to the main theme of this section, we shall establish the following:

LEMMA 5.1. *Let R be a right Noetherian ring, N a semiprime ideal such that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$. Then, for each natural number k, $\tilde{N}^k/h(N^k)$ is N-torsion.*

https://doi.org/10.4153/CJM-1974-099-4 Published online by Cambridge University Press

Proof. For $k = 1$ this follows from the definition of $\tilde{N}$. Assume the result for k; we shall prove it for $k + 1$. Given any element $q \in \tilde{N}^{k+1}$, we seek an element $c \in \mathscr{C}(N)$ such that $qc \in h(N^{k+1})$. Without loss in generality, we may consider $q = q_1q_2$, where $q_1 \in \tilde{N}^k$ and $q_2 \in \tilde{N}$. (For $\mathscr{D}_N$ is closed under finite intersections.) By inductional assumption, $q_1c_1 \in h(N^k)$. Also $q_2c_2 = h(n)$, $n \in N$. By the right Ore condition, we can find $c' \in \mathscr{C}(N)$ and $n' \in R$ such that $c_1n' = nc' \in N$, hence also $n' \in N$. Therefore

$$qc_2c' = q_1nc' = q_1c_1n' \in h(N^k)N = h(N^{k+1}).$$

Given a semiprime ideal N in the right Noetherian ring R, we define a closure operation ρ_N on right ideals and a closure operation λ_N on left ideals.

For any right ideal E, $\rho_N E/E = T_N(R/E)$. In other words, $\rho_N E = \{r \in R | r^{-1}E \in \mathscr{D}_N\}$.

For any left ideal F, $\lambda_N F$ is the smallest left ideal F' containing F such that

$$\mathscr{C}(N) \subseteq \mathscr{C}(F') = \{c \in R | \forall_{r \in R} cr \in F' \Rightarrow r \in F'\}.$$

LEMMA 5.2. *Let R be right Noetherian, N a semiprime ideal, and A any ideal of R. Then $\rho_N A$ and $\lambda_N A$ are ideals, and $\lambda_N\rho_N\lambda_N A = \rho_N\lambda_N A$. If N has the right Ore property, then $\rho_N\lambda_N A = \rho_N A$.*

Proof. (1) Suppose $r \in \rho_N A$, $s \in R$. Then there exists $D \in \mathscr{D}_N$ such that $rD \subseteq A$, hence $srD \subseteq A$, hence $sr \in \rho_N A$.

(2) Let $B = \{r \in R | rR \subseteq \lambda_N A\}$, then B is an ideal and $A \subseteq B \subseteq \lambda_N A$. We will show that $\mathscr{C}(N) \subseteq \mathscr{C}(B)$. Indeed, suppose $c \in \mathscr{C}(N)$, $r \in R$, and $cr \in B$. Then $crR \subseteq \lambda_N A$, hence $rR \subseteq \lambda_N A$, hence $r \in B$. Thus $\lambda_N A = B$.

(3) We will show that $\mathscr{C}(N) \subseteq \mathscr{C}(\rho_N\lambda_N A)$. Suppose $c \in \mathscr{C}(N)$, $r \in R$, $cr \in \rho_N\lambda_N A$. Then, for all $s \in R$, there exists $c' \in \mathscr{C}(N)$ such that $crsc' \in \lambda_N A$, hence $rsc' \in \lambda_N A$, hence $r \in \rho_N\lambda_N A$. Therefore $c \in \mathscr{C}(\rho_N\lambda_N A)$.

(4) Now assume the right Ore condition for $\mathscr{C}(N)$. We will first show that $\mathscr{C}(N) \subseteq \mathscr{C}(\rho_N A)$.

For this argument we may as well assume that $A = \rho_N A$. Suppose $c \in \mathscr{C}(N)$, $r \in R$, and $cr \in A$. Pick a natural number n such that $c^{-n}A = \{r \in R | c^n r \in A\}$ is maximal. By the Ore condition, there exists $c' \in \mathscr{C}(N)$ and $r' \in R$ such that $c^n r' = rc'$, hence $c^{n+1}r' = crc' \in A$. But $c^{-(n+1)}A = c^{-n}A$, hence $c^n r' \in A$, that is, $rc' \in A$. Thus $r \in \rho_N A$.

Now it follows that $\lambda_N\rho_N A = \rho_N A$. Therefore $\lambda_N A \subseteq \rho_N A$, hence $\rho_N\lambda_N A \subseteq \rho_N A$, and our proof is complete.

Part (4) of the above proof is essentially the same as that of the implication (*) $\Rightarrow$ (**) in Remark 2.6.

THEOREM 5.3. *Let R be a right Noetherian ring, N a semiprime ideal, $I = E(R/N)$. Then the following statements are equivalent:*

(1) *For each right ideal E of R there exists a natural number n such that $E \cap \lambda_N N^n \subseteq \rho_N(EN)$.*

(2) *For each element* $i \in I$ *there exists a natural number* n *such that* $i\lambda_N N^n = 0$.

(3) $\tilde{N}$ *is an ideal of* R_N *and* $(R_N, \tilde{N})$ *is a classical right semilocal ring.*

Moreover these conditions imply that R *satisfies the right Ore condition with respect to* $\mathscr{C}(N)$.

One might put condition (1) into words by saying that N has the *right symbolic Artin-Rees property.*

Proof. Assume (1). Let V be an essential submodule of I for which $VN = 0$, and let $E = i^{-1}V = \{r \in R | ir \in V\}$. Pick n such that $E \cap \lambda_N N^n \subseteq \rho_N(EN)$. Suppose $i\lambda_N N^n \neq 0$; then $i\lambda_N N^n \cap V \neq 0$. Thus there exists $r \in \lambda_N N^n$ such that $0 \neq ir \in V$, hence $r \in \lambda_N N^n \cap E \subseteq \rho_N(EN)$. Therefore there exists a right ideal D such that R/D is N-torsion and $rD \subseteq EN$, hence $irD = 0$, and so $ir = 0$, a contradiction. Thus (1) $\Rightarrow$ (2).

Assume (2). We shall prove first of all that N has the right Ore property. Take any $c \in \mathscr{C}(N)$; we wish to show that R/cR is torsion. Suppose $icR = 0$, we will show that $iR = 0$. We know that $i(cR + \lambda_N N^n) = 0$, so it suffices to show that $cR + \lambda_N N^n \in \mathscr{D}_N$.

Consider the ring $R/\lambda_N N^n$. Clearly N is nilpotent modulo $\lambda_N N^n$. Moreover, all elements of $\mathscr{C}(N)$ are right regular modulo $\lambda_N N^n$, that is, all elements of $\mathscr{C}(N/\lambda_N N^n)$ are right regular. By Corollary 3.4 (Small's Theorem), $N/\lambda_N N^n$ has the right Ore property. Take any $r \in R$; then there exist $c' \in \mathscr{C}(N)$ and $r' \in R$ such that $cr' - rc' \in \lambda_N N^n$. Therefore $r^{-1}(cR + \lambda_N N^n)$ meets $\mathscr{C}(N)$, as was to be shown.

Thus N has the right Ore property. By Theorem 2.7, $\tilde{N}$ is the Jacobson radical and $R_N/\tilde{N}$ is semisimple Artinian. Also R_N is right Noetherian. It remains to verify one of the equivalent conditions of Proposition 4.3 for $(R_N, \tilde{N})$. We shall verify condition (b).

Let I' be the injective hull of $R_N/\tilde{N}$ as an R_N-module. It is an essential extension of $R_N/\tilde{N}$ also as an R-module, hence $I' \subseteq I = E(R/N)$. Take any $i \in I'$; then by (2) there exists a natural number n such that $i\lambda_N N^n = 0$. *A fortiori*, $iN^n = 0$. But, by Lemma 5.1, $\tilde{N}^n/h(N^n)$ is N-torsion, hence $i\tilde{N}^n = 0$. Thus (2) $\Rightarrow$ (3).

Assume (3). Then $\tilde{N}$ is the Jacobson radical of R_N, by Proposition 4.3, and R satisfies the right Ore condition with respect to $\mathscr{C}(N)$, by Theorem 2.7.

Let E be any right ideal of R. By Lemma 1.3, $\tilde{E} = ER_N$, $\tilde{N} = NR_N$, $\widetilde{EN} = ENR_N$. By (3), there exists n such that

$$\tilde{E} \cap \tilde{N}^n \subseteq \tilde{E}\tilde{N} = ER_N NR_N = \widetilde{EN}.$$

For the moment, let σ_N denote "N-closure in R_N", that is, for any right ideal A of R_N, $\sigma_N A/A = T_N(R_N/A)$. Then clearly

$$\sigma_N(A \cap B) = \sigma_N A \cap \sigma_N B,\ h^{-1}\sigma_N A = \rho_N h^{-1}A,\ \tilde{E} = \sigma_N hE.$$

Thus we have

$$\tilde{E} \cap \sigma_N \tilde{N}^n \subseteq \sigma_N(\tilde{E} \cap \tilde{N}^n) \subseteq \sigma_N \widetilde{EN} = \widetilde{EN}.$$

Apply h^{-1} to this and note that

$$E \subseteq \rho_N E = \rho_N h^{-1}\tilde{E} = h^{-1}\sigma_N \tilde{E} = h^{-1}\tilde{E}$$

and, by Lemma 5.2, that

$$\lambda_N N^n \subseteq \rho_N \lambda_N N^n = \rho_N N^n \subseteq \rho_N h^{-1}\tilde{N}^n = h^{-1}\sigma_N \tilde{N}^n,$$

hence

$$E \cap \lambda_N N^n \subseteq h^{-1}(E\tilde{N}) = h^{-1}\sigma_N(E\tilde{N}) = \rho_N h^{-1}(E\tilde{N}) = \rho(EN).$$

Thus (3) $\Rightarrow$ (1).

Remark 5.4. Condition (3) of Theorem 5.3 can be relaxed as follows:

(3′) *$\tilde{N}$ is an ideal, $R_N/\tilde{N}$ is semisimple Artinian and, for every finitely generated right ideal E of R_N, there exists a natural number n such that* $E \cap \tilde{N}^n \subseteq \widetilde{EN}$.

This is almost the same as (3), except that we do not assert that R_N is right Noetherian, and that $\tilde{N}$ is the Jacobson radical of R_N. We shall deduce from (3′) that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$.

Put $N_k = h^{-1}(\tilde{N}^k)$. We claim that N/N_k has the right Ore property in R/N_k. This follows from 3.4, since N/N_k is clearly nilpotent (because $N^k \subseteq N_k$) and semiprime, and since $\mathscr{C}(N/N_k) \subseteq \mathscr{C}(0)$, as we shall now show.

Suppose $[c] \in \mathscr{C}(N/N_k)$, then one easily calculates that $c \in \mathscr{C}(N)$. We prove by induction on k that $R_N c + \tilde{N}^k = R_N$. This is so for $k = 1$, since $R_N/\tilde{N} = Q_{cl}(R/N)$, by Proposition 2.3. Assume the result for k; then

$$R_N = R_N c + \tilde{N}^k = R_N c + \tilde{N}^k(R_N c + \tilde{N}) = R_N c + \tilde{N}^{k+1}.$$

Now suppose $r \in R$ and $[c][r] = 0$, that is $cr \in N_k$, then $h(r) \in R_N cr + \tilde{N}^k \subseteq \tilde{N}^k$, hence $r \in N_k$, and so $[r] = 0$. Thus $[c] \in \mathscr{C}(0)$, as claimed.

We shall now establish the right Ore condition for $\mathscr{C}(N)$ in R. Let $r \in R$, $c \in \mathscr{C}(N)$. By the above, for each natural number k, there exist $r_k \in R$, $c_k \in \mathscr{C}(N)$ such that $rc_k - cr_k = u_k \in N_k$. Let F be the right ideal of R generated by the u_k. Since R is right Noetherian, there exists a natural number m such that

$$F = u_1 R + \ldots + u_m R.$$

By (3′), there exists a natural number n such that

$$FR_n \cap \tilde{N}^n \subseteq F\tilde{N}.$$

Now $h(u_n) \in FR_N \cap \tilde{N}^n$, hence

$$h(u_n) = \sum_{k=1}^{m} u_k q_k$$

where $q_k \in \tilde{N}$. Pick $d \in \mathscr{C}(N)$ such that all $q_k d \in \tilde{N} \cap h(R) = h(N)$, say $q_k d = h(n_k)$, $n_k \in N$.

Now, following an idea of Goldie's, put

$$c' = c_n d - \sum_{k=1}^{m} c_k n_k, \quad r' = r_n d - \sum_{k=1}^{m} r_k n_k.$$

Then $h(cr' - rc') = 0$, hence there exists $d' \in \mathscr{C}(N)$ such that $(cr' - rc')d' = 0$, that is, $c(r'd') = r(c'd')$, and our proof is complete.

References

1. C. Faith, *Orders in semilocal rings*, Bull. Amer. Math. Soc. *77* (1971), 960–962.
2. P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France *90* (1962), 323–448.
3. A. W. Goldie, *Semiprime rings with maximum condition*, Proc. London Math. Soc. *10* (1960), 201–220.
4. ——— *Localization in non-commutative rings*, J. Algebra *5* (1967), 89–105.
5. ——— *The structure of Noetherian rings*, Springer Verlag, Lecture Notes in Mathematics *246* (1972), 214–321.
6. O. Goldman, *Rings and modules of quotients*, J. Algebra *13* (1969), 10–47.
7. A. G. Heinicke, *On the ring of quotients at a prime ideal of a right Noetherian ring*, Can. J. Math. *24* (1972), 703–712.
8. J. Lambek, *Lectures on rings and modules* (Waltham, Toronto, London, 1966).
9. ——— *Torsion theories, additive semantics and rings of quotients*, Springer Verlag, Lecture Notes in Mathematics *177* (Berlin, Heidelberg, New York, 1971).
10. ——— *Bicommutators of nice injectives*, J. Algebra *21* (1972), 60–73.
11. J. Lambek and G. Michler, *The torsion theory at a prime ideal of a right Noetherian ring*, J. Algebra *25* (1973), 364–389.
12. ——— *Completions and classical localizations of right Noetherian rings*, Pacific J. Math. *48* (1973), 133–140.
13. L. Lesieur and R. Croisot, *Extension au cas non-commutatif d'un théorème de Krull et d'un lemme d'Artin-Rees*, J. Reine Angew. Math. *204* (1960,) 216–220.
14. E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. *8* (1958), 511–528.
15. ——— *Some properties of Noetherian domains of Dimension 1*, Can. J. Math. *13* (1967), 569–586.
16. G. Michler, *Right symbolic powers and classical localization in right Noetherian rings*, Math. Z. *127* (1972), 57–69.
17. L. Small, *Orders in Artinian rings*, J. Algebra *4* (1966), 13–41.
18. ——— *Orders in Artinian rings, II*, J. Algebra *9* (1968), 266–273.
19. ——— *The embedding problem for Noetherian rings*, Bull. Amer. Math. Soc. *75* (1969), 147–148.
20. C. L. Walker and E. Walker, *Quotient categories and rings of quotients*, Rocky Mountain J. Math. *2* (1972), 513–555.

McGill University,
Montreal, Quebec;
Justus Liebig Universität,
Giessen, Germany