Canad. Math. Bull. Vol. **23** (4), 1980

ARITHMETIC PROGRESSIONS CONTAINED IN SEQUENCES WITH BOUNDED GAPS

BY
MELVYN B. NATHANSON

Van der Waerden [1, 4, 5] proved that if the nonnegative integers are partitioned into a finite number of sets, then at least one set in the partition contains arbitrarily long finite arithmetic progressions. This is equivalent to the result that a strictly increasing sequence of integers with bounded gaps contains arbitrarily long finite arithmetic progressions. Szemerédi [3] proved the much deeper result that a sequence of integers of positive density contains arbitrarily long finite arithmetic progressions. The purpose of this note is a quantitative comparison of van der Waerden's theorem and sequences with bounded gaps.

Denote by $[a, b)$ (resp. $[a, b]$) the interval of integers $a \leq n < b$ (resp. $a \leq n \leq b$). Let $k \geq 3$ and $r \geq 2$.

Let $W(k, r)$ denote the least integer such that, if $w \geq W(k, r)$ and if $[0, w) = \bigcup_{j=0}^{r-1} A_j$ is a partition into r pairwise disjoint sets, then at least one of the sets A_j contains a k-term arithmetic progression. The theorem of van der Waerden asserts the existence of the numbers $W(k, r)$.

Let $M(k, r)$ denote the least integer such that if $m \geq M(k, r)$ and if $A = \{a_n\}_{n=0}^{m-1}$ is a sequence with $a_n \in [nr, (n+1)r)$ for $n \in [0, m)$, then A contains a k-term arithmetic progression.

Let $G(k, r)$ denote the least integer such that, if $g \geq G(k, r)$ and if $A = \{a_n\}_{n=0}^{g-1}$ is a strictly increasing sequence of integers with bounded gaps $a_n - a_{n-1} \leq r$ for $n \in [1, g)$, then A contains a k-term arithmetic progression.

The existence of the numbers $M(k, r)$ and $G(k, r)$ follows from van der Waerden's theorem (Rabung [2]) and also from the results below.

THEOREM 1. $M(k, r) \leq W(k, r) \leq M((k-1)r+1, r)$.

Proof. Let $w = W(k, r)$ and let $A = \{a_n\}_{n=0}^{w-1}$ satisfy $a_n \in [nr, (n+1)r)$ for $n \in [0, w)$. Then $a_n = nr + \varepsilon_n$, where $\varepsilon_n \in [0, r)$. Partition $[0, w)$ into r sets $A_0, A_1, \ldots, A_{r-1}$ as follows: $n \in A_j$ if and only if $\varepsilon_n = j$. Since $w = W(k, r)$, it follows that some set in the partition contains a k-term arithmetic progression. Suppose that $n_i = c + id \in A_t$ for some $d \geq 1$ and all $i \in [0, k)$. Then $\varepsilon_{n_i} = t$ and

$$a_{n_i} = n_i r + \varepsilon_{n_i} = (c + id)r + t = (cr + t) + idr$$

Received by the editors July 9, 1979.

https://doi.org/10.4153/CMB-1980-074-x Published online by Cambridge University Press

for $i \in [0, k)$, hence $a_{n_0} < a_{n_1} < \cdots < a_{n_{k-1}}$ is a k-term arithmetic progression in A. Thus, $M(k, r) \leq W(k, r)$.

Conversely, let $m = M((k-1)r+1, r)$ and let $[0, m) = \bigcup_{j=0}^{r-1} A_j$ be a partition into r sets. For $n \in [0, m)$, define $\varepsilon_n \in [0, r)$ by $\varepsilon_n = j$ if and only if $n \in A_j$. Construct the sequence $A = \{a_n\}_{n=0}^{m-1}$ by setting $a_n = nr + \varepsilon_n \in [nr, (n+1)r)$. Since $m = M((k-1)r+1, r)$, it follows that A contains an arithmetic progression of length $(k-1)r+1$, say, $a_{n_0} < a_{n_1} < \cdots < a_{n_{(k-1)r}}$, where $a_{n_i} - a_{n_{i-1}} = d$ for some $d \geq 1$ and all $i \in [1, (k-1)r]$. Then $a_{n_0} < a_{n_r} < a_{n_{2r}} < \cdots < a_{n_{(k-1)r}}$ is a k-term arithmetic progression with difference dr, and

$$dr = a_{n_{ir}} - a_{n_{(i-1)r}} = (n_{ir} - n_{(i-1)r})r + \varepsilon_{n_{ir}} - \varepsilon_{n_{(i-1)r}}$$

for all $i \in [1, k)$. This implies that $\varepsilon_{n_{ir}} = \varepsilon_{n_{(i-1)r}}$ and so $n_{ir} - n_{(i-1)r} = d$ for $i \in [1, k)$. If $\varepsilon_{n_0} = t$, then $\varepsilon_{n_{ir}} = t$ for $i \in [0, k)$, and so $n_0 < n_r < n_{2r} < \cdots < n_{(k-1)r}$ is a k-term arithmetic progression contained in the set A_t. Thus, $W(k, r) \leq M((k-1)r+1, r)$.

THEOREM 2. $G(k, r) \leq rM(k, r)$.

Proof. Let $m = M(k, r)$. Let $B = \{b_i\}_{i=0}^{mr-1}$ be a strictly increasing sequence of mr integers with bounded gaps $b_i - b_{i-1} \leq r$ for $i \in [1, mr)$. Replacing each b_i by $b_i - b_0$, we can assume without loss of generality that $b_0 = 0$. Then for each $n \in [0, m)$, the interval $[nr, (n+1)r)$ contains at least one element of B. Choose $a_n \in B \cap [nr, (n+1)r)$. Since $m = M(k, r)$, the sequence $\{a_n\}_{n=0}^{m-1}$ contains a k-term arithmetic progression. Thus, B contains a k-term arithmetic progression, and so $G(k, r) \leq rM(k, r)$.

THEOREM 3. $M(k, r) \leq G(k, 2r-1)$.

Proof. Let $g = G(k, 2r-1)$ and let $A = \{a_n\}_{n=0}^{g-1}$ satisfy $a_n \in [nr, (n+1)r)$. Then A has bounded gaps $a_n - a_{n-1} \leq 2r-1$. Thus, A contains a k-term arithmetic progression and $M(k, r) \leq G(k, 2r-1)$.

THEOREM 4. $G(k, r) \leq W(k, r) \leq G((k-1)r+1, 2r-1)$.

Proof. Let $w = W(k, r)$ and let $A = \{a_i\}_{i=0}^{w-1}$ be a strictly increasing sequence of integers satisfying $a_i - a_{i-1} \leq r$ for $i \in [1, w)$. Replacing each a_i by $a_i - a_0$, we can assume that $a_0 = 0$. Clearly, $a_{w-1} \geq w-1$. Partition the interval $[0, w)$ into r pairwise disjoint sets $A_0, A_1, \ldots, A_{r-1}$ as follows: $n \in A_j$ if and only if $j = \min\{a_i - n \mid a_i \geq n\}$. If $n \in [0, w)$, then $a_i - n \in [0, r)$ for some $a_i \in A$. If $n \in A_t$, then $t + n = a_i$ for some $a_i \in A$. Since $w = W(k, r)$, at least one set in the partition $[0, w) = \bigcup_{j=0}^{r-1} A_j$ contains a k-term arithmetic progression. If $c + id \in A_t$ for $i \in [0, k)$, then $(t+c) + id = a_i \in A$ is a k-term arithmetic progression in A. Thus, $G(k, r) \leq W(k, r)$.

Finally, by Theorems 1 and 3,

$$W(k, r) \leq M((k-1)r+1, r) \leq G((k-1)r+1, 2r-1).$$

This completes the proof.

References

1. R. L. Graham and B. L. Rothschild, *A short proof of van der Waerden's theorem on arithmetic progressions*, Proc. Amer. Math. Soc. **42** (1974), 385–386.
2. J. R. Rabung, *On applications of van der Waerden's theorem*, Math. Mag. **48** (1975), 142–148.
3. E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arithmetica **27** (1975), 199–245.
4. B. L. van der Waerden, *Beweis einer Baudet'schen Vermutung*, Nieuw Arch. Wiskunde **15** (1927), 212–216.
5. B. L. van der Waerden, *How the proof of Baudet's conjecture was found, Studies in Pure Mathematics*, L. Mirsky, ed., Academic Press, New York, 1971, 251–260.

Department of Mathematics
Southern Illinois University
Carbondale
Illinois 62901

https://doi.org/10.4153/CMB-1980-074-x Published online by Cambridge University Press