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SOME STRUCTURAL PROPERTIES OF THE SET OF REMOTE POINTS OF A METRIC SPACE

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A *remote point* of a metric space X is a point in $\beta X\backslash X$ not in the βX-closure of any discrete subset of X. Remote points have been studied by Fine and Gillman [**2**], Plank [**4**], Robinson [**6**], Woods [**9, 10**], Van Douwen [**7**] and others. The main results concerning the existence of remote points are listed in Section 1. In this paper we determine some structural properties of the set of remote points of a metric space which has no isolated points. The notation of Gillman and Jerison [**3**] and Walker [**8**] will be used. All spaces are assumed to be Tychonoff.

Let X be a space. The *general remote points* of X, denoted TX, are those points in $\beta X\backslash X$ which are not in the βX-closure of any nowhere dense sets of X. If X is a metric space without isolated points, then TX is precisely the set of remote points of X as defined in Walker [**8**]. In Section 2, it is shown that if Y is a normal space which is the image of a space X under a closed irreducible continuous mapping then the set of general remote points of X and that of Y are homeomorphic. Thus it follows that coabsolute normal spaces have homeomorphic sets of general remote points. In Section 3 we show that an arbitrary normal space has a "decomposition" which carries over to the set of general remote points and reduces the study of the sets of general remote points for normal spaces to two types of spaces: (i) those spaces for which the set of locally compact points is dense, and (ii) the nowhere locally compact spaces. In Section 4, machinery and properties are developed for the study of the general remote points of a space by using the absolute of the space as a subspace of its Stone space. Using this approach, it is easy to see that when X is a dense subset of Y where X and Y are normal spaces, TY is homeomorphic to a subset of TX. In Section 5, the results of Section 4 are used to investigate the structure of the set of remote points of a metric space for which the set of locally compact points is dense. It is shown that the set of remote points for these spaces is made entirely of homeomorphic copies of the set of remote points of the real numbers.

1. Preliminaries.

In this section we list the known results about remote points which will be used in this paper. Although the original proofs of

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Theorems 1.1, 1.2, and 1.4 assumed the continuum hypothesis ($\aleph_1 = 2^{\aleph_0}$), it has recently been shown (e.g. van Douwen [7]) that this is not necessary.

THEOREM 1.1. (Plank) *If X is a non-compact separable metric space in which the set of isolated points has compact closure, then βX contains 2^c remote points which form a dense subspace of $\beta X \backslash X$.*

THEOREM 1.2. (Robinson) *Every locally compact metric space X without isolated points contains a set of remote points which is dense in $\beta X \backslash X$.*

THEOREM 1.3. (Van Douwen) *If X is a normal non-pseudocompact space of countable π-weight, then there are 2^c remote points in βX; if in addition each closed pseudo-compact subspace of X is compact, then TX is dense in $\beta X \backslash X$.*

For a space X, dX denotes the density of X.

THEOREM 1.4. (Woods) *Let X and Y be two locally compact, non-compact metric spaces without isolated points. If $dX = dY$ then TX and TY are homeomorphic.*

The next lemma is useful in some of the straightforward proofs which have been omitted in this paper. It is a special case of Lemma 3.2 [7]. Let $\text{Bd}_X S = \text{cl}_X S \backslash \text{int}_X S$ where $S \subseteq X$ for any space X and let $\text{Ex}_X U = \beta X \backslash \text{cl}_{\beta X}(X \backslash U)$ where U is any open subset of a space X. Van Douwen [7] has shown that $\text{Bd}_{\beta X}(\text{Ex}_X U) = \text{cl}_{\beta X}(\text{Bd}_X U)$ where U is any open set in X and X is any completely regular space. For our purposes we use the following version of this fact.

LEMMA 1.5. *Let $R \subseteq X$ be regular closed. Then $\text{Ex}_X(\text{int}_X R) = \text{int}_{\beta X}\text{cl}_{\beta X} R$, so*

$$\text{cl}_{\beta X} R \backslash \text{int}_{\beta X}\text{cl}_{\beta X} R = \text{cl}_{\beta X}(R \backslash \text{int}_X R);$$

therefore, if $p \in \text{cl}_{\beta X} R \backslash \text{int}_{\beta X}\text{cl}_{\beta X} R$ for some regular closed subset R of X, then $p \in TX$.

2. General remote points are preserved by closed irreducible mappings.

A continuous mapping f from a space X onto a space Y is *closed irreducible* if the image under f of every proper closed subset of X is a proper closed subset of Y. Some properties of closed irreducible mappings are given in the following lemma.

LEMMA 2.1. *Let $f: X \to Y$ be closed irreducible and continuous.*

(a) *If $S \subseteq Y$ is dense, then $f^{\angle}(S)$ is dense in X.*

(b) *If $C \subseteq X$ is closed nowhere dense in X, then $f(C)$ is closed nowhere dense in Y.*

(c) *If A and B are disjoint closed subsets of X, then $f(A) \cap f(B)$ is closed nowhere dense in Y.*

(d) *$f^\beta: \beta X \to \beta Y$ is closed irreducible where f^β is the Stone extension of f.*

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(e) *If $x \in X$ is an isolated point in X, then $f(x)$ is an isolated point in Y. If $y \in Y$ is an isolated point in Y, then $f^{\angle}(y)$ is an isolated point of X.*

Proof. See [**9**] for (a). For (b): If $U \subseteq f(C)$ and U is a non-empty open set in Y, then $f^{\angle}(U) \cap X \backslash C \neq \emptyset$, so $X \backslash f^{\angle}(U) \cup C$ is a proper closed subset of X which maps onto Y, contradicting the assumption that f is closed irreducible. So $U = \emptyset$.

(c) If $U \subseteq f(A) \cap f(B)$ is a non-empty open set in Y, then $f^{\angle}(U) \cap A \neq \emptyset$ so $X \backslash f^{\angle}(U) \cup B$ is a proper closed subset of X. But $f(X \backslash f^{\angle}(U) \cup B) = Y$ which is a contradiction of the assumption that f is closed irreducible. So $U = \emptyset$.

(d) Since βX is compact, f^{β} is closed and onto. Let $C \subseteq \beta X$ be a proper closed subset of βX and U be open in βX such that $C \subseteq U \subseteq \mathrm{cl}_{\beta X} U \subsetneq \beta X$. Then

$$f^{\beta}(C) \subseteq f^{\beta}(\mathrm{cl}_{\beta X} U) = f^{\beta}(\mathrm{cl}_{\beta X}(U \cap X)) = \mathrm{cl}_{\beta Y} f(U \cap X).$$

If $f^{\beta}(C) = \beta Y$ then $f(U \cap X)$ is dense in βY and also in Y; so

$$Y = \mathrm{cl}_Y f(U \cap X) = f(\mathrm{cl}_X(U \cap X)).$$

Since f is closed irreducible, $\mathrm{cl}_X(U \cap X) = X$ and this contradicts $\mathrm{cl}_{\beta X} U \neq \beta X$. Therefore $f^{\beta}(C) \neq \beta Y$.

(e) The proof is straightforward.

The next two lemmas are needed for the proof that closed irreducible functions "preserve" general remote points.

LEMMA 2.2. *Let X and Y be spaces with Y normal, and let $f:X \rightarrow Y$ be a closed irreducible continuous mapping. If $f^{\beta}:\beta X \rightarrow \beta Y$ is the Stone extension of f, then $f^{\beta\angle}(TY) = TX$.*

Proof. Suppose $p \in X$ but $p \notin TX$. If $p \notin \beta X \backslash X$, then $f^{\beta}(p) \notin \beta Y \backslash Y$ so $f^{\beta}(p) \notin TY$. If $p \in \beta X \backslash X$, then there exists a closed nowhere dense $F \subseteq X$ such that $p \in \mathrm{cl}_{\beta X} F$. Then

$$f^{\beta}(p) \in f^{\beta}(\mathrm{cl}_{\beta X} F) = \mathrm{cl}_{\beta Y} f(F)$$

since f^{β} is closed and continuous. Since $f(F)$ is closed nowhere dense in Y by Lemma 2.1(b), then $f^{\beta}(p) \notin TY$. So $f^{\beta\angle}(TY) \subseteq TX$.

On the other hand suppose $p \in \beta X$ and $p \notin f^{\beta\angle}(TY)$. If $f^{\beta}(p)$ is an isolated point of Y, p is an isolated point in X so $p \notin TX$. Otherwise there exists a closed nowhere dense set F in Y such that $f^{\beta}(p) \in \mathrm{cl}_{\beta Y} F$. If $p \notin \mathrm{cl}_{\beta X}(f^{\angle}(F))$, then since βX is regular, there exists a set U open in βX such that $p \in U$ and $\mathrm{cl}_{\beta X} U \cap \mathrm{cl}_{\beta X}(f^{\angle}(F)) = \emptyset$. Since $p \in \mathrm{cl}_{\beta X} U$,

$$f^{\beta}(p) \in f^{\beta}(\mathrm{cl}_{\beta X} U) = \mathrm{cl}_{\beta Y} f(\mathrm{cl}_X(U \cap X)).$$

So

$$f^{\beta}(p) \in \mathrm{cl}_{\beta Y} f(\mathrm{cl}_X(U \cap X)) \cap \mathrm{cl}_{\beta Y} F.$$

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But $\mathrm{cl}_X(U \cap X) \cap f^{\angle}(F) = \emptyset$ so $f(\mathrm{cl}_X(U \cap X)) \cap F = \emptyset$; and since Y is normal, disjoint closed sets in Y are completely separated so

$$\mathrm{cl}_{\beta Y} f(\mathrm{cl}_X(U \cap X)) \cap \mathrm{cl}_{\beta Y} F = \emptyset,$$

which is a contradiction. So $p \in \mathrm{cl}_{\beta X}(f^{\angle}(F))$. By Lemma 2.1(a), $f^{\angle}(F)$ is closed nowhere dense so $p \notin TX$. Thus $TX \subseteq f^{\beta\angle}(TY)$.

LEMMA 2.3. *Let X and Y be spaces and let $f:X \to Y$ be a closed irreducible continuous mapping with $f^{\beta}:\beta X \to \beta Y$ the Stone extension of f. If $p \in \beta Y$ and $|f^{\beta\angle}(p)| > 1$, then $p \notin TY$.*

Proof. Suppose $q_1 \neq q_2$ but q_1 and q_2 belong to $f^{\beta\angle}(p)$ where $p \in \beta Y$. Let U_1 and U_2 be open subsets of βX with disjoint closures such that $q_i \in U_i$, $i = 1, 2$; then

$$f^{\beta}(\mathrm{cl}_{\beta X} U_1) \cap f^{\beta}(\mathrm{cl}_{\beta X} U_2) = \mathrm{cl}_{\beta Y} f^{\beta}(U_1) \cap \mathrm{cl}_{\beta Y} f^{\beta}(U_2)$$

has empty interior by Lemma 2.1(d) and (c). Thus there is an open subset $U \subseteq \beta X$ (e.g. U_1 or U_2) with

$$p \in \mathrm{cl}_{\beta X} f^{\beta}(U) \backslash \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} f^{\beta}(U).$$

Now

$$\beta Y \backslash f^{\beta}(\beta X \backslash U) \subseteq f^{\beta}(U) \subseteq \mathrm{cl}_{\beta Y}(\beta Y \backslash f^{\beta}(\beta_X \backslash U))$$

where the last inclusion is justified by Lemma 10.49 [**8**] since f^{β} is closed irreducible. So if $W = \beta Y \backslash f^{\beta}(\beta X \backslash U)$, $\mathrm{cl}_{\beta Y} W = \mathrm{cl}_{\beta Y} f^{\beta}(U)$; and thus

$$p \in \mathrm{cl}_{\beta Y} W \backslash \mathrm{int}_{\beta Y} \mathrm{cl}_{\beta Y} W.$$

Now $\mathrm{cl}_{\beta Y} W = \mathrm{cl}_{\beta Y}(\mathrm{cl}_{\beta Y} W \cap Y)$ and $\mathrm{cl}_{\beta Y} W \cap Y$ is regular closed in Y, so we have precisely the situation of Lemma 1.5. Thus $p \notin TY$.

THEOREM 2.4. *Let X and Y be spaces with Y normal, and let $f:X \to Y$ be a closed irreducible continuous mapping with $f^{\beta}:\beta X \to \beta Y$ the Stone extension of f. Then $f^{\beta}|TX : TX \to TY$ is a homeomorphism with $f^{\beta\angle}(TY) = TX$.*

Proof. $f^{\beta}|TX$ is onto by Lemma 2.2, one-to-one by Lemma 2.3, and continuous. Furthermore, since $TX = f^{\beta\angle}(TY)$ by Lemma 2.2 and since any closed set in TX is of the form $C \cap TX$ for some closed C in βX,

$$(f^{\beta}|TX)(C \cap TX) = f^{\beta}(C \cap TX) = f^{\beta}(C) \cap TY$$

which is closed in TY. So $f^{\beta}|TX$ is closed. Thus it is a homeomorphism.

Recall that a space is extremally disconnected if the closure of every open set is open.

COROLLARY 2.5. *Let X be an extremally disconnected space and let Y be a metric space without isolated points and let $f:X \to Y$ be closed irreducible and*

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continuous with $f^\beta:\beta X \to \beta Y$ *the Stone extension of* f. *Then* TY *is precisely those points of* βY *whose inverse image under* f^β *is a single point.*

Proof. Suppose $p \in \beta Y$ and $p \notin TY$. Since Y has no isolated points, if $p \notin TY$ then for some closed nowhere dense subset Z of Y, $p \in \mathrm{cl}_{\beta Y} Z$. Since Z is nowhere dense and Y is metric without isolated points, $Z \subseteq R \cap Y \backslash \mathrm{int}_Y R$ for some regular closed subset R of Y ([**5**], Lemma 1.2). So

$$p \in \mathrm{cl}_{\beta Y} R \cap \mathrm{cl}_{\beta Y}(Y \backslash \mathrm{int}_{\beta Y} R).$$

Now

$$\mathrm{cl}_{\beta Y} R = f^\beta(\mathrm{cl}_{\beta X}(f^{\angle}(\mathrm{int}_Y R))) \text{ and } \mathrm{cl}_{\beta Y}(Y \backslash \mathrm{int}_Y R) = f^\beta(\mathrm{cl}_{\beta X}(f^{\angle}(Y \backslash R)));$$

and since $f^{\angle}(\mathrm{int}_Y R)$ and $f^{\angle}(Y \backslash R)$ are disjoint open subsets of X which is an extremally disconnected space, the βX-closures of these two sets are also disjoint ([**3**], Exercise 1H4). So there are at least two distinct points in $f^{\beta\angle}(p)$. Thus $|f^{\beta\angle}(p)| > 1$. This result, along with Lemma 2.3, shows that $p \in TY$ if and only if $|f^{\beta\angle}(p)| = 1$.

Van Douwen [**7**] defined a space X to be *extremally disconnected* at a point p if $p \notin \mathrm{cl}_X U \cap \mathrm{cl}_X V$ for any two disjoint open subsets U and V in X. Using this terminology, the proofs of Corollary 2.5 and Lemma 2.3 show the following: Let X be an extremally disconnected space and let Y be a completely regular space, and let $f:X \to Y$ be closed irreducible and continuous with $f^\beta:\beta X \to \beta Y$ the Stone extension of f. Then the points where βY is extremally disconnected are precisely those points of βY whose inverse image under f^β is a singleton.

For each space X, there is a unique (up to a homeomorphism) extremally disconnected space EX called the *absolute of* X that can be mapped onto X by a closed irreducible perfect continuous mapping. If X and Y are spaces for which EX and EY are homeomorphic, then X and Y are said to be *coabsolute.* The following corollary to Theorem 2.4 is immediate. This corollary appears, in part, in Theorem 4.3 [**9**].

COROLLARY 2.6. *If* X *is normal then* TEX *and* TX *are homeomorphic where* EX *is any homeomorphic copy of the absolute of* X. *Thus, if two normal spaces are coabsolute, their sets of general remote points are homeomorphic.*

The second statement in Corollary 2.6 indicates that a sufficient condition for two normal spaces to have homeomorphic sets of general remote points is that the spaces be coabsolute. The following example shows that this condition is not necessary.

Example 2.7. Let I be the closed unit interval, $\mathbf{Q}$ be the space of rational numbers and X be the disjoint union of I and $\mathbf{Q}$. Then $TX = T\mathbf{Q}$ since I is compact. But EX and $E\mathbf{Q}$ are not homeomorphic because $E\mathbf{Q}$ is nowhere locally compact but EX is not. This follows since the inverse image of a no-

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where locally compact space under a closed irreducible continuous mapping is nowhere locally compact and the inverse image of a locally compact space under a perfect continuous mapping is locally compact [**1**].

3. Decomposition of TX. If X is a space then LX will denote the subspace of locally compact points of X, i.e., all those points $x \in X$ for which there exists an open neighborhood U in X such that $\mathrm{cl}_X U$ is compact; NX will denote $X \backslash \mathrm{cl}_X LX$. Clearly NX and LX have the following properties: NX and LX are disjoint open subsets of X with NX regular open, and $\mathrm{cl}_X NX$ and $\mathrm{cl}_X LX$ are regular closed subsets of X; $X = \mathrm{cl}_X NX \cup \mathrm{cl}_X LX$; $\mathrm{cl}_X NX \cap \mathrm{cl}_X LX$ is closed nowhere dense in X and so $\mathrm{cl}_X NX$ and $\mathrm{cl}_X LX$ have disjoint interiors. Some other properties of $\mathrm{cl}_X NX$ and $\mathrm{cl}_X LX$ are given in Lemma 3.1.

LEMMA 3.1. *Let X be a space.*

(a) *If $V \subseteq X$ is open and $V \subseteq NX$ then $\mathrm{cl}_X V$ is a nowhere locally compact space. In particular, $\mathrm{cl}_X NX$ is a nowhere locally compact space.*

(b) *The locally compact points of $\mathrm{cl}_X LX$ are dense in $\mathrm{cl}_X LX$.*

LEMMA 3.2. *Let X be a space and let $j : \mathrm{cl}_X LX \,\dot{\cup}\, \mathrm{cl}_X NX \to X$ be the map with $j|\mathrm{cl}_X LX$ and $j|\mathrm{cl}_X NX$ the identity maps on $\mathrm{cl}_X LX$ and $\mathrm{cl}_X NX$ respectively. Then j is a closed irreducible perfect continuous mapping.*

Proof. This follows from a more general statement: Let $\{R_a : a \in A\}$ be a locally finite cover of regular closed subsets of X with pairwise disjoint interiors. Then the mapping $f : \Sigma R_a \to X$ where $f\,|R_a$ is the identity mapping is a closed irreducible perfect continuous mapping. The proof of this assertion is straightforward.

By making appropriate assumptions on the space X, a decomposition for TX is obtained using Lemma 3.2 along with Theorem 2.4 as shown in the following theorem.

THEOREM 3.3. *Let X be a normal space. Then TX and $T(\mathrm{cl}_X LX) \,\dot{\cup}\, T(\mathrm{cl}_X NX)$ are homeomorphic. Thus TX has a decomposition into disjoint clopen sets A and B such that $A = TY$ for some normal space Y for which the set of locally compact points is dense and $B = TW$ for some nowhere locally compact normal space W.*

Proof. Since $\beta(\mathrm{cl}_X LX \,\dot{\cup}\, \mathrm{cl}_X NX) = \beta(\mathrm{cl}_X LX) \,\dot{\cup}\, \beta(\mathrm{cl}_X NX)$,

$$T(\mathrm{cl}_X LX \,\dot{\cup}\, \mathrm{cl}_X NX) = T(\mathrm{cl}_X LX) \,\dot{\cup}\, T(\mathrm{cl}_X NX),$$

and by Lemma 3.2 and Theorem 2.4, TX and $T(\mathrm{cl}_X LX \,\dot{\cup}\, \mathrm{cl}_X NX)$ are homeomorphic. The last statement of the theorem follows from Lemma 3.1 by choosing $Y = \mathrm{cl}_X LX$ and $W = \mathrm{cl}_X NX$.

Previous studies of remote points of metric spaces have focused primarily on locally compact metric spaces or nowhere locally compact metric spaces. Theorem 3.3 shows that, in fact, the remote points for any metric space without isolated points may be studied by considering (i) nowhere locally compact metric spaces and (ii) perfect metric spaces with a dense subset which is locally compact. We show in the next theorem that Theorem 1.2 generalizes to spaces of type (ii).

The following lemma is proved by Robinson.

LEMMA 3.4. ([**6**], Lemma p. 338) *Let X be a metric space and let R be a regular closed subset of X. Every remote point of R is a remote point of X, i.e., every remote point of R is contained in $\mathrm{cl}_{\beta X}R \cap TX$ (since $\beta R = \mathrm{cl}_{\beta X}R$* ([**3**], 6.9(a))).

LEMMA 3.5. *Let Y be a metric space, and let X be open and dense in Y with the property that whenever R is a regular closed subset of Y and $R \subseteq X$, then R is compact. Then Y is a compact space.*

The proof of Lemma 3.5 is straightforward.

THEOREM 3.6. *Let X be a non-compact metric space without isolated points which has a dense locally compact subset. Then TX is dense in $\beta X \backslash X$.*

Proof. Let $W \subseteq X$ be dense and locally compact and let $U \subseteq \beta X$ be open with $U \cap \beta X \backslash X \neq \emptyset$. We must show that $U \cap TX \neq \emptyset$. Let V be open in βX with $V \subseteq \mathrm{cl}_{\beta X} V \subseteq U$ and $V \cap \beta X \backslash X \neq \emptyset$. Since

$$\mathrm{cl}_X(V \cap W) = \mathrm{cl}_{\beta X} V \cap X \neq \mathrm{cl}_{\beta X} V,$$

$\mathrm{cl}_X(V \cap W)$ is not compact so by Lemma 3.5 there is a non-compact regular closed subset R of $\mathrm{cl}_X(V \cap W)$ (and thus of X) with $R \subseteq V \cap W$. Since R is non-compact, locally compact metric, $TR \neq \emptyset$. By Lemma 3.4, $TR \subseteq \mathrm{cl}_{\beta X}R \cap TX$ and since $\mathrm{cl}_{\beta X}R \subseteq \mathrm{cl}_{\beta X} V \subseteq U$, $TR \subseteq U \cap TX$. Thus $U \cap TX \neq \emptyset$.

4. General remote points of the absolute. According to Corollary 2.6, we can use the absolute of a normal space to study the general remote points of the space. In this section we use this approach on a particular representation of the absolute. The results are rather technical and some details which are routine to verify are omitted. The advantage of such a study is that some structural properties of the set of remote points become quite transparent due to the simple nature of the absolute and its Stone-Cech compactification.

The reader is assumed to be familiar with Boolean algebras and the Stone Representation Theorem. For a space X, the family of regular closed subsets of X, denoted by $R(X)$, is a complete Boolean algebra under the following operations:

Let $A, B, A_a \in R(X)$.

(i) $A \leqq B$ if and only if $A \subseteq B$.

(ii) $\vee_a A_a = \mathrm{cl}_X(\bigcup_a \mathrm{int}_X A_a) = \mathrm{cl}_X \bigcup_a A_a$.

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(iii) $\wedge_a A_a = \text{cl}_X \text{int}_X(\bigcap_a A_a)$.
(iv) $A' = \text{cl}_X(X \backslash A)$.

The following version of the Stone Representation Theorem can be found in [**9**].

THEOREM 4.1. *Let X be a space and $S(R(X))$ be the set of Boolean algebra ultrafilters on $R(X)$. For each $A \in R(X)$, let $\lambda(A) = \{a \in S(R(X)) : A \in a\}$. Then $\{\lambda(A) : A \in R(X)\}$ can be used as a base for a topology on $S(R(X))$; and $S(R(X))$, so topologized, is a compact extremally disconnected space. The map $A \to \lambda(A)$ is a Boolean algebra isomorphism from $R(X)$ onto the clopen subsets of $S(R(X))$. If $k_{\beta X} : S(R(X)) \to \beta X$ is defined by $k_{\beta X}(a) = \bigcap_{A \in a} \text{cl}_{\beta X} A$, then $k_{\beta X}$ is a well-defined closed irreducible continuous mapping from $S(R(X))$ onto βX and $k_{\beta X}(\lambda(A)) = \text{cl}_{\beta X} A$ for any $A \in R(X)$.*

Since $S(R(X))$ is extremally disconnected, the absolute of X may be identified with $k_{\beta X}^{\angle}(X) \subseteq S(R(X))$. This identification is convenient since then $\beta EX = S(R(X))$ and so $TEX \subseteq S(R(X)) \backslash EX$. For the remainder of this section this identification of EX will be used to obtain topological properties of TEX and consequently of TX.

LEMMA 4.2. *Let X be a normal space and let $EX = k_{\beta X}^{\angle}(X)$. Then $k_{\beta X}^{\angle}(\text{cl}_{\beta X} A \cap TX) = \lambda(A) \cap TEX$ for $A \in R(X)$, so $\text{cl}_{\beta X} A \cap TX$ and $\lambda(A) \cap TEX$ are homeomorphic. Furthermore, $\lambda(A) \cap TEX = TEA$ where $\lambda(A) \cap EX$ is identified with EA (and thus $\beta EA = \lambda(A)$) for any space X, so $\text{cl}_{\beta X} A \cap TX$ is homeomorphic to TA when X is normal.*

The proof of this lemma follows from Theorem 4.1 and Theorem 2.4.

When $A \in R(X)$, it is sometimes convenient to write TA when the actual set being referred to is $\text{cl}_{\beta X} A \cap TX$; this will be indicated by saying that *the canonical representation of TA* is being used.

The following structural information concerning the set of general remote points is a consequence of Lemma 4.2 and properties of the space $S(R(X))$.

THEOREM 4.3. *Let X be a normal space. Then TX is a zero-dimensional space with $\{TA : A \in R(X)\}$ as a clopen basis, where the sets $\{TA : A \in R(X)\}$ have the canonical representation.*

We are interested in the relationship between TX and TY when X is a dense subspace of Y. In general, βX and βY are different spaces which complicates efforts to compare TX and TY. However, the absolutes of X and Y and their Stone-Cech compactifications can be obtained within a single space, namely $S(R(Y))$.

Since X is dense in Y, $R(X)$ and $R(Y)$ are isomorphic Boolean algebras under the Boolean algebra isomorphism of $R(X)$ onto $R(Y)$ defined by $A \to \text{cl}_Y A$ where $A \in R(X)$. The inverse of this isomorphism, mapping $R(Y)$

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onto $R(X)$, is defined by $B \to B \cap X$ where $B \in R(Y)$. Thus $S(R(X)) = S(R(Y))$ where $a \in S(R(X))$ is identified with $\{\mathrm{cl}_Y A : A \in a\} \in S(R(Y))$, or equivalently, $a^* \in S(R(Y))$ is identified with $\{B \cap X : B \in a^*\} \in S(R(X))$. Note that, under this identification,

$$k_{\beta X}(a) = \cap_{A \in a} \mathrm{cl}_{\beta X} A \text{ if } a \in S(R(X)) \text{ and}$$
$$k_{\beta X}(a^*) = \cap_{A \in a^*} \mathrm{cl}_{\beta X}(A \cap X) \text{ if } a^* \in S(R(Y)).$$

It is straightforward to check that the following diagram commutes.

$$\begin{array}{ccc} S(R(Y)) & \xrightarrow{k_{\beta Y}} & \beta Y \\ \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\ k_{\beta Y}{}^{\angle}(Y) & \xrightarrow{k_{\beta Y}|k_{\beta Y}{}^{\angle}(Y)} & Y \\ \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\ k_{\beta Y}{}^{\angle}(X) & \xrightarrow{k_{\beta Y}|k_{\beta Y}{}^{\angle}(X)} & X \end{array}$$

Each containment is dense, each space on the left is extremally disconnected, and each mapping indicated is closed irreducible perfect and continuous; so the spaces on the left are homeomorphic to the absolutes of the corresponding spaces on the right. Using these representations, $EX \subseteq EY \subseteq S(R(Y))$ and $\beta EX = \beta EY = S(R(Y))$, so TEY and TEX are subsets of $S(R(Y))$. In addition, the following diagrams commute. Here $i:X \to Y$ is the inclusion map.

$$\begin{array}{ccc} S(R(X)) & \xrightarrow{k_{\beta X}} & \beta X \\ \| & & \text{\rotatebox{90}{$\subseteq$}} \\ S(R(Y)) & & X \\ \text{\rotatebox{90}{$\subseteq$}} & \nearrow_{k_{\beta Y}|k_{\beta Y}{}^{\angle}(X)} & \\ k_{\beta Y}{}^{\angle}(X) & & \end{array} \qquad \begin{array}{ccc} S(R(Y)) & \searrow^{k_{\beta Y}} & \\ \| & & \beta Y \\ S(R(X)) & & \\ \downarrow k_{\beta X} & \nearrow_{i^{\beta}} & \\ \beta X & & \end{array}$$

LEMMA 4.4. *Let Y be a space and let X be a dense subset of Y. Let $EX = k_{\beta Y}{}^{\angle}(X)$ and $EY = k_{\beta Y}{}^{\angle}(Y)$.*

(a) *$TEY \subseteq TEX$ as subspaces of $S(R(Y))$.*

(b) *If $A \in R(Y)$ and $A \subseteq X$, then $\lambda(A) \cap TEY = \lambda(A) \cap TEX$.*

(c) *Let X be an open dense subset of Y and Y be normal.*

If $H(X) = \{A \in R(Y) : A \subseteq X\}$, then

$$TEY = \cup_{H(X)} (\lambda(A) \cap TEY) = \cup_{H(X)} (\lambda(A) \cap TEX)$$

and so TEY is an open subset of TEX. Furthermore, $\{\lambda(A) \cap TEY : A \in H(X)\}$ is a clopen basis for TEY.

Proof. (a) Suppose $p \in S(R(Y)) \backslash TEX$. If $p \in EY$ then $p \notin TEY$. If $p \in S(R(Y)) \backslash EY$, since $p \notin TEX$ there exists a closed nowhere dense subset

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F of EX with $p \in \mathrm{cl}_{S(R(Y))}F$. Thus

$$p \in \mathrm{cl}_{S(R(Y))}F = \mathrm{cl}_{S(R(Y))}(\mathrm{cl}_{EY}F).$$

But $\mathrm{cl}_{EY}F$ is closed nowhere dense in EY since EX is dense in EY, and so $p \notin TEY$. Therefore $TEY \subseteq TEX$.

(b) Let $A \in R(Y)$ and $A \subseteq X$. Then $\lambda(A) \cap EX = \lambda(A) \cap EY$. By (a), $\lambda(A) \cap TEY \subseteq \lambda(A) \cap TEX$. To show the other inclusion, suppose $p \in \lambda(A)$ and $p \notin TEY$. If $p \in EY$, then $p \in EX$ since $\lambda(A) \cap EY = \lambda(A) \cap EX$. If $p \notin EY$, there exists a closed nowhere dense subset F of EY with $p \in \mathrm{cl}_{S(R(Y))}F$. Now $p \in \lambda(A) \cap \mathrm{cl}_{S(R(Y))}F$ and since $\lambda(A)$ is clopen in $S(R(Y))$,

$$\lambda(A) \cap \mathrm{cl}_{S(R(Y))}F = \mathrm{cl}_{S(R(Y))}(\lambda(A) \cap F).$$

Furthermore, since $\lambda(A) \cap EX = \lambda(A) \cap EY$ and $F \subseteq EY$, $\lambda(A) \cap F \subseteq \lambda(A) \cap EX$ and thus $\lambda(A) \cap F$ is a closed nowhere dense subset of EX. So $p \notin TEX$.

(c) By (b),

$$\cup_{H(X)} (\lambda(A) \cap TEX) = \cup_{H(X)} (\lambda(A) \cap TEY)$$

and clearly $\cup_{H(X)} (\lambda(A) \cap TEX)$ is an open subset of TEX. It is also clear that

$$\cup_{H(X)} (\lambda(A) \cap TEY) \subseteq TEY.$$

Thus the conclusions of (c) will follow provided we show that $\{\lambda(A) \cap TEY: A \in H(X)\}$ is an open basis for TEY. Let $p \in TEY \cap \lambda(B)$ where $B \in R(Y)$. Since Y is normal, $k_{\beta Y}(p) \in TY$; so $k_{\beta Y}(p) \notin \mathrm{cl}_{\beta Y}(Y\backslash X)$ since $Y\backslash X$ is a closed nowhere dense subset of Y. Thus we can find a regular closed subset C of βY with $k_{\beta Y}(p) \in \mathrm{int}_{\beta Y}C$ and $C \cap Y\backslash X = \emptyset$; so $C \cap Y \subseteq X$ and $p \in \lambda(C \cap Y)$. Thus $p \in \lambda(C \cap Y) \cap \lambda(B) = \lambda(T)$ where $T = (C \cap Y) \wedge B \subseteq C \cap Y \subseteq X$ so $T \in H(X)$. Finally then, $p \in \lambda(T) \cap TEY \subseteq \lambda(B) \cap TEY$ and $\lambda(T) \cap TEY \in \{\lambda(A) \cap TEY: A \in H(X)\}$. So $\{\lambda(A) \cap TEY: A \in H(X)\}$ is an open basis for TEY.

If X is a proper open dense subset of Y with Y metric and if X is locally compact, the embedding of TEY into TEX of Lemma 4.4 is not a dense embedding as Lemma 4.7 will indicate. Lemma 4.5 is proved in [**9**].

LEMMA 4.5. *Let X be a metric space, or nowhere locally compact, or realcompact and let $EX = k_{\beta X}{}^{\angle}(X)$. Then*

$$k_{\beta X}|S(R(X))\backslash EX : S(R(X))\backslash EX \to \beta X\backslash X$$

is closed irreducible.

LEMMA 4.6. *Let Y be a space which is first countable at each point of $Y\backslash X$ where X is a locally compact dense subset of Y. For each $y \in Y\backslash X$, there exists an $A \in R(Y)$ such that $y \in A$, $A \cap Y\backslash X = \{y\}$, and A is compact.*

LEMMA 4.7. *Let Y be a first countable space of countable π-weight with X a locally compact dense normal realcompact subset of Y, or Y metric without isolated points and X a locally compact dense subset of Y. Then for each $y \in Y\backslash X$ there exists an $A \in R(Y)$ such that $y \in A$ and $\lambda(A) \cap TEY = \emptyset$ but $\lambda(A) \cap TEX \neq \emptyset$ where $EX = k_{\beta X}{}^{\angle}(X)$, $EY = k_{\beta Y}{}^{\angle}(Y)$ and $S(R(X))$ is identified with $S(R(Y))$. Thus TEY is not dense in TEX.*

Proof. Let $y \in Y\backslash X$. By Lemma 4.6 there is an $A \in R(Y)$ such that $A \cap Y\backslash X = \{y\}$ and A is compact. Thus $\lambda(A) \subseteq EY$ so $\lambda(A) \cap TEY = \emptyset$. However $A \cap X = A\backslash\{y\} \in R(X)$ is not compact, so $\lambda(A) \cap S(R(Y))\backslash EX \neq \emptyset$. Since TX is dense in $\beta X\backslash X$ by Theorem 1.2 or 1.3, then $k_{\beta X}{}^{\angle}(TX) = TEX$ is dense in $S(R(Y))\backslash EX$ by Lemmas 4.5 and 2.1(a). Thus $TEX \cap \lambda(A) \neq \emptyset$. Since $TEX \cap \lambda(A)$ is a non-empty subset of TEX and $TEY \cap \lambda(A) = \emptyset$, it follows that TEY is not dense in TEX.

The following proposition is a restatement of Lemmas 4.4 and 4.7 in terms of the remote points of the spaces rather than the general remote points of their absolutes.

PROPOSITION 4.8. *Let Y be a normal space and let X be a dense subset of Y.*

(a) *If X is normal, then TY is homeomorphic to a subset of TX.*

(b) *If X is open in Y, then TY can be represented as $\bigcup_{A \in R(Y), A \subseteq X} TA$ where $\{TA : A \subseteq X, A \in R(Y)\}$ is a clopen basis for TY.*

(c) *If X is open in Y and normal, then TY is homeomorphic to an open subset of TX.*

(d) *If Y is normal, first countable, and of countable π-weight and X is locally compact, normal and realcompact, or if Y is metric without isolated points and X is locally compact, then TY is homeomorphic to a non-dense subset of TX.*

5. Structural properties of the set of remote points of a metric space with $\mathrm{cl}_X LX = X$. $\mathbf{R}$ will denote the metric space of real numbers with the usual topology. In this section we use the machinery of the preceding section to analyze the set of remote points for metric spaces X for which $\mathrm{cl}_X LX = X$. First we give some examples of metric spaces with this property. Let

$$X = \{(x, y) : 0 \leqq x < 1, 0 \leqq y < 1\} \subseteq \mathbf{R}^2, X_1 = X \cup \{(1, 0)\},$$
$$X_2 = X \cup \{(1, 1/n) : n \in \mathbf{N}\}, \text{ and } X_3 = X_2 \cup \{(1, 0)\}.$$

Each of these is a metric space as a subspace of $\mathbf{R}^2$ and $LX_i = X$ for $i = 1, 2, 3$ with $\mathrm{cl}_{X_i} LX_i = X_i$. $X_1\backslash LX_1$ is a singleton; $X_2\backslash LX_2$ is non-compact; $X_3\backslash LX_3$ is compact, but not a singleton.

Let X be a metric space without isolated points. When $X = \mathrm{cl}_X LX$, LX is an open dense subset of X so we know (Lemma 4.4(c)) that TX is homeomorphic to a union of clopen sets homeomorphic to $\{TA : A \in R(X), A \subseteq LX\}$, and these sets form a clopen basis for TX. A π-basis of a space is a collection B of non-empty open sets of the space such that every non-empty

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open set of the space contains a member of B. We now show that $\{TA: A \in R(X), A \subseteq LX, A$ is separable and non-compact$\}$ is a π-basis for TX.

LEMMA 5.1. (Robinson) *Let X be a locally compact, non-compact metric space. There there is an $A \in R(X)$ such that A is separable and non-compact.*

THEOREM 5.2. *Let X be a non-compact metric space without isolated points. If $X = \mathrm{cl}_X LX$ then TX has a π-basis of clopen sets each homeomorphic to $T\mathbf{R}$.*

Proof. Let $B = \{TA : A \in R(X),\ A$ is separable and non-compact and $A \subseteq LX$ where TA is represented canonically$\}$. If $B \neq \emptyset$, by Theorems 1.1 and 1.4, B consists of non-empty sets and they are clopen subsets of TX each homeomorphic to $T\mathbf{R}$. Let U be open in βX with $U \cap TX \neq \emptyset$. Since βX is regular, there exists V open in βX with $V \subseteq \mathrm{cl}_{\beta X} V \subseteq U$ and $V \cap TX \neq \emptyset$. Now $\mathrm{cl}_X(V \cap LX) \in R(X)$ and is not compact since $\mathrm{cl}_{\beta X}(\mathrm{cl}_X(V \cap LX)) = \mathrm{cl}_{\beta X} V$ and $\mathrm{cl}_{\beta X} V \cap TX \neq \emptyset$, and $V \cap LX$ is non-empty open and dense in $\mathrm{cl}_X(V \cap LX)$; so by Lemma 3.5, there is an $A \in R(\mathrm{cl}_X(V \cap LX))$ such that $A \subseteq V \cap LX$ and A is not compact. Since A is a locally compact, non-compact metric space, there exists an $S \in R(A)$ such that S is separable and non-compact, by Lemma 5.1. Now $S \in R(A)$ and $A \in R(X)$ so $S \in R(X)$ and $S \subseteq A \subseteq LX$, so $TS \in B$. Further, $\mathrm{cl}_{\beta X} S \subseteq \mathrm{cl}_{\beta X} A \subseteq U$ and by Lemma 3.4, $TS \subseteq \mathrm{cl}_{\beta X} S \cap TX$ so $TS \subseteq U \cap TX$. Thus B is a π-basis for TX.

Since a π-basis of a space is a cover for an open dense subspace, it follows from Theorem 5.2 that when $X = \mathrm{cl}_X LX$, TX contains an open dense subspace which is the union of clopen sets each homeomorphic to $T\mathbf{R}$. When it is also assumed that X is separable, then the π-basis B of TX defined in the proof of Theorem 5.2 is a cover of TX. So TX is the union of clopen sets each homeomorphic to $T\mathbf{R}$, and, in addition, this characterizes the property that $X = \mathrm{cl}_X LX$ when X is separable. This fact is included in the next theorem.

THEOREM 5.3. *Let X be a non-compact separable metric space without isolated points. The following are equivalent:*

(a) $X = \mathrm{cl}_X LX$.

(b) *TX is a union of open sets each homeomorphic to $T\mathbf{R}$.*

(c) *TX is a union of clopen sets each homeomorphic to $T\mathbf{R}$.*

Proof. First it will be shown that (a) implies (c). By Proposition 4.8, if $X = \mathrm{cl}_X LX$ then TX is homeomorphic to $\bigcup_{A \in R(X), A \subseteq LX} TA$ with TA clopen. If $A \in R(X)$, $A \subseteq LX$ and A is non-compact, then TA is homeomorphic to $T\mathbf{R}$ since A is locally compact and separable. If A is compact, then $TA = \emptyset$.

Clearly (c) implies (b). To show that (b) implies (a), suppose TX is a union of open subsets each homeomorphic to $T\mathbf{R}$. It is known that $T\mathbf{R}$ has no non-empty open extremally disconnected subsets (since $T\mathbf{R}$ is a dense subset of $\beta\mathbf{N}\backslash\mathbf{N}$), so the same property is true of TX. Now

$$TX = T(\mathrm{cl}_X LX) \cup T(\mathrm{cl}_X NX)$$

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by Theorem 3.3 and since $\mathrm{cl}_X NX$ is a nowhere locally compact separable metric space, $T(\mathrm{cl}_X NX)$ is extremally disconnected [**9**]; thus $T(\mathrm{cl}_X NX) = \emptyset$ so $\mathrm{cl}_X NX = \emptyset$ and $X = \mathrm{cl}_X LX$.

When X is a normal space, we have already seen that $TX = \bigcup_{A \in R(X)} TA$, and when $X = \mathrm{cl}_X LX$ then $TX = \bigcup_{A \in H(LX)} TA$ where $H(LX)$ is the subcollection of $R(X)$ of regular closed subsets of X contained in LX. Next we show that when $X \backslash LX$ is non-empty and compact and the quotient space obtained by identifying $X \backslash LX$ to a point p is first countable at p, then we can find a countable subcollection D of $H(LX)$ such that $TX = \sum_{A \in D} TA$. First it will be shown that when $X \backslash LX$ is non-empty and compact with $X = \mathrm{cl}_X LX$ it suffices to assume that $X \backslash LX$ is a single point when studying TX.

LEMMA 5.4. *Let Y be a normal space and let X be a proper open dense subset of Y. Let $Y_X = X \cup \{Y \backslash X\}$ be the quotient space of Y with $Y \backslash X$ identified to a point, and let $f: Y \rightarrow Y_X$ be the quotient mapping.*

(a) *The quotient space Y_X is normal, and f is a closed irreducible, continuous mapping.*

(b) *If Y is a metric space and $Y \backslash X$ is compact, then Y_X is a metric space.*

(c) *If $Y \backslash X$ is compact and $LY = X$, then $L(Y_X) = X$.*

Proof. (a) Since $Y \backslash X$ is closed in Y, Y_X is normal and f is closed and continuous. To see that f is closed irreducible, let $C \subseteq Y$ be a closed subset such that $f(C) = Y_X$. Since $X \subseteq Y_X$ and f is one-to-one on X, $X \subseteq C$; and since C is closed in Y, $C = Y$.

(b) By part (a), f is closed and continuous. Furthermore, since $Y \backslash X$ is compact, f is perfect. So Y_X is metric since perfect mappings preserve metrizability ([**1**], XI, Theorem 5.2).

(c) Since $Y_X = X \cup \{Y \backslash X\}$ with the identification topology, and since $LY = X$, it is clear that $X \subseteq LY_X$. But Y_X is not locally compact since f is perfect, and the inverse image of a locally compact space under a perfect mapping is locally compact ([**1**], XI, 6.6). Thus $L(Y_X) \neq Y_X$, so $L(Y_X) = X$.

By Lemma 5.4, if X is normal and $\mathrm{cl}_X L_X = X$ with $X \backslash LX$ non-empty and compact, then X_{LX} is normal with $L(X) = L(X_{LX})$ and such that TX and $T(X_{LX})$ are homeomorphic. Further, if X is metric, then X_{LX} is metric. The following two lemmas give some technical results which will be used for the case that X is dense in Y and $Y \backslash X$ is a single point.

LEMMA 5.5. *Let X be a normal space which is first countable at $p \in X$. Let $\{U_n : n \in \mathbf{N}\}$ be a neighborhood base for p of regular open sets such that $\mathrm{cl}_X U_{n+1} \subseteq U_n$ for each $n \in \mathbf{N}$. If $S_1 = X \backslash U_1$ and $S_i = \mathrm{cl}_X(U_{i-1} \backslash \mathrm{cl}_X U_i)$ for $i \geqq 2$, then $TX = \sum_{i=1}^{\infty} TS_i$ where the representation on the right is canonical.*

Proof. Clearly $S_i \in R(X)$ and $p \notin S_i$ for each $i \in \mathbf{N}$. Since $k_{\beta X}$ is a homeomorphism between TEX and TX where $k_{\beta X}{}^{\angle}(X) = EX$ and also between $\lambda(S_i) \cap TEX$ and $\mathrm{cl}_{\beta X} S_i \cap TX$ for each $i \in \mathbf{N}$ by Theorem 2.4 and Lemma

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4.2, it suffices to show that (i) $\lambda(S_i) \cap \lambda(S_j) \cap TEX = \emptyset$ for $i \neq j$ and (ii) $TEX = \bigcup_{i=1}^{\infty} (\lambda(S_i) \cap TEX)$. Since $U_{i-1}\backslash \text{cl}_X U_i \cap U_{j-1}\backslash \text{cl}_X U_j = \emptyset$ for $i \neq j$, $\text{int}_X S_i \cap \text{int}_X S_j = \emptyset$ for $i \neq j$; so $S_i \wedge S_j = \emptyset$ and $\lambda(S_i) \cap \lambda(S_j) = \emptyset$ and thus (i) is true. It is clear that

$$\bigcup_{i=1}^{\infty} (\lambda(S_i) \cap TEX) \subseteq TEX.$$

So suppose $\alpha \in S(R(X))$ but $\alpha \notin \bigcup_{i=1}^{\infty} \lambda(S_i)$. Then for any $n \in \mathbf{N}$, $\alpha \notin \lambda(X\backslash U_n)$ since $\lambda(X\backslash U_n) = \lambda(S_1) \cup \ldots \cup \lambda(S_n)$. So $\alpha \in \lambda(\text{cl}_X U_n)$ for each $n \in \mathbf{N}$. Thus it follows that $\alpha \in k_{\beta X}{}^{\angle}(p) \subseteq EX$ since $\{U_n : n \in \mathbf{N}\}$ is a neighborhood base for p. So $\alpha \notin TEX$. Thus $TEX \subseteq \bigcup_{i=1}^{\infty} \lambda(S_i)$ and (ii) follows.

LEMMA 5.6. *Let X be a space without isolated points with X first countable at $p \in X$ and such that p has no compact neighborhood. Then for any countable neighborhood base $\{U_n : n \in \mathbf{N}\}$ for p of regular open sets such that $\text{cl}_X U_{n+1} \subseteq U_n$, there is a subsequence $\{n_k\}$ of $\mathbf{N}$ such that $\text{cl}_X(U_{n_k}\backslash \text{cl}_X U_{n_{k+1}})$ is not compact for each k.*

Proof. Suppose $\{U_n : n \in \mathbf{N}\}$ is a neighborhood base for p of regular open sets for which there is no subsequence $\{n_k\} \subseteq \mathbf{N}$ such that $\text{cl}_X(U_{n_k}\backslash \text{cl}_X U_{n_{k+1}})$ is not compact for each k. Then $\text{cl}_X(U_n\backslash \text{cl}_X U_{n+1})$ is compact for each $n \geqq K$ for some $K \in \mathbf{N}$. Let $M > K$. Since $p \in U_M \subseteq \text{cl}_X U_M$, $\text{cl}_X U_M$ is not compact; so, there exists a family F of closed subsets of X which are contained in $\text{cl}_X U_M$ and which have non-empty finite intersections, but $\bigcap F = \emptyset$. Thus, for some $C \in F$, $p \notin C$, so $C \cap U_n = \emptyset$ for some $n > M$; and so

$$C \subseteq \text{cl}_X U_M\backslash U_n \subseteq \bigcup_{k=M}^{n=1} \text{cl}_X(U_k\backslash \text{cl}_X U_{k+1})$$

which is a contradiction.

The following proposition describes the structure of TX when $X\backslash LX$ is non-empty and compact and X_{LX} is first countable at the point to which $X\backslash LX$ is identified.

PROPOSITION 5.7. *Let X be a non-compact normal space such that $\text{cl}_X LX = X$ where $X\backslash LX$ is non-empty and compact, and for which there is a countable collection $\{U_n : n \in \mathbf{N}\}$ of open sets such that if $X\backslash LX \subseteq W$ where W is open in X then there exists an $n \in \mathbf{N}$ for which $X\backslash LX \subseteq U_n \subseteq W$. Then TX is homeomorphic to $\sum_{\mathbf{N}} TX_n$ where the disjoint countable sum can be represented canonically for a collection $\{X_n : n \in \mathbf{N}\} \subseteq R(X)$ with $X_n \subseteq LX$. Further;*

(a) *if X has countable π-weight and each closed pseudo-compact subspace of X is compact, then the X_n's can be chosen so that $TX_i \neq \emptyset$ for each i;*

(b) *if X is a metric space without isolated points, then TX is homeomorphic to $\sum_{\mathbf{N}} T(\sum_{d_n} \mathbf{R})$ where $dX \geqq d_n > 0$ for $n \in \mathbf{N}$.*

Proof. As indicated before Lemma 5.4, it suffices to consider the case where $X\backslash LX = \{p\}$. The hypothesis implies that X is first countable at p. By Lemmas

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5.5 and 5.6, $TX = \sum_{\mathbf{N}} TX_i$ where X_i is a non-compact regular closed subset of X such that $p \notin X_i$ for each $i \in \mathbf{N}$ and the representation is canonical. Since $p \notin X_i$, $X_i \subseteq LX$. For (a), by Theorem 1.3, $TX_i \neq \emptyset$ since X_i can be chosen to be non-compact by Lemma 5.6. For (b), $TX_i = T(\sum_{d_i} \mathbf{R})$ where $d_i = dX_i$ by Theorem 1.4.

COROLLARY 5.8. *Any two separable metric spaces without isolated points for which the set of non-locally compact points is non-empty and compact have homeomorphic sets of remote points.*

Proof. By Proposition 5.7, $TX = \sum_{\mathbf{N}} T\mathbf{R}$.

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