Appendix D The beta function in supersymmetric Yang–Mills theory

We have seen that holomorphy is a powerful tool with which to understand the dynamics of supersymmetric field theories. But one can easily run into puzzles and paradoxes. One source of confusion is the holomorphy of the gauge coupling. At tree level, the gauge coupling arises from a term in the action of the form

$$\int d^2\theta S W_\alpha^2, \tag{D1}$$

where $S = -1/4g^2 + ia$. This action, in perturbation theory, has a symmetry

$$S \to S + i\alpha. \tag{D2}$$

This is just an axion shift symmetry. Combined with holomorphy, it greatly restricts the form of the effective action. The only allowed terms are:

$$\mathcal{L}_{\text{eff}} = \int d^2\theta \, (S + \text{constant}) W_\alpha^2. \tag{D3}$$

The constant term corresponds to a one-loop correction. But higher-loop corrections are forbidden.

However, it is well known that there are two-loop corrections to the beta function in supersymmetric Yang–Mills theories (higher-loop corrections have also been computed). Does this represent an inconsistency? This puzzle can be stated – and has been stated – in other ways. For example, the axial anomaly lies in a supermultiplet with the conformal anomaly – the anomaly in the trace of the stress tensor. One usually says that the axial anomaly is not renormalized but that the trace anomaly is proportional to the beta function.

The resolution to this puzzle was provided by Shifman and Vainshtein; we will present it in a form developed by Arkani-Hamed and Murayama and updated by Dine and Festuccia. The idea is to exploit the finiteness of $N = 4$ supersymmetric Yang–Mills to use it as a regulator for the pure $N = 1$ gauge theory (this can be generalized to a variety of other theories as well). We take the $N = 4$ theory and add masses for the adjoint chiral fields. Calling these masses M, the low-energy theory is just pure Yang–Mills. The ultraviolet divergences of the theory necessarily become logarithms of M.

In the language of $N = 1$, the $N = 4$ theory is often presented in a way which makes the $SU(4)$ R-symmetry manifest:

$$\mathcal{L} = \int d^4\theta \frac{1}{g^2}\Phi_i^\dagger \Phi_i - \frac{1}{32\pi^2}\int d^2\theta \left(\frac{8\pi^2}{g^2} + i\theta\right) W_\alpha^2 + \int d^2\theta \, \frac{1}{g^2}\Phi_1\Phi_2\Phi_3. \tag{D4}$$

https://doi.org/10.1017/9781009290883.043 Published online by Cambridge University Press

It is helpful to present the theory in a fashion which is holomorphic in the gauge coupling, $\tau = 8\pi^2/g^2 + i\theta$. This is achieved by the rescaling $\Phi_i \to g^{2/3}\Phi_i$. Then, including a holomorphic mass term for the Φ_is,

$$\mathcal{L} = \int d^4\theta \frac{1}{g^{2/3}} \Phi_i^\dagger \Phi_i - \frac{1}{32\pi^2} \int d^2\theta \left(\frac{8\pi^2}{g^2} + i\theta \right) W_\alpha^2 + \int d^2\theta \left(\Phi_1\Phi_2\Phi_3 + m_{\text{hol}}\Phi_i\Phi_i \right). \tag{D5}$$

Now, consider integrating out the physics between two scales, m_1 and m_2. Since there are no infrared divergences and we have written the Lagrangian in a manifestly holomorphic form, the coupling renormalization is necessarily holomorphic:

$$\frac{8\pi^2}{g^2(m_2)} = \frac{8\pi^2}{g^2(m_1)} + b_0 \log(m_{\text{hol}}^{(2)}/m_{\text{hol}}^{(1)}). \tag{D6}$$

From the Lagrangian, Eq. (D5) we see that, at the classical level, the physical mass (i.e. the actual mass of the Φ_i particles) and the holomorphic mass are related by

$$m_{\text{hol}} = g^{-2/3} m. \tag{D7}$$

So, defining the beta function by

$$\beta(g) = \frac{\partial g}{\partial \log m} \tag{D8}$$

and differentiating Eq. (D6) yields

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{3N}{1 - 2Ng^2/(16\pi^2)}. \tag{D9}$$

This expression is known as the Novikov–Shifman–Vainstein–Zakharov (NSVZ) beta function. It is, in some sense, exact since the holomorphic expression is exact. However, if we insist, for example, that m should be the physical ("pole") mass of the Φ_i particles then the relation between m and m_{hol} is corrected in each order of perturbation theory. Indeed, the scheme in which the NSVZ beta function is exact is precisely that in which one insists that m and g are related as in Eq. (D7). So, such exact relations must be used with care. In any case this analysis is readily extended to gauge theories with matter, which can be embedded in finite $N = 2$ theories.

Suggested reading

The use of finite theories as regulators was developed in Arkani-Hamed and Murayama (2000); the presentation described here appears in Dine *et al.* (2011).

https://doi.org/10.1017/9781009290883.043 Published online by Cambridge University Press

Exercise

Starting with the finite $N - 2$ theories discussed in Chapter 16, proceed as we did for the $N - 4$ theories (Eqs. (D4)–(D6)) to derive the beta function for $N - 1$ theories with matter, Eq. (16.35). Note that the analysis is valid for only a restricted number of flavors.

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