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THE MOD 2 HOMOLOGY OF $Sp(n)$ INSTANTONS AND THE CLASSIFYING SPACE OF THE GAUGE GROUP

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We study the mod 2 homology of the moduli space of instantons associated with the principal $Sp(n)$ bundle over the four-sphere and the classifying space of the gauge group using the Serre spectral sequence and the homology operations.

1. INTRODUCTION

Let G be a compact, connected simple Lie group. The fact that $\pi_3(G) = \pi_4(BG) = Z$ leads to the classification of principal G bundles P_k over S^4 by the integer k in Z. For a given P_k, the orbit spaces of connections up to based gauge equivalence is homotopy equivalent to the triple loop space of G [**2**]. That is, $\mathcal{C}_k = \mathcal{A}_k/\mathcal{G}^b(P_k) \simeq \Omega^3_k G$ where $\mathcal{A}_k$ is the space of all connections in P_k and $\mathcal{G}^b(P_k)$ is the based gauge group which consists of all base point preserving automorphisms on P_k. Let $\mathcal{M}_k$ be the space of based gauge equivalence classes of all connections in P_k satisfying the Yang-Mills self-duality equations, which we call the moduli space of G instantons. Then there is a natural inclusion map $i : \mathcal{M}_k \to \mathcal{C}_k \simeq \Omega^3_k G$ and the inclusion map $i : \mathcal{M}_\infty \to \mathcal{C}_\infty$ induces a homotopy equivalence [**6**] where $\mathcal{M}_\infty$ and $\mathcal{C}_\infty$ are the direct limits under the inclusions.

While $\Omega^3_k G$ is infinite dimensional and each $\Omega^3_k G$ is homotopy equivalent to $\Omega^3_0 G$ for any component k, $\mathcal{M}_k$ is finite dimensional and the dimension of $\mathcal{M}_k$ increases as k increases. Hence whenever k increases, more elements of the homology of $\Omega^3_0 G$ are contained in the homology of $\mathcal{M}_k$. So it is reasonable to study the homology of $\Omega^3_0 G$ to get information about the homology of the instanton space .

Let $Sp(n)$ denote the symplectic group, that is, the group of $n \times n$ quaternionic unitary matrices. Much work has been done on the moduli space of $Sp(1)$ ($\cong SU(2)$ $\cong Spin(3)$) instantons [**2, 3, 4**]. In this paper we first study the mod 2 homology of the moduli space of $Sp(2)$ instantons exploiting the inclusion map into the triple loop space of $Sp(2)$ with the aid of the Dyer–Lashof operations. Then we study the rational type of the classifying space of the gauge group and we compute the mod 2 homology

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of the classifying space of the gauge group via the Serre spectral sequence. Finally we study the general $Sp(n)$ case by the same method.

Since the computations are 2-primary, all coefficients of homology are assumed to be $Z/(2)$ unless otherwise mentioned.

2. THE $Sp(2)$ CASE

Let $E(x)$ be the exterior algebra on x and $P(x)$ be the polynomial algebra on x and $\Gamma(x)$ be the divided power Hopf algebra on x which is free over $\gamma_i(x)$ as a $Z/(2)$ module with the product $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$ and with the the coproduct $\Delta(\gamma_n(x)) = \sum_{i=0}^{n} \gamma_{n-i}(x) \otimes \gamma_i(x)$.

For an $(n+1)$-fold loop space, there are homology operations, called the Dyer–Lashof operations,

$$Q_i : H_q(\Omega^{n+1}X) \longrightarrow H_{2q+i}(\Omega^{n+1}X)$$

defined for $0 \leqslant i \leqslant n$ which are natural for an $(n+1)$ fold loop space. Let Q_i^a be the iterated operation $Q_i \dots Q_i$ (α times). Since S^3 and $Sp(2)$ are Lie groups, $\Omega^3 S^3 \simeq \Omega^4 BS^3$ and $\Omega^3 Sp(2) \simeq \Omega^4 BSp(2)$ where $\simeq$ means homotopy equivalence. So we can define Q_i for $0 \leqslant i \leqslant 3$. In particular the moduli space of instantons behaves like C_4–space up to homotopy, so we can define the homology operations Q_i for $0 \leqslant i \leqslant 3$ [4]. Throughout this paper, the subscript of an element always means the degree, for example the degree of the element x_i is i.

It is well known that

$$H_*(Sp(2)) = E(x_3) \otimes E(x_7).$$

Since $\pi_3(Sp(2)) = Z$, $\pi_0(\Omega^3 Sp(2)) = Z$. Let $\Omega_0^3 Sp(2)$ be the zero component of $\Omega^3 Sp(2)$. We first compute the homology of $\Omega_0^3 Sp(2)$, that is, $\mathcal{M}_\infty$, the direct limit of $\mathcal{M}_k$ for $Sp(2)$. Let us recall the following facts.

$$H_*(\Omega_0^3 S^3) = P\big(Q_1^a Q_2^b[1] * [-2^{a+b}] : a \geqslant 0,\ b \geqslant 0\big),$$
$$H_*(\Omega^3 S^{2n+1}) = P\big(Q_1^a Q_2^b z_{2n-2} : a \geqslant 0,\ b \geqslant 0\big) \quad \text{for} \quad n > 1.$$

Here [1] is the image of the generator in $\widetilde{H}_0(S^0)$ for the map: $S^0 \to \Omega^3 S^3$ and $*$ is the loop sum Pontryagin product. If $x \in H_*(\Omega_s^3 S^3)$ and $y \in H_*(\Omega_t^3 S^3)$, $x * y \in H_*(\Omega_{s+t}^3 S^3)$ and $Q_i(x) \in H_*(\Omega_{2s}^3 S^3)$ where $\Omega_r^3 S^3$ means the r-component of $\Omega^3 S^3$.

THEOREM 2.1.

$$H_*(\Omega_0^3 Sp(2)) = P\big(Q_1^a Q_2^b[1] * [-2^{a+b}] : a \geqslant 0,\ b \geqslant 0\big) \otimes P\big(Q_1^a Q_2^b z_4 : a \geqslant 0,\ b \geqslant 0\big).$$

PROOF: We have the following map of fibrations:

$$\begin{array}{ccccc} \Omega^4 S^7 & \longrightarrow & * & \longrightarrow & \Omega^3 S^7 \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_0^3 S^3 & \xrightarrow{\Omega^3 \iota} & \Omega_0^3 Sp(2) & \xrightarrow{\Omega^3 p} & \Omega^3 S^7 \end{array}$$

Consider the Serre spectral sequence for the bottom row fibration with

$$E^2 = H_*(\Omega^3 S^7) \otimes H_*(\Omega_0^3 S^3).$$

Since the Dyer–Lashof operation satisfies naturality and commutes with the homology suspension, the differentials for the above spectral sequence are completely determined by the first differential from z_4 where $H_*(\Omega^3 S^7) = P(Q_1^a Q_2^b z_4 : a \geqslant 0,\ b \geqslant 0)$. If this differential is non-trivial, then target of the differential will be $Q_1 Q_1[1] * [-4]$ because of the uniqueness of the primitive element in that dimension. Note that the target of the first non-trivial differential is a primitive element in the spectral sequence of a Hopf algebra. Here this element is the image of the lowest-dimensional element in $H_*(\Omega^4 S^7)$ for the first column map which is, in fact, the Hurewicz image of $S^3 \subset \Omega^4 S^7$ into $\Omega_0^3 S^3$. However $Sq_*^1 Q_1 Q_1[1] * [-4]$ is the non-zero element $(Q_1[1] * [-2])^2$ in $H_*(\Omega_0^3 S^3)$. This is a contradiction to the naturality of the Steenrod actions. So the differential from z_4 is trivial. Hence the above spectral sequence collapses from the E^2-term. ∎

Note that $(\Omega^3 \iota)_*$ is one to one in the mod 2 homology. We shall use this fact later.

Let $\mathcal{M}_k(G)$ denote the based moduli space of all G instantons with instanton number k. Let $\mathcal{M}'_k(G)$ be the moduli space of all G instantons with instanton number k, that is, the space of all G instantons with instanton number k modulo the full gauge group. Let $C_G(SU(2))$ be the centraliser of $SU(2)$ in G.

THEOREM 2.2. [5, Proposition 3.1] *Let G be a compact simple simply connected Lie group. Then the based moduli space $\mathcal{M}_1(G)$ fibers trivially with the fiber $G/C_G(SU(2))$ over $\mathcal{M}'_1(G)$ which is homeomorphic to the five ball. Furthermore, the composition of maps*

$$G/C_G(SU(2)) \xrightarrow{j} \mathcal{M}_1(G) \xrightarrow{i_1} \mathcal{C}_1(G) \xrightarrow{\theta} \Omega_1^3 G$$

is given by the map $J = \theta \circ i_1 \circ j$:

$$J(C_G(SU(2))g) = [x \to g^{-1} i(x) g]$$

where j and i_1 are natural inclusions, i is a fixed embedding of $SU(2)$ into G, and θ is the Atiyah-Jones equivalence. ∎

Now $q \in Sp(1)$ can be imbedded into $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \in Sp(2)$. Since the centre of $Sp(1)$ is 1 or -1, $Sp(1)/C(Sp(1)) = RP^3$ and $C_{Sp(2)}(Sp(1)) = Z/2 \times Sp(1)$. Just considering the first column vector (q_1, q_2) which is the same class as $(-q_1, -q_2)$ with the condition $q_1^2 + q_2^2 = 1$, we get that $SP(2)/C_{Sp(2)}(Sp(1)) = RP^7$. Hence $\mathcal{M}_1(Sp(1)) \simeq RP^3 \simeq SO(3)$ and $\mathcal{M}_1(Sp(2)) \simeq RP^7$. We have the following fibration:

$$Sp(1) \xrightarrow{\iota} Sp(2) \xrightarrow{p} S^7$$

We know that

$$H_*(SO(3)) = E(x_1, x_2).$$

Moreover we have the following facts (see [**5**, Corollary 5.17]):

$$\begin{aligned} J_*(x_1) &= Q_1[1] * [-1], \\ J_*(x_2) &= Q_2[1] * [-1], \\ J_*(x_1x_2) &= Q_3[1] * [-1]. \end{aligned}$$

Now we shall calculate the homology of $Sp(2)$-instantons by the study of the map J. If $J_*(x) \neq 0$ in $H_*\big(\Omega_1^3 Sp(2)\big)$ for some element x and $Q_i(J_*(x)) \neq 0$ for some i in $H_*\big(\Omega_k^3 Sp(2)\big)$ for the corresponding component k of $\Omega^3 Sp(2)$, by the naturality of the Dyer–Lashof operation $Q_i(j_*(x))$ is also not zero in $H_*(\mathcal{M}_k(Sp(2)))$. Hence we can get the rich non-trivial homology elements in $H_*(\mathcal{M}_k(Sp(2)))$ by the actions of the Dyer–Lashof operations on the special elements such that the images of J_* for those elements are not zero.

We have the following map:

$$\begin{array}{ccc} H_*(\mathcal{M}_1(Sp(2))) & \xrightarrow{\theta_* \circ (i_1)_*} & H_*\big(\Omega_1^3 Sp(2)\big) \\ {[1]} & \mapsto & {[1]} \end{array}$$

Then we can apply the Dyer-Lashof operations Q_i for $0 \leqslant i \leqslant 3$ on the element $[1]$. Remember that $Q_i^a[1]$ is the homology element in the 2^a component. By analysing non-zero Dyer-Lashof actions on $[1]$ in $H_*\big(\Omega_1^3 Sp(2)\big)$, we can get the following non-trivial homology elements.

Proposition 2.3. *There are the following non-zero elements in $H_*(\mathcal{M}_k(Sp(2)))$. For any $a, b, c \geqslant 0$,*

$$Q_0^a Q_1^b Q_2^c[1] \in H_{((2^{c+1}-1)2^b-1)2^a}(\mathcal{M}_{2^{a+b+c}}(Sp(2))).$$

We have the following commutative diagram up to homotopy:

$$\begin{array}{ccc} RP^3 \simeq Sp(1)/C(Sp(1)) & \xrightarrow{\iota} & Sp(2)/C_{Sp(2)}(Sp(1)) \simeq RP^7 \\ {\scriptstyle J}\downarrow & & {\scriptstyle J}\downarrow \\ \Omega_1^3 Sp(1) & \xrightarrow{\Omega^3\iota} & \Omega_1^3 Sp(2) \end{array} \tag{2.4}$$

Exploiting the fact that $J_*(x_1)$, $J_*(x_2)$ and $J_*(x_1x_2)$ are not zero in $H_*(\Omega_1^3 Sp(1))$, Boyer and Mann got the following theorem. Let z_i be the element in $H_*(\mathcal{M}_1(Sp(1)))$ such that $(\theta_* \circ (i_1)_*(z_i)) * [1] = Q_i([1])$ for $i = 1, 2, 3$.

Theorem 2.5. [4, Theorem 9.7] *$H_*(\mathcal{M}_k(Sp(1)))$ contains elements of the form*

$$z = z(I_1, \cdots, I_n, j_1, \cdots, j_n) = Q_{I_1}(z_{j_1}) * \cdots * Q_{I_n}(z_{j_n})$$

for all sequences $(I_1, \cdots, I_n, j_1, \cdots, j_n)$ such that $\sum_{m=1}^{n} 2^{l(I_m)} \leqslant k$. Here each $I_m = (i_1, \cdots, i_{l(I_m)})$ is an admissible sequence $0 \leqslant i_1 \leqslant \cdots \leqslant i_{l(I_m)} \leqslant 3$ and $0 \leqslant j_a \leqslant 3$ for all $1 \leqslant a \leqslant n$.

Corollary 2.6. *Every element in Theorem 2.5 is also non-zero in $H_*(\mathcal{M}_k(Sp(2)))$.*

Proof: Since the map ι_* is one to one and $(\Omega^3\iota)_*$ is also one to one by Theorem 2.1, each element in Theorem 2.5 is also non-zero in $H_*(\Omega_k^3 Sp(2))$ and so in $H_*(\mathcal{M}_k(Sp(2)))$. ∎

For the homology information, we shall try to find more elements whose images under J_* are not zero in $H_*(\Omega_1^3 Sp(2))$. It is well known that

$$H^*(RP^7) = P(z_1)/(z_1^8).$$

Then $H_*(RP^7)$ is free on generators $x_1, x_2, \ldots, x_7$ such that

$$\langle z^i, x_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

So we have the following coproduct structure:

$$\Delta\,(x_i) = \sum_{k=0}^{i} x_k \otimes x_{i-k}.$$

We consider the above diagram (2.4) again. Since $J_*(x_1)$, $J_*(x_2)$ are not zero,

$$\begin{aligned} \Delta\,(J_*(x_4)) &= J_*(\Delta\,(x_4)) \\ &= \sum_{k=0}^{k=4} J_*(x_k) \otimes J_*(x_{4-k}) \\ &\neq 0. \end{aligned}$$

Hence $J_*(x_4) \neq 0$. So there exists an element, say v_4, in $H_*(\mathcal{M}_1(Sp(2)))$ such that

$$\begin{array}{ccc} H_*(\mathcal{M}_1(Sp(2))) & \xrightarrow{\theta_* \circ (i_1)_*} & H_*(\Omega^3_1 Sp(2)) \\ v_4 & \longrightarrow & J_*(x_4) \end{array}$$

Since there does not exist a non-trivial 4-dimensional homology class in $H_*(RP^7; Z/(p))$ for odd prime p, we cannot apply this method for the odd prime case. Note that every element in $H_*(\Omega^3_0 Sp(2))$ becomes a stable element in $H_*(\Omega^3_0 Sp) = H_*(BO)$. From the coproduct structure, $J_*(x_4) = z_4 * [1]$. Now we get

PROPOSITION 2.7. *There are the following non-zero elements in $H_*(\mathcal{M}_k(Sp(2)))$. For any $a, b, c \geqslant 0$,*

$$Q_0^a Q_1^b Q_2^c(v_4) \in H_{((3\cdot 2^{c+1}-1)2^b-1)2^a}(\mathcal{M}_{2^{a+b+c}}(Sp(2))).$$

Now we consider the loop sum product $*$ for the elements in $H_*(\mathcal{M}_k(Sp(2)))$. Remember that if x and y are homology classes in the s and t components, then $x * y$ is the homology class in the $s+t$ component.

THEOREM 2.8. *The loop sum products of any elements in Proposition 2.3, Corollary 2.6 and Proposition 2.7 are also non-trivial homology elements of the space $\mathcal{M}_k(Sp(2))$ for the corresponding k.*

Now we turn to the computation of the homology for the classifying space of the gauge group. Let $\mathcal{G}_k$ be the gauge group of the principal $Sp(2)$ bundle P_k over S^4 with the instanton number k. From [**1**, Proposition 2.4] we can get

$$B\mathcal{G}_k \simeq Map_{p_k}(S^4, BSp(2))$$

where the subscript p_k denotes the component of a map of S^4 into $BSp(2)$ which induces P_k. First we shall study the rational type of $B\mathcal{G}_k$. We have the following theorem.

THEOREM 2.9. [**1**, Theorem 2.6] *Suppose that X is any finite complex. Let $\pi_q(Y) = 0$ for $q \neq n$ and $\pi_n(Y) = \pi$, that is, $Y = K(\pi, n)$. Then*

$$Map(X, Y) \simeq \prod_q K(H^q(X; \pi), n-q).$$

PROPOSITION 2.10. *Over the rationals,*

$$B\mathcal{G}_k \simeq_Q K(Z,4) \times K(Z,4) \times K(Z,8).$$

PROOF: Since $BSp(2) \simeq_Q K(Z,4) \times K(Z,8)$,

$$Map(S^4, BSp(2)) \simeq_Q Map(S^4, K(Z,4)) \times Map(S^4, K(Z,8)).$$

Applying the above Theorem, we get

$$\begin{aligned} Map(S^4, BSp(2)) &\simeq_Q Map(S^4, K(Z,4)) \times Map(S^4, K(Z,8)) \\ &\simeq \prod_q K(H^q(S^4; Z), 4-q) \times \prod_q K(H^q(S^4; Z), 8-q) \\ &\simeq Z \times K(Z,4) \times K(Z,4) \times K(Z,8). \end{aligned}$$

Since $Map(S^4, BSp(2)) \simeq Map_{P_k}(S^4, BSp(2)) \times Z$,

$$\begin{aligned} B\mathcal{G}_k &\simeq Map_{P_k}(S^4, BSp(2)) \\ &\simeq_Q K(Z,4) \times K(Z,4) \times K(Z,8). \end{aligned}$$

∎

Then we also get

COROLLARY 2.11.

$$\begin{aligned} H_*(B\mathcal{G}_k; Q) &= H_*(K(Z,4); Q) \otimes H_*(K(Z,4); Q) \otimes H_*(K(Z,8); Q) \\ &= P(a_4) \otimes P(b_4) \otimes P(c_8). \end{aligned}$$

THEOREM 2.12. *As a vector space,*

$$H_*(B\mathcal{G}_k) = H_*(\Omega_0^3 Sp(2)) \otimes H_*(BSp(2)).$$

PROOF: There is a fibration:

$$Map^*(S^4, BSp(2)) \longrightarrow Map(S^4, BSp(2)) \longrightarrow BSp(2)$$

where $*$ means the base point preserving maps. Since $Map^*(S^4, BSp(2)) = \Omega_0^3 Sp(2) \times Z$, we get the following fibration:

$$\Omega_0^3 Sp(2) \longrightarrow Map_{P_k}(S^4, BSp(2)) \longrightarrow BSp(2).$$

Note that this fibration is not an H–fibration. Consider the Serre spectral sequence converging to $H_*(Map_{P_k}(S^4, BSp(2)))$ with

$$E^2 = H_*(BSp(2)) \otimes H_*(\Omega_0^3 Sp(2)).$$

The possible first non-zero differential is the transgression from some $4n$–dimensional element, say, χ_{4n} where $H_*(BSp(2)) = \Gamma(x_4, x_8)$ as a coalgebra. Since the target of the first non-zero differential is primitive, the target will be a $(4n-1)$ dimensional primitive element. But in $H_*(\Omega_0^3 Sp(2))$, the Sq_*^1 action on every $(4n-1)$ dimensional primitive element is non trivial. In fact every $(4n-1)$ dimensional primitive element, say y_{4n-1}, in $H_*(\Omega_0^3 Sp(2))$ is $Q_1 y_{2n-1}$ for some $(2n-1)$ dimensional primitive element, y_{2n-1}. So from the Nishida relation

$$Sq_*^1 y_{4n-1} = Sq_*^1 Q_1(y_{2n-1}) = (y_{2n-1})^2.$$

Since χ_{4n} is transgressive, $Sq_*^1(\tau(\chi_{4n})) = \tau(Sq_*^1 \chi_{4n})$ where τ is the transgression. Since the Sq_*^1 action on every element in $H_*(BSp(2))$ is trivial, this leads a contradiction.

Hence the Serre spectral sequence collapses from the E^2-term. So $E^2 = E^\infty$ and we get the conclusion. ∎

3. The $Sp(n)$ case

In this section we study the mod 2 homology of the moduli space of $Sp(n)$ instantons and the classifying space of the gauge group.

Theorem 3.1.

$$H_*(\Omega_0^3 Sp(n)) = P\big(Q_1^a Q_2^b[1] * [-2^{a+b}] : a \geqslant 0,\ b \geqslant 0\big) \otimes P\big(Q_1^a Q_2^b z_{4m} : 1 \leqslant m \leqslant n-1,\ a \geqslant 0,\ b \geqslant 0\big).$$

Proof: We prove this inductively. It is true for $n = 2$ by Theorem 2.1 . Assume that it is true for $n = k$. There is the following map of fibrations:

$$\begin{array}{ccccc} \Omega^4 S^{4k+3} & \longrightarrow & * & \longrightarrow & \Omega^3 S^{4k+3} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_0^3 Sp(k) & \xrightarrow{\Omega^3 \iota} & \Omega_0^3 Sp(k+1) & \xrightarrow{\Omega^3 p} & \Omega^3 S^{4k+3} \end{array}$$

Consider the Serre spectral sequence for the bottom row fibration with

$$E^2 = H_*(\Omega^3 S^{4k+3}) \otimes H_*(\Omega_0^3 Sp(k)).$$

Like in Theorem 2.1, the differentials for this spectral sequence are completely determined by the first differential from z_{4k} where $H_*(\Omega^3 S^{4k+3}) = P\big(Q_1^a Q_2^b z_{4k} : a \geqslant 0,\ b \geqslant 0\big)$. If the differential from z_{4k} is non-trivial, then the target of the differential will be a $(4k-1)$ primitive element. But the Sq_*^1 action of a $(4k-1)$ primitive element is non–trivial in $H_*(\Omega_0^3 Sp(k))$ by the same argument as Theorem 2.12. So the spectral sequence collapses from the E^2 term and we get the conclusion. ∎

We can prove the following two Propositions in the same manner as the $Sp(2)$ case.

Proposition 3.2. *There are the following non-zero elements in* $H_*(\mathcal{M}_k(Sp(n)))$. *For any* $a, b, c \geqslant 0$,

$$Q_0^a Q_1^b Q_2^c[1] \in H_{((2^{c+1}-1)2^b-1)2^a}(\mathcal{M}_{2^{a+b+c}}(Sp(n))).$$

Proposition 3.3. *There are the following non-zero elements in* $H_*(\mathcal{M}_k(Sp(n)))$. *For any* $a, b, c \geqslant 0$,

$$Q_0^a Q_1^b Q_2^c(v_{4m}) \in H_{(((2m+1)2^{c+1}-1)2^b-1)2^a}(\mathcal{M}_{2^{a+b+c}}(Sp(n))), \quad 1 \leqslant m \leqslant n-1.$$

We now compute the homology of the classifying space of the gauge group.

Proposition 3.4. *Over the rationals,*

$$B\mathcal{G}_k \simeq_Q \prod_{m=1}^{n-1} (K(Z,4m) \times K(Z,4m)) \times K(Z,4n).$$

Proof: Since $BSp(n) \simeq_Q \prod_{m=1}^{n} K(Z,4m)$,

$$\begin{aligned} Map(S^4, BSp(n)) &\simeq_Q \prod_{m=1}^{n} Map(S^4, K(Z,4m)) \\ &\simeq Z \times \prod_{m=1}^{n-1} (K(Z,4m) \times K(Z,4m)) \times K(Z,4n). \end{aligned}$$

Hence $B\mathcal{G}_k \simeq_Q \prod_{m=1}^{n-1} (K(Z,4m) \times K(Z,4m)) \times K(Z,4n)$. ∎

Theorem 3.5. *As a vector space,*

$$H_*(B\mathcal{G}_k) = H_*(\Omega_0^3 Sp(n)) \otimes H_*(BSp(n)).$$

Proof: We have the following fibration:

$$\Omega_0^3 Sp(n) \longrightarrow Map_{P_k}(S^4, BSp(n)) \longrightarrow BSp(n).$$

Consider the Serre spectral sequence converging to $H_*(Map_{P_k}(S^4, BSp(n)))$ with

$$E^2 = H_*(BSp(n)) \otimes H_*(\Omega_0^3 Sp(n)).$$

The possible first non-zero differential is the transgression from some $4k$-dimensional element where $H_*(BSp(n)) = \Gamma(x_{4m} : 1 \leqslant m \leqslant n)$ as a coalgebra. But by the same reason as in the proof of Theorem 2.12, there is no non-trivial differential. Hence the Serre spectral sequence collapses from the E^2-term and we obtain the conclusion. ∎

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