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NUMERICAL INVARIANTS IN HOMOTOPICAL ALGEBRA, II-APPLICATIONS

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Introduction. This paper deals with some applications in the results obtained in "Numerical Invariants in Homotopical Algebra" [**7**]. The applications are mainly concerned with the homotopy theory of modules developed by P. J. Hilton [**4**]. However we have to restrict the class of rings because we want to obtain a situation where the axioms of Quillen [**6**] hold good. This paper is organised as follows.

§ 1 deals with the injective homotopy (i-homotopy) theory of modules over a Dedekind domain $\wedge$. Defining Cofibrations, Weak Equivalences and Fibrations to be respectively the collection of monomorphisms, i-homotopy equivalences and maps satisfying the lifting property (abbreviated as L.P.) mentioned below we prove in § 1 that the category $\mathscr{C}$ of $\wedge$-modules satisfies the axioms of Quillen [**6**].

> ($L.P.$) A map $p : E \rightarrow M$ in $\mathscr{C}$ satisfies ($L.P.$) if given any injective module J and any map $f : J \rightarrow M$ there exists a lift $g : J \rightarrow E$ (i.e., $p \circ g = f$) of f.

It is easily seen that all the objects in $\mathscr{C}$ are fibrant and cofibrant simultaneously. It will turn out that both the notions of left homotopy and right homotopy considered by Quillen [**6**] agree in this case with the notion of i-homotopy introduced by Hilton [**4**; **5**, Chapter 13].

§ 2 deals with the projective homotopy (p-homotopy) theory of finitely generated $\wedge$ modules where $\wedge$ is a principal ideal domain (PID). Defining Fibrations, Weak Equivalences and Cofibrations to be respectively the collection of epimorphisms, projective homotopy equivalences and maps having the extension property ($E.P.$) mentioned below we prove in § 2 that the category $\mathscr{F}$ of finitely generated modules over a PID satisfies the axioms of Quillen.

> ($E.P.$) A map $q : M \rightarrow N$ is said to have the $E.P$ if given any finitely generated free $\wedge$-module F and any map $\alpha : M \rightarrow F$ there exists a map $\beta : N \rightarrow F$ satisfying $\beta \circ q = \alpha$.

Again it turns out that all the objects in $\mathscr{F}$ are simultaneously fibrant and cofibrant. The notions of left homotopy and right homotopy in the case of $\mathscr{F}$ coincide with the notion of p-homotopy of Hilton [**4**].

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It is easily checked that the categories $\mathscr{C}$ and $\mathscr{F}$ satisfy the axioms A_1 to A_5 and W of [7]. Hence all the results obtained in [7] are valid for $\mathscr{C}$ and $\mathscr{F}$. Following the general development of Quillen [6] we see that in the categories $\mathscr{C}$ and $\mathscr{F}$ for each object M it is possible to associate objects ΣM and ΩM which are well-determined up to homotopy type. This means that in the case of $\mathscr{C}$ the objects ΣM and ΩM are determined unique up to i-homotopy type and in the case of $\mathscr{F}$ they are determined unique up to p-homotopy type. It will turn out that for any M in $\mathscr{C}$ any candidate for ΣM following Quillen is automatically a candidate to be the suspension of M as defined by Hilton in [4] and vice versa. Similarly in the case of $\mathscr{F}$ for any M in $\mathscr{F}$ any candidate for ΩM in the sense of Quillen is automatically a candidate for the loop space of M in the sense of Hilton's p-homotopy theory.

In the category $\mathscr{C}$, namely for the i-homotopy theory, it turns out that a module M is contractible if and only if it is injective. In § 3 we mention candidates that can serve as a cylinder object and respectively as a path object for a given M in $\mathscr{C}$. It will follow from this as an easy consequence that for any M in $\mathscr{C}$ the objects ΣM and ΩM are contractible. The main result proved in § 3 is Theorem 3.4. In the case of a connected CW complex X it is known [2; 3] that $W - \text{Cat } X = \text{Ind Cat } X$. From Theorem 3.4 it follows that in the category $\mathscr{C}$ the equality $W - \text{Cat } M = \text{Ind Cat } M$ is not valid in general. In fact for any non injective $\wedge$-module M we have $W - \text{Cat } M = 1$ whereas $\text{Ind Cat } M = \infty$.

In § 4 we look at the category $\mathscr{F}$, namely we look at the projective homotopy theory of finitely generated modules over a PID. In this case it turns out that an object P of $\mathscr{F}$ is contractible if and only if it is free. In the case of $\mathscr{F}$ also it will turn out that for any M in $\mathscr{F}$ the objects ΣM and ΩM will be contractible. The main result in § 4 is Theorem 4.4 which is the analogue of Theorem 3.4.

Finally I wish to thank Professor Hilton for sending me his articles pertaining to general homotopy theory. They proved to be of immense value in the above study of numerical invariants.

1. Injective homotopy theory. The main references for § 1 and § 2 are [4] and [5, Chapter 13]. Let $\wedge$ be a Dedekind domain and $\mathscr{C}$ the category of $\wedge$-modules. Let us choose

(a) all monomorphisms in $\mathscr{C}$ as cofibrations;

(b) i-homotopy equivalences in the sense of [5] as weak equivalences; and

(c) homomorphisms $p : E \to M$ have the lifting property $(L.P)$ stated in the introduction as fibrations.

THEOREM 1.1. *With the above choice of cofibrations, weak equivalences and fibrations $\mathscr{C}$ is a model category in the sense of Quillen* [6, Chapter I].

$\mathscr{C}$ is clearly closed under direct and inverse limits. Thus axiom M_0 of Quillen is trivially valid for $\mathscr{C}$. Thus we have only to check axioms M_1 to M_5. Throuughout this paper $j_M : M \to M \oplus N$, $\delta_M : M \oplus N \to M$ will denote the canonical inclusion and projection respectively. Before taking up the verification of axioms M_1 to M_5 we state some results that we need.

PROPOSITION 1.2. (i) *A monomorphism $\mu : A \to B$ is an i-homotopy equivalence if and only if there exist an injective module K and an isomorphism $\varphi : A \oplus K \simeq B$ satisfying $\mu = \varphi \circ j_A$.*

(ii) *A homomorphism $f : A \to B$ is an i-homotopy equivalence if and only if there exist injective modules K and J, and an isomorphism $\gamma : A \oplus K \simeq B \oplus J$ satisfying $f = \delta_B \varphi j_A$.*

This is Theorem 13.7 of [**5**]. Actually Hilton proves (i) first and uses it to prove (ii).

PROPOSITION 1.3. *Let $\mu : A \to B$ and $f : B \to M$, $\theta : A \to M$ be such that μ is a monomorphism and $\theta \simeq_i f \circ \mu$. Then there exists a homomorphism $\psi : B \to M$ satisfying $\psi \circ \mu = \theta$ and $\psi \simeq_i f : B \to M$.*

This is Theorem 13.6 of [**5**]. Actually Propositions 1.2 and 1.3 are valid over any ring.

LEMMA 1.4. *Given any module M over any ring Γ there exists a unique maximal divisible submodule D of M. If C is any divisible submodule of M then $C \subset D$.*

PROPOSITION 1.5. *A module M over a Dedekind domain $\wedge$ is injective if and only if M is divisible.*

This is well-known. Refer to Propositions 1.2 and 5.1 in Chapter VII of [**1**].

PROPOSITION 1.6. *Let M, N be arbitrary modules over $\wedge$ and $f : N \to M$ any homomorphism. Let D be the maximal divisible submodule of M. Let $h : N \oplus D \to M$ be defined by $h(x, u) = f(x) + u$ for any $x \in N$, $u \in D$. Then $h : N \oplus D \to M$ is a fibration in $\mathscr{C}$.*

Proof. Let $\theta : J \to M$ be any homomorphism with J injective. Then J is divisible and hence $\theta(J)$ is divisible. By 1.4, we have $\theta(J) \subset D$. Let $\varphi : J \to N \oplus D$ be given by $\varphi(y) = (0, \theta(y))$ for any $y \in J$. Then clearly $h \circ \varphi = \theta$. This proves that h is a fibration.

Proof of M_1. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\mu"'] & E \arrow[d, "p"] \\ B \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 1

be a commutative diagram in $\mathscr{C}$ with μ a monomorphism and p a fibration.

Case (1). Let μ be a *w.e.* By (i) Proposition 1.2 there exist an injective module K and an isomorphism $\varphi : A \oplus K \simeq B$ satisfying $\mu = \varphi \circ j_A$. Since K is injective and p a fibration there exists a $\gamma : K \to E$ making Diagram 2 below commutative.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ K \arrow[ur, "\gamma"] \arrow[r, "f\varphi j_K"'] & M \end{tikzcd}$$

Diagram 2

The map $h = k\varphi^{-1}$ where $k : A \oplus K \to E$ is given by $k(a, u) = g(a) + \gamma(u)$ clearly makes Diagram 3 commutative.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\mu"'] & E \arrow[d, "p"] \\ B \arrow[ur, dashed, "h"] \arrow[r, "f"'] & M \end{tikzcd}$$

Diagram 3

Case (2). Let p be a *w.e.* By (ii) Proposition 1.2 there exist injective modules K, J and an isomorphism $\varphi : E \oplus K \to M \oplus J$ such that $p = \delta_M \varphi j_E$. Now, $\delta_M \varphi j_E g = pg = f\mu$ (from Diagram 1). Writing $\lambda : A \to J$ for the map $\delta_J \varphi j_E g$ we have $\varphi j_E g(a) = (f\mu(a), \lambda(a))$ for all $a \in A$. Since J is injective and $\mu : A \to B$ a monomorphism, there exists a map $\alpha : B \to J$ satisfying $\alpha\mu = \lambda$. Define $\theta : B \to M \oplus J$ by $\theta(b) = (f(b), \alpha(b))$. Then $\delta_M \theta = f$. Since $\delta_M \varphi j_E = p$ we get

(1) $\quad \delta_M \theta - \delta_M \varphi j_E \delta_E \varphi^{-1} \theta = f - p \delta_E \varphi^{-1} \theta.$

However,

(2) $\quad \theta - \varphi_E \delta_E \varphi^{-1} \theta = (\varphi 1_{E \oplus K} \varphi^{-1}) \theta - \varphi j_E \delta_E \varphi^{-1} \theta = \varphi(1_{E \oplus K} - j_E \delta_E) \varphi^{-1} \theta.$

Clearly, $(1_{E \oplus K} - j_E \delta_E)\ (E \oplus K) \subset K$. This together with (2) and (1) yields

(3) $\quad \{f - p\delta_E \varphi^{-1} \theta\}\ (B) \subset \delta_M \varphi(K).$

Since K is injective it follows that $\delta_M \varphi(K)$ is divisible and hence by 1.5 $\delta_M \varphi(K)$ is injective. Let $\nu : \delta_M \varphi(K) \to M$ denote the inclusion. Since p is a fibration there exists a map $t : \delta_M \varphi(K) \to E$ satisfying $pt = \nu$. Define $h : B \to E$ by

$h(b) = t\{f - p\delta_E\varphi^{-1}\theta\}(b) + \delta_E\varphi^{-1}\theta(b)$. Then

$$\begin{aligned} ph(b) &= pt\{f - p\delta_E\varphi^{-1}\theta\}(b) + p\delta_E\varphi^{-1}\theta(b) \\ &= \{f - p\delta_E\varphi^{-1}\theta\}(b) + p\delta_E\varphi^{-1}\theta(b), \end{aligned}$$

for pt is the inclusion of $\delta_M\varphi(K)$ in M. Hence

$$(4) \qquad ph(b) = f(b) - p\delta_E\varphi^{-1}\theta(b) + p\delta_E\varphi^{-1}\theta(b) = f(b).$$

Also, for any $a \in A$,

$$(5) \qquad \theta(\mu(a)) = (f\mu(a), \alpha\mu(a)) = (f\mu(a), \lambda(a) = \varphi j_E g(a).$$

Hence

$$\begin{aligned} \{f - p\delta_E\varphi^{-1}\theta\}(\mu(a)) &= f\mu(a) - p\delta_E\varphi^{-1}\theta\,\mu(a) \\ &= f\mu(a) - p\delta_E\varphi^{-1}\varphi j_E g(a) \quad \text{from (5)} \\ &= f\mu(a) - p\delta_E j_E g(a) \\ &= f\mu(a) - pg(a) \quad \text{since } \delta_E j_E = 1_E \\ &= 0 \quad \text{from the commutativity of Diagram 1.} \end{aligned}$$

Thus

$$\begin{aligned} h(\mu(a)) &= t\{f - p\delta_E\varphi^{-1}\theta\}(\mu(a)) + \delta_E\varphi^{-1}\theta(\mu(a)) \\ &= 0 + \delta_E\varphi^{-1}\theta(\mu(a)) \\ &= \delta_E\varphi^{-1}\varphi j_E g(a) \quad \text{from (5)} \\ &= \delta_E j_E g(a) = g(a). \end{aligned}$$

(4) and (6) imply that Diagram 4 below is commutative.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\mu"'] & E \arrow[d, "p"] \\ B \arrow[r, "f"] \arrow[ur, dashed, "h"] & M \end{tikzcd}$$

Diagram 4

This completes the proof of M_1.

Proof of M_2. Let $f : A \to B$ be any map in $\mathscr{C}$. Let D be the maximal divisible submodule of B. Since D is injective $j_A : A \to A \oplus D$ is an i-homotopy equivalence. Let $p : A \oplus D \to F$ be given by $p(a, u) = f(a) + u$. Then p is a fibration by Proposition 1.6 clearly $f = p \circ j_A$ and j_A is a cofibration *w.e.*

Let $\bar{A}$ be any injective module containing A as a submodule. Let $\mu : A \to \bar{A} \oplus B$ be given by $\mu(a) = (a, f(a))$. Then $f = \delta_B \circ \mu$, μ is a monomorphism and δ_B an i-homotopy equivalence. Also if $\alpha : J \to B$ is any homomorphism, $\beta : J \to \bar{A} \oplus B$ defined by $\beta(u) = (0, \alpha(u))$ clearly satisfies $\delta_B \circ \beta = \alpha$. This shows that δ_B is a fibration. In the decomposition $f = \delta_B \circ \mu$, δ_B is a *w.e.* fibration and μ a cofibration.

This completes the proof of M_2.

Proof of M_3. That every isomorphism is a fibration as well as a cofibration is trivial to see. It is equally trivial to see that composition of cofibrations (respectively fibrations) is a cofibration (respectively a fibration).

Let

$$\begin{tikzcd} H \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ A \arrow[r, dash, "f"'] & M \end{tikzcd}$$

Diagram 5

be a pull-back in $\mathscr{C}$ with p a fibration. Let $\alpha : J \to A$ be any map with J injective. Since p is a fibration there exists a $\beta : J \to E$ satisfying $p \circ \beta = f\alpha$. Since Diagram 5 is a pull-back it follows that there exists a $\gamma : J \to H$ (unique, once β is chosen) such that Diagram 6 below is commutative.

$$\begin{tikzcd} J \arrow[dr, dashed, "\gamma"'] \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\ & H \arrow[r, "g"'] \arrow[d, "q"'] & E \arrow[d, "p"] \\ & A \arrow[r, dash, "f"] & M \end{tikzcd}$$

Diagram 6

Then $q \circ \gamma = \alpha$. This proves that q is a fibration.

A push-out of

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mu"'] & M \\ B & \end{tikzcd}$$

in $\mathscr{C}$ is got as follows. Let $L = \{(f(a), -\mu(a) \in M \oplus B | a \in A\}$. L is a submodule of $M \oplus B$. Let

$$\eta : M \oplus B \to \frac{M \oplus B}{L}$$

be the quotient map. Then

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mu"'] & M \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & \dfrac{M \oplus B}{L} \end{tikzcd}$$

Diagram 7

where $\alpha = \eta \circ j_M$ and $\beta = j_B$ is a push-out diagram in $\mathscr{C}$. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mu"'] & M \arrow[d, "\lambda"] \\ B \arrow[r, "h"'] & N \end{tikzcd}$$

Diagram 8

be any push-out diagram in $\mathscr{C}$. Then there exists a unique isomorphism $\theta : N \rightarrow M \oplus B/L$ making Diagram 9 commutative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mu"'] & M \arrow[d, "\lambda"'] \arrow[ddr, "\alpha"] & \\ B \arrow[r, "h"] \arrow[drr, "\beta"'] & N \arrow[dr, "\theta", "\cong"'] & \\ & & \dfrac{M \oplus B}{L} \end{tikzcd}$$

Diagram 9

Now suppose μ is a monomorphism. To show that λ is a monomorphism it suffices to prove that α is. Let $x \in M$ be such that $\alpha(x) = 0$. But $\alpha(x) = \eta(x, 0)$ and $\eta(x, 0) = 0 \Leftrightarrow (x, 0) \in L$. Let $a \in A$ be such that $(x, 0) = (f(a), -\mu(a))$. Then $\mu(a) = 0$. Since μ is a monomorphism, this gives $a = 0$. Hence $x = f(a) = 0$. This proves that α is a monomorphism. This means that a push-out of any cofibration is a cofibration.

This completes the proof of M_3.

Axiom M_5. That $\mathscr{C}$ satisfies axiom M_5 is well-known and is actually an immediate consequence of the characterisation of i-homotopy equivalences given by Theorem 13.12 of [**5**].

In the proof of axiom M_4 we will be making use of the following:

PROPOSITION 1.7. *Let $\theta : J \to M$ be any homomorphism with J injective. Let $h : M \oplus J \to M$ be given by $h(m, u) = m + \theta(u)$ and let*

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "\pi"'] & M \oplus J \arrow[d, "h"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 10

be a pull-back diagram. Then $\pi : N \to A$ is an i-homotopy equivalence.

Proof. Since every isomorphism is an i-homotopy equivalence and composition of i-homotopy equivalences is an i-homotopy equivalence for proving proving Proposition 1.7 we can without loss of generality assume that

$$N = \{(m, u, a) \in M \oplus J \oplus A | M + \theta(u) = f(a)\};$$

$\pi(m, u, a) = a; g(m, u, a) = (m, u)$. For any $a \in A$ we have clearly $f(a), 0, a)$ $\in N$ and $\lambda : A \to N$ defined by $\lambda(a) = (f(a), 0, a)$ satisfies $\pi \circ \lambda(a) = a$ for all $a \in A$. The composite

$$N \xrightarrow{\pi} A \xrightarrow{\lambda} N$$

is given by $\lambda\pi(m, u, a) = (f(a), 0, a)$. Hence $\{1_N - \lambda\pi\}(m, u, a) = (m - f(a), u, 0)$. But whenever $(m, u, a) \in N$ we have $m + \theta(u) = f(a)$. Thus $\{1_N - \lambda\pi\}(m, u, a) = (-\theta(u), u, 0)$. If $\alpha : N \to J$ and $\beta : J \to N$ are given by $\alpha(m, u, a) = u$, $\beta(u) = (-\theta(u), u, 0)$ then clearly $1_N - \lambda\pi = \beta\alpha$. Since J is injective it follows that $1_N - \lambda\pi \simeq_i 0$. Thus $\pi\lambda = 1_A$ and $\pi\lambda \simeq_i 1_N$. This proves that π is an i-homotopy equivalence with λ as an i-homotopy inverse, thus completing the proof of Proposition 1.7.

Proof of M_4. Let

$$\begin{tikzcd} C \arrow[r, "t"] \arrow[d, "w"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 11

be a pull-back diagram with p a $w.e.$ fibration. Since $p : E \to M$ is an i-homotopy equivalence by (ii) Proposition 1.2 there exist injective modules K and J and an isomorphism $\varphi : E \oplus K \simeq M \oplus J$ satisfying $p = \delta_M \varphi j_E$. Consider $\alpha = \delta_M \varphi : E \oplus K \to M$. Since K is injective and $p : E \to M$ is a fibration we see that there exists a map $\theta : K \to E$ satisfying $p \circ \theta = \alpha \circ j_K$. Define $h : E \oplus K \to E$ by $h(e, u) = e + \theta(u)$ for any $(e, u) \in E \oplus K$. Then clearly $p \circ h = \alpha$. Let

$$\begin{tikzcd} N \arrow[r, "q"] \arrow[d, "\pi"'] & E \oplus K \arrow[d, "h"] \\ C \arrow[r, "t"] & E \end{tikzcd}$$

Diagram 12

be a pull-back diagram. From the fact that Diagrams 11 and 12 are pull-back diagrams we see that

$$\begin{tikzcd} N \arrow[r, "q"] \arrow[d, "w\pi"'] & E \oplus K \arrow[d, "ph = \alpha"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 13

is a pull-back.

Our aim is to show that w is an i-homotopy equivalence. By Proposition 1.7, $\pi : N \to C$ is an i-homotopy equivalence. By axiom M_5 it suffices to prove that $w\pi$ is an i-homotopy equivalence. Clearly

$$\begin{tikzcd} A \oplus J \arrow[r, "f \oplus 1_J"] \arrow[d, "\delta_A"'] & M \oplus J \arrow[d, "\delta_M"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 14

is a pull-back. Let

$$\begin{tikzcd} N' \arrow[r, "q'"] \arrow[d, "\psi"'] & E \oplus K \arrow[d, "\varphi"] \\ A \oplus J \arrow[r, "f \oplus 1_J"'] & M \oplus J \end{tikzcd}$$

Diagram 15

be a pull-back. From Diagrams 14 and 15 we see that

$$\begin{tikzcd} N' \arrow[r, "q'"] \arrow[d, "\delta_A\psi"'] & E \oplus K \arrow[d, "\delta_M\varphi = \alpha"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 16

is a pull-back. Since both Diagrams 13 and 16 are pull-backs we see that there exists an isomorphism $\gamma : N \rightarrow N'$ satisfying $\delta_A\psi\gamma = w\pi$ and $q'\gamma = q$. Since φ is an isomorphism the pull-back $\psi : N' \rightarrow A \oplus J$ is also an isomorphism. $\delta_A : A \oplus J \rightarrow A$ is an i-homotopy equivalence since J is injective. Hence $w\pi = \delta_A\psi\gamma$ is an i-homotopy equivalence.

Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mu"'] & M \arrow[d, "q"] \\ B \arrow[r, "g"'] & N \end{tikzcd}$$

Diagram 17

be a push-out diagram with $\mu : A \rightarrow B$ a *w.e.* cofibration (namely a monomorphism which is also an i-homotopy equivalence). By (i) Proposition 1.2 we can without loss of generality assume $\mu = j_A : A \rightarrow A \oplus K$ with K injective. Since

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j_A"'] & M \arrow[d, "j_M"] \\ A \oplus K \arrow[r, "f \oplus 1_K"'] & M \oplus K \end{tikzcd}$$

Diagram 18

is a push-out we can assume without loss of generality that $q = j_M$. Clearly j_M is an i-homotopy equivalence.

This completes the proof of M_4.

Remark 1.8. For any $A \in \mathscr{C}$ the map $0 \rightarrow A$ is clearly a cofibration and $A \rightarrow 0$ a fibration. Thus all the objects in $\mathscr{C}$ are simultaneously cofibrant and fibrant. It follows that the notions of left homotopy and right homotopy introduced by Quillen [**6**] coincide in this case.

PROPOSITION 1.9. *Let* $f, g \in \mathrm{Hom}\,(A, B)$. *Then*

$$f \sim^l g \Leftrightarrow f \simeq_i g \Leftrightarrow f \sim^r g.$$

Proof. From the observation at the end of Remark 1.8 it suffices to prove the implications $f \sim^l g \Rightarrow f \simeq_i g \Rightarrow f \sim^r g$. Observe that in $\mathscr{C}$ for any two objects C, D the union $C \vee D$ is the same as $C \oplus D$. We will however write $C \vee D$ itself thus conforming to the notation in [**6**]. Let $f \sim^l g$. Then there exists a commutative diagram

$$\begin{tikzcd} A \vee A \arrow[r, "f+g"] \arrow[d, "\nabla"'] \arrow[dr, "\partial_0 + \partial_1"] & B \\ A & \tilde{A} \arrow[l, "\sigma"'] \arrow[u, "h"'] \end{tikzcd}$$

Diagram 19

with σ a *w.e.* (namely an i-homotopy equivalence). From Diagram 19 we get $\sigma\partial_0 = 1_A = \sigma\partial_1$; $h\partial_0 = f$, $h\partial_1 = g$. Since σ is an i-homotopy equivalence it follows that ∂_0, ∂_1 are both i-homotopy inverses to σ. In particular $\partial_0 \simeq_i \partial_1$. This immediately yields $h\,\partial_0 \simeq_i h\partial_1$; in other words $f \simeq_i g$.

Let us now assume $f \simeq_i g$. Then there exist an injective module J and maps $\alpha : A \to J$, $\beta : J \to B$ such that $\beta\alpha = g - f$. Denoting the map $B \oplus J \to B \times B$ carrying (b, u) of $B \oplus J$ into the elements $(b, b + \beta(u))$ of $B \times B$ by $(1 + 0, 1 + \beta)$ it is clear that

$$\begin{tikzcd} B \oplus J \arrow[dr, "(1+0,\,1+\beta)"] & B \arrow[l, "j_B"'] \arrow[d, "\Delta"] \\ A \arrow[u, "(f,\,\alpha)"] \arrow[r, "(f,\,g)"'] & B \times B \end{tikzcd}$$

Diagram 20

is commutative. Here j_B is an i-homotopy equivalence. This proves $f \sim^r g$.

The proof of Proposition 1.9 is complete.

2. Projective homotopy theory. The results obtained here are in a "certain sense" dual to the results obtained in § 1. However, we have to restrict our attention to the category of finitely generated modules over a PID.

Let $\mathscr{F}$ denote the category of finitely generated modules over a principal ideal domain $\wedge$. Let us choose

(a) all epimorphisms in $\mathscr{F}$ as fibrations,

(b) p-homotopy equivalences as weak equivalences, and

(c) maps $q : M \to N$ satisfying the extension property $(E.P)$ stated in the introduction as cofibrations.

THEOREM 2.1. *With the above choice of fibrations, weak equivalences and cofibrations $\mathscr{F}$ is a model category in the sense of Quillen.*

Axiom M_0 of Quillen requires that the category under consideration be closed under finite projective and finite inductive limits. Clearly the category of finitely generated modules over a Noetherian ring (in particular over a PID) is closed under finite limits. Though the proofs in this section are in a certain sense dual to the proofs of results in § 1, the fact that we are considering finitely generated modules over a PID will play a special role. We therefore prefer to give proofs. In a way this helps in understanding the duality better.

PROPOSITION 2.2. *Let $\wedge$ be any ring and $\nu : M \to N$ a homomorphism of $\wedge$ modules.*

(i) *Suppose $\nu : M \to N$ is an epimorphism. Then ν is a p-homotopy equivalence if and only if there exist a projective module P and an isomorphism $\varphi : M \to N \oplus P$ satisfying $\nu = \delta_N \varphi$.*

(ii) *In the general case* (when ν is not necessarily an epimorphism) *ν is a p-homotopy equivalence if and only if there exist projective modules P and R and an isomorphism $\varphi : M \oplus R \to N \oplus P$ satisfying $\nu = \delta_N \varphi j_M$. Moreover any projective ancestor of N could be chosen for R.*

(iii) *In case M, N are finitely generated P is automatically finitely generated in case* (i) *and R* (hence P) *could be chosen to be finitely generated in case* (ii).

PROPOSITION 2.3. *Let $\nu : M \to N$ be any epimorphism and $\theta : A \to N$, $f : A \to M$ homomorphisms satisfying $\theta \simeq_p \nu \circ f$. Then there exists a homomorphism $\psi : A \to M$ satisfying*

$$\nu \circ \psi = \theta \quad \text{and} \quad \psi \simeq_p f : A \to M.$$

Propositions 2.2 and 2.3 are well-known [**4**] and are the duals of Propositions 1.2 and 1.3.

PROPOSITION 2.4. *Let $f : M \to N$ be any map in $\mathscr{F}$. Let $T(M)$ be the tension submodule of M and $\eta : M \to M/T(M)$ the canonical quotient map. Then the map*

$$(f, \eta) : M \to N \oplus \frac{M}{T(M)}$$

is a cofibration in $\mathscr{F}$.

Proof. Let $\alpha : M \to R$ be any map into a finitely generated free module R.

Then $\alpha(T(M)) = 0$. Hence α yields by passage to quotient a map

$$\bar{\alpha}: \frac{M}{T(M)} \to R.$$

Define

$$\beta: N \oplus \frac{M}{T(M)} \to R \quad \text{by } \beta(x, u) = \bar{\alpha}(u) \text{ for any } x \in N, u \in \frac{M}{T(M)}.$$

Then it is clear that $\beta \circ (f, \eta) = \alpha$. This proves that

$$(f, \eta): M \to N \oplus \frac{M}{T(M)}$$

is a cofibration in $\mathscr{F}$.

We now take up the proof of the axioms M_1 to M_5 for the category $\mathscr{F}$.

Proof of M_1. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "\nu"] \\ B \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 21

be a commutative diagram in $\mathscr{F}$ with ν an epimorphism and q a cofibration.

Case (1). Let ν be a p-homotopy equivalence. By (iii) Proposition 2.2 there exist a finitely generated projective $\wedge$-module P and an isomorphism $\varphi : E \to M \oplus P$ such that $\nu = \delta_M \circ \varphi$. Since $\wedge$ is a PID it follows that P is also free. Since q is a cofibration it follows that there exists a map $\gamma : B \to P$ such that Diagram 22 below is commutative.

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "\delta_P \varphi g"] & \\ B \arrow[r, "\gamma"'] & P \end{tikzcd}$$

Diagram 22

The map $h = \varphi^{-1}k$ where $k : B \to M \oplus P$ is given by $k = (f, \gamma)$ clearly makes Diagram 23 below commutative.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "\nu"] \\ B \arrow[r, "f"'] \arrow[ur, dashed, "h"] & M \end{tikzcd}$$

Diagram 23

Case (2). Let q be a p-homotopy equivalence. By (iii) Proposition 2.2 and the fact that $\wedge$ is a PID we see that there exist finitely generated free modules P and R and an isomorphism $\varphi : A \oplus R \to B \oplus P$ satisfying $q = \delta_B \varphi j_A$. Since R is free and $\nu : E \to M$ is onto we see that there exists a map $\alpha : R \to E$ satisfying $\nu\alpha = f\delta_B\varphi j_R$. Consider the map $\theta : A \oplus R \to E$ defined by $\theta(a, u) = g(a) + \alpha(u)$ for all $a \in A$, $u \in R$. Then $\theta j_A = g$. Also from $q = \delta_B \varphi j_A$ we get that

(7) $\quad \theta j_A - \theta\varphi^{-1} j_B \delta_B j_A = g - \theta\varphi^{-1} j_B q.$

But,

(8) $$\begin{aligned} \theta - \theta\varphi^{-1} j_B \delta_B \varphi &= \theta\varphi^{-1} 1_{B\oplus P}\varphi - \theta\varphi^{-1} j_B \delta_B \varphi \\ &= \theta\varphi^{-1}(1_{B\oplus P} - j_B\delta_B)\varphi. \end{aligned}$$

Clearly, $(1_{B\bigcirc P} - j_B\delta_B)(B \oplus P) \subset P$. Since $\varphi(A) \subset B \oplus P$ we get $(1_{B\oplus P} - j_B\delta_B)\varphi(A) \subset P$. Submodule of a (finitely generated) free module over a PID being itself (finitely generated) free we see that $(1_{B\oplus P} - j_B\delta_B)\varphi(A)$ is a finitely generated free module. Since φ is an isomorphism it follows that $F = \varphi^{-1}(1_{B\oplus P} - j_B\delta_B)\varphi(A)$ is a finitely generated free module. Since $q : A \to B$ is a cofibration in $\mathscr{F}$ it follows that there exists a map $t : B \to F$ such that Diagram 24 is commutative.

$$\begin{tikzcd} A \arrow[rr, "\varphi^{-1}(1_{B\oplus P} - j_B\delta_B)\varphi j_A"] \arrow[dr, "q"'] & & F \\ & B \arrow[ur, "t"'] & \end{tikzcd}$$

Diagram 24

Observe that F is a finitely generated free submodule of $A \oplus R$ satisfying $\varphi(F) = (1_{B\oplus P} - j_B\delta_B)\varphi(A) \subset P$. Since $\delta_B(P) = 0$, it follows that

(9) $\quad \delta_B(F) = 0.$

Define

(10) $\quad h : B \to E$ by $h(b) = \theta t(b) + \theta\varphi^{-1} j_B(b).$

Then

(11) $\quad \nu \circ h(b) = \nu\theta t(b) + \nu\theta\varphi^{-1} j_B(b).$

By the very definition of θ we have

$$\begin{aligned} \nu\theta(a, u) &= \nu g(a) + \nu\alpha(u) \quad \text{for all } (a, u) \in A \oplus R \\ &= fq(a) + f\delta_B\varphi j_R(u) \\ &= f\delta_B\varphi j_A(a) + f\delta_B\varphi j_R(u) \\ &= f\delta_B\varphi(a, 0) + f\delta_B\varphi(0, u) \\ &= f\delta_B\varphi(a, u). \end{aligned}$$

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Hence $\nu\theta = f\delta_B\varphi$. Substituting $f\delta_B\varphi$ for $\nu\theta$ in (11) we get

$$\nu h(b) = f\delta_B\varphi(t(b)) + f\delta_B\varphi\varphi^{-1}j_B(b).$$

Since $t(b) \in F$ by (9) we have $\delta_B\varphi(t(b)) = 0$. Hence

$$\text{(12)} \quad \nu h(b) = f\delta_B\varphi\varphi^{-1}j_B(b) = f\delta_B j_B(b) = f(b).$$

Also, $hq(a) = \theta t(q(a)) + \theta\varphi^{-1}j_B(q(a))$. But $tq = \varphi^{-1}(1_{B\oplus P} - j_B\delta_B)\varphi j_A$ from Diagram 24. Hence

$$\begin{aligned} \theta tq &= \theta\varphi^{-1}(1_{B\oplus P} - j_B\delta_B)\varphi j_A \\ &= \theta j_A - \theta\varphi^{-1}j_B\delta_B\varphi j_A \\ &= g - \theta\varphi^{-1}j_B q \quad \text{by (7).} \end{aligned}$$

Therefore

$$hq(a) = g(a) - \theta\varphi^{-1}j_B q(a) + \theta\varphi^{-1}j_B(q(a)) = g(a).$$

(12) and (13) together imply that Diagram 25 below is commutative.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "\nu"] \\ B \arrow[r, "f"'] \arrow[ur, dashed, "h"] & M \end{tikzcd}$$

Diagram 25

This completes the proof of M_1.

Proof of M_2. Let $f : M \to N$ be any map in $\mathscr{F}$. Let $\eta : M \to M/T(M)$ be the canonical quotient map where $T(M)$ is the torsion submodule of M. By Proposition 2.4, $(f, \eta)M \to N \oplus M/T(M)$ is a cofibration. Since M is finitely generated and $\wedge$ a PID it follows that $M/T(M)$ is free. Hence $\delta_N : N \oplus M/T(M) \to N$ is a p-homotopy equivalence. Denoting $(f, \eta) : M \to N \oplus M/T(M)$ by q we have $\delta_N q = f$, q a cofibration and δ_N a *w.e.* fibration.

Let

$$R \xrightarrow{\epsilon} N$$

be an epimorphism with R a finitely generated free module. Let $\nu : M \oplus R \to N$ be given by $\nu(x, u) = f(x) + \epsilon(u)$. Then ν is an epimorphism and $f = \nu j_M$. If $\alpha : M \to F$ is any map in $\mathscr{F}$ the map $\beta : M \oplus R \to F$ given by $\beta = \alpha\delta_M$ clearly satisfies $\beta j_M = \alpha$. Hence j_M is a cofibration. Clearly $j_M : M \oplus R$ is a p-homotopy equivalence.

This completes the proof of M_2.

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Proof of M_3. Clearly any isomorphism is an epimorphism and the composition of epimorphisms is an epimorphism. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\lambda"'] & E \arrow[d, "\nu"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 26

be a pull-back with ν an epimorphism. We can identify A with

$$\{(e, n) \in E \oplus N | \nu(e) = f(n)\},$$

$g : A \to E$ and $\lambda : A \to N$ with the maps $g(e, n) = e$, $\lambda(e, n) = n$. For any $n \in N$ there exists an $e \in E$ such that $\nu(e) = f(n)$. Then $(e, n) \in A$ and $\lambda(e, n) = n$. This proves that $\lambda : A \to N$ is an epimorphism. Hence the pull-back of any fibration in $\mathscr{F}$ is a fibration.

It is trivial to see that any is omorphism is a cofibration and that the composition of cofibrations is a cofibration in $\mathscr{F}$. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & C \arrow[d, "u"] \\ B \arrow[r, "g"] & D \end{tikzcd}$$

Diagram 27

be a push-out diagram with q a cofibration. Let

$$C \xrightarrow{\alpha} F$$

be any map with F a finitely generated free module. Since q is a cofibration there exists a map $\beta : B \to F$ satisfying $\beta q = \alpha f : A \to F$. Since Diagram 27 is a push-out there exists a map $\gamma : D \to F$ (unique once β is chosen) such that Diagram 28 is commutative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & C \arrow[d, "u"'] \arrow[ddr, "\alpha"] & \\ B \arrow[r, "g"] \arrow[drr, "\beta"'] & D \arrow[dr, dashed, "\gamma"'] & \\ & & F \end{tikzcd}$$

Diagram 28

Then $\gamma \circ u = \alpha$. This proves that u is a cofibration. The proof of M_3 is now complete.

Axiom M_5. This is again well-known and is an immediate consequence of the characterisation of p-homotopy equivalences.

For the proof of axiom M_4 we need the following.

PROPOSITION 2.5. *Let* $\wedge$ *be any ring and* $\theta : B \to P$ *any map with* P *projective. Let* $h = (1_B, \theta) : B \to B \oplus P$ *and*

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "h"'] & N \arrow[d, "w"] \\ B \oplus P \arrow[r, "k"] & C \end{tikzcd}$$

Diagram 29

be a push-out diagram. Then $w : N \to C$ *be a* p*-homotopy equivalence.*

Proof. Without loss of generality we can assume

$$C = \frac{B \oplus P \oplus N}{L}$$

where $L = \{(b, \theta(b), -f(b)) \in B \oplus P \oplus N | b \in B\}$ and $w = \eta \circ j_N$, $k = \eta \circ j_{B \oplus P}$ where

$$\eta : B \oplus P \oplus N \to \frac{B \oplus P \oplus N}{L}$$

is the canonical quotient map. Consider the map $\alpha : B \oplus P \oplus N \to N$ given by $\alpha(b, u, n) = n + f(b)$. For any $(b, \theta(b), -f(b)) \in L$ we have $\alpha(b, \theta(b), -f(b)) = -f(b) + f(b) = 0$. Hence α passes down to quotient to induce $\lambda : C \to N$ satisfying $\alpha = \lambda \circ \eta$. We have $\lambda \circ w(n) = \lambda \circ \eta \circ j_N(n) = \alpha \circ j_N(n) = \alpha(0, 0, n) = n$. Hence the composite

$$N \xrightarrow{w} C \xrightarrow{\lambda} N$$

is 1_N. Consider the map $\gamma : B \oplus P \oplus N \to P$ given by $\gamma : (b, u, n) = u - \theta(b)$. For any $(b, \theta(b), -f(b)) \in L$ we have $\gamma(b, \theta(b), -f(b)) = \theta(b) - \theta(b) = 0$. Hence γ induces a map $\mu : C \to P$ satisfying $\gamma = \mu \circ \eta$.

Let $x \in C$ and $(b, u, n) \in B \oplus P \oplus N$ be such that $\eta(b, u, n) = x$. We have

$$\begin{aligned} (14) \quad \eta j_P \mu(x) &= \eta j_P \mu \eta(b, u, n) = \eta j_P \gamma(b, u, n) = \eta j_P(u - \theta(b)) \\ &= \eta(0, u - \theta(b), 0). \end{aligned}$$

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Also

$$\begin{aligned}(1_C - w\lambda) &= x - w\lambda(x) = \eta(b, u, n) - w\lambda\eta(b, u, n)\\ &= \eta(b, u, n) - \eta j_N\alpha(b, u, n) = \eta(b, u, n) - \eta j_N(n + f(b))\\ &= \eta(b, u, n) - \eta(0, 0, n + f(b)) = \eta(b, u, -f(b)).\end{aligned}$$

However $(b, u, -f(b)) - (0, u - \theta(b), 0) = (b, \theta(b), -f(b) \in L$. Hence $\eta(b, u, -f(b)) = \eta(0, u - \theta(b), 0)$. This shows that $\eta j_P\mu(x) = (1_C - w\lambda)(x)$ and $x \in C$ is arbitrary. Hence $(1_C - w\lambda) = \eta j_P\mu$. This means Diagram 30 below is commutative.

$$\begin{tikzcd} & P \arrow[dr, "\eta j_P"] & \\ C \arrow[ur, "\mu"] \arrow[rr, "1_C - w\lambda"'] & & C \end{tikzcd}$$

Diagram 30

Since P is projective we see that $1_C - w\lambda \simeq_p 0$.

Thus $w\lambda \simeq_p 1_C$ and $\lambda w - 1_N$. This shows that $w : N \to C$ is a p-homotopy equivalence.

Proof of M_4. Let

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\lambda"'] & E \arrow[d, "\nu"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 31

be a pull-back diagram with ν a $w.e$ fibration. Then ν is an epimorphism which is also a p-homotopy equivalence. By (i) Proposition 2.2 we can assume $E = M \oplus P$ with P finitely generated projective and $\nu = \delta_M : M \oplus P \to M$. Since

$$\begin{tikzcd} A \oplus P \arrow[r, "f \oplus 1_P"] \arrow[d, "\delta_A"'] & M \oplus P \arrow[d, "\delta_M"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

Diagram 32

is a pull-back it follows that $C \overset{\lambda}{\to} A$ can be replaced by

$$A \oplus P \overset{\delta_A}{\to} A$$

which is clearly a p-homotopy equivalence. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q"'] & M \arrow[d, "u"] \\ B \arrow[r, "f"] & N \end{tikzcd}$$

Diagram 33

be a push-out with q a *w.e.* cofibration. Since $q : A \to B$ is a p-homotopy equivalence by (iii) Proposition 2.2 there exist finitely generated projective modules P and R and an isomorphism $\varphi : A \oplus R \to B \oplus P$ satisfying $q = \delta_B \varphi j_A$. Since $\wedge$ is a PID both P and R are free. Let $\alpha = \varphi j_A$. Since P is a finitely generated free module and $q : A \to B$ is a cofibration we see that there exists a map $\theta : B \to P$ satisfying $\theta \circ q = \delta_p \alpha$. Let $h = (1_B, \theta) : B \oplus P$. We have $\alpha = \varphi j_A = (\delta_B \varphi j_A, \delta_P \varphi j_A) = (q, \delta_P \alpha)$ and $hq = (q, \theta q) = (q, \delta_P \alpha)$. Hence $hq = \alpha$. Let

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "h"'] & N \arrow[d, "\pi"] \\ B \oplus P \arrow[r, "k"] & D \end{tikzcd}$$

Diagram 34

be a push-out diagram. Since Diagrams 33 and 34 are push-outs it follows that

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\alpha = hq"'] & M \arrow[d, "\pi u"] \\ B \oplus P \arrow[r, "k"] & D \end{tikzcd}$$

Diagram 35

is a push-out.

Our aim is to show that $u : M \to N$ is a p-homotopy equivalence. By Proposition 2.5, $\pi : N \to D$ is a p-homotopy equivalence. By axiom M_5 it suffices to show that πu is a p-homotopy equivalence.

Clearly

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "j_A"] & M \arrow[d, "j_M"] \\ A \oplus R \arrow[r, "g \oplus 1_R"'] & M \oplus R \end{tikzcd}$$

Diagram 36

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is a push-out diagram. Let

$$\begin{tikzcd} A \oplus R \arrow[r, "g \oplus 1_R"] \arrow[d, "\varphi"'] & M \oplus R \arrow[d, "\psi"] \\ B \oplus P \arrow[r, "k'"] & D' \end{tikzcd}$$

Diagram 37

be a push-out. It follows that Diagram 38 below is a push-out.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\alpha = \varphi j_A"'] & M \arrow[d, "\psi j_M"] \\ B \oplus P \arrow[r, "k'"] & D' \end{tikzcd}$$

Diagram 38

Comparison of Diagrams 35 and 38 yields an isomorphism

$$D' \xrightarrow{\gamma} D,$$

satisfying $\pi u = \gamma \psi j_M$. Since φ is an isomorphism the push-out ψ is also an isomorphism. $j_M : M \oplus R$ is a p-homotopy equivalence (R being free) and $\gamma : D' \to D$ is an isomorphism. Hence πu is a p-homotopy equivalence.

This completes the proof of M_4 for $\mathscr{F}$.

Remark 2.6. It is easily seen that every object in $\mathscr{F}$ is simultaneously fibrant and cofibrant. Hence the notions of left homotopy and right homotopy of Quillen agree in $\mathscr{F}$ as well.

PROPOSITION 2.7. *Let $f, g \in \text{Hom}(A, B)$. Then $f \sim^l g \Leftrightarrow f \simeq_p g \Leftrightarrow f \sim^r g$.*

Proof. We first observe that when $f \simeq_p g$ there exists a finitely generated projective module P and maps $\alpha : A \to P$, $\beta : P \to B$ satisfying $\beta\alpha = g - f$. This is because a projective ancestor of B could be chosen to be finitely generated.

The proof of Proposition 2.7 is got from the proof of Proposition 1.9 by replacing J by a finitely generated projective module P and "i-homotopy" by "p-homotopy".

3. Numerical invariants in injective homotopy theory. This section is devoted to the study of the numerical invariants defined in [7] for the model category $\mathscr{C}$ of § 1. From Proposition 1.9 it follows that for any A, B in $\mathscr{C}$ the set $\pi(A, B) = [A, B]$ of homotopy classes of maps in the sense of Quillen

[**6**] is the same as the set $\bar{\pi}(A, B)$ of injective homotopy classes of maps as introduced by Hilton [**5**]. It is easily seen that the contractible objects according to [**7**, Definition 2.16] in $\mathscr{C}$ are precisely the injective modules.

Remarks. 3.1. A module M dominates N if and only if there exist maps $f : N \to M$, $g : M \to N$ such that $gf \simeq_i 1_N$. If N is dominated by an injective module (a contractible object) J then N itself is contractible and hence injective.

3.2. Let $\bar{A}$ be any injective module containing A and let

$$\mu : A \vee A = A \oplus A \to A \oplus \bar{A}$$

be given by $\mu(x, y) = (x + y, y)$. Then $\delta_A \circ \mu(x, y) = x + y = \nabla(x, y)$. Clearly μ is a monomorphism and δ_A an i-homotopy equivalence. Thus ∇ factors into

$$A \vee A \xrightarrow{\mu} A \oplus A \xrightarrow{\delta_A} A$$

with μ a cofibration and δ_A a *w.e.* It follows that $A \oplus \bar{A}$, along with the maps $\mu : A \oplus A \to A \oplus \bar{A}$, $\delta_A : A \oplus \bar{A} \to A$, is a cylinder object of A in the sense of [**6**, Chapter I]. Hence a candidate for ΣA in the sense of Quillen is the cofibre of the map $\mu : A \oplus A \to A \oplus \bar{A}$ which is the quotient module $A \oplus \bar{A}/\mu(A \oplus A)$. Let $\eta : A \oplus \bar{A} \to (A \oplus \bar{A})/\mu(A \oplus A)$ be the quotient map. It is clear that η carries $\bar{A}$ onto $(A \oplus \bar{A})/\mu(A \oplus A)$ and that Ker $(\eta/\bar{A}) = \bar{A} \cap \mu(A \oplus A) = 0 \oplus A$. Hence

$$(A \oplus \bar{A})/\mu(A \oplus A) \simeq \bar{A}/A.$$

But $\bar{A}/A$ is precisely the definition of ΣA in Hilton's set up of i-homotopy theory [**4**]. Since $\wedge$ is a Dedekind domain it follows from Proposition 1.5 that $\bar{A}/A$ is injective.

3.3. Let D be the maximal divisible submodule of A. Let $p : A \oplus D \oplus D \to A \oplus A = A \times A$ be given by $\rho(a, x, y) = (a + x, a + y)$. Then $p\, j_A(a) = p(a, 0, 0) = (a, a) = \Delta(a)$. Then $p\, j_A = \Delta$. $D \oplus D$ is the maximal divisible submodule of $A \oplus A$. Hence by Proposition 1.6 the map $p : A \oplus D \oplus D \to A \oplus A$ is a fibration. Also $D \oplus D$ being injective we see that $j_A : A \to A \oplus D \oplus D$ is an i-homotopy equivalence. It follows that $A \oplus D \oplus D$ along with the maps $j_A : A \to A \oplus D \oplus D$, $p : A \oplus D \oplus D \to A \times A$ yields a path object of A in the sense of [**6**, Chapter I]. Hence a candidate for $\Omega(A)$ is the fibre (namely the kernel) of the map $p : A \oplus D \oplus D \to A \oplus A$. Now, $p(a, x, y) = 0$ if and only if $a + x = 0 = a + y \Leftrightarrow x = -a = y$. The map $\lambda : D \to A \oplus D \oplus D$ given by $\lambda(x) = (-x, x, x)$ clearly maps D isomorphically onto the kernel of p. Hence a candidate for $\Omega(A)$ is D, and D is injective. Since $\Omega(A)$ is determined unique up to homotopy type (which in this case happens to be i-homotopy type) it follows that every candidate for $\Omega(A)$ is injective.

Thus we see that for every A in $\mathscr{C}$ both ΣA and $\Omega(A)$ are contractible.

Let W-Cat A, Ind Cat A, Cocat A, Nil A and Conil A be the invariant associated to A as in [**7**]. The invariant W-Cat A is the Lusternik-Schnirelmann category of A defined along the methods of G. W. Whitehead. The main result of this section can be stated as follows:

THEOREM 3.4. (1) W-Cat $M = 0 =$ Ind Cat $M =$ Cocat M *whenever* M *is an injective* $\wedge$*-module.*

(2) W-Cat $M = 1$, Ind Cat $M = \infty =$ Cocat M *whenever* M *is a* $\wedge$*-module which is not injective.*

(3) Nil $M = 0 =$ Conil M *for all* M *in* $\mathscr{C}$.

In the proof of this theorem we will be using the following.

LEMMA 3.5. *Let* $M \in \mathscr{C}$. *If* Ind Cat $M < \infty$ (respectively Cocat $M < \infty$) *then* M *is injective and hence* Ind Cat $M = 0$ (respectively Cocat $M = 0$).

Proof. For any integer $k \geqq 0$ let (S_k) and (T_k) be the statements given below:

(S_k): Ind Cat $M \leqq k \Rightarrow M$ is injective.

(T_k): Cocat $M \leqq k \Rightarrow M$ is injective.

We will prove both these statements by induction k. That will establish Lemma 3.5.

Clearly the statement (S_0) is true by the very definition. Let now $k \geqq 1$ and assume (S_l) is true for all $l \leqq k - 1$. By definition there exists a cofibration $A \xrightarrow{f} Y$ with Ind Cat $Y \leqq k - 1$ and the cofibre C of f dominating M. By (S_{k-1}) we see that Y is injective. The cofibre of f is a quotient of Y and hence C is injective. Since C dominates M by 3.1 we see that M is injective.

The statement (T_0) is also true by the very definition. Let $k \geqq 1$ and assume (T_l) be true for all $l \leqq k - 1$. Since Cocat $M \leqq k$, there exists a fibration

$$E \xrightarrow{p} B$$

with Cocat $E \leqq k - 1$ and the fiber C of p dominating M. By (T_{k-1}) we see that E is injective. Hence $p(E)$ is divisible and hence injective. Let $j : p(E) \to B$ denote the inclusion of $p(E)$ in B. Since $p : E \to B$ is a fibration by the very definition of a fibration there exists a map $g : p(E) \to E$ such that

$$\begin{tikzcd} & E \arrow[d, "p"] \\ p(E) \arrow[ur, "g"] \arrow[r, "j"'] & B \end{tikzcd}$$

Diagram 39

is commutative. This means, for the map $p : E \to p(E)$ we have a splitting $E \to p(E)$. In other words

$$0 \to \text{Ker } p \to E \to p(E) \to 0$$

is a split exact sequence. Hence $C = \operatorname{Ker} p$ is a direct summand of the injective module E and hence C itself is injective. Since M is dominated by 3.1 we see that M is injective. This completes the proof of Lemma 3.5.

Proof of Theorem 3.4. (1) is immediate from the definition of these invariants and the fact that the contractible objects of $\mathscr{C}$ are precisely the injective modules.

(2) For any $M \in \mathscr{C}$ we have $M \vee M = M \oplus M = M \times M$. It is therefore clear that the diagonal map $\Delta : M \to M \times M$ factors through $M \vee M$. Hence W-Cat $M \leqq 1$. Also W-Cat $M = 0$ implies that 1_M is homotopic to zero; in other words $1_M \simeq_i 0$. This means M is a direct summand of an injective module and hence injective. This proves that W-Cat $M = 1$ for any module M which is not injective.

That Ind Cat $M = \infty =$ Cocat M whenever M is not injective is an immediate consequence of Lemma 3.5.

(3) By definition

$$\operatorname{Nil} M = \sup_{A \in \mathscr{L}} \operatorname{nil}\, [\Sigma A, M] \quad \text{and} \quad \operatorname{Conil} M = \sup_{A \in \mathscr{L}} \operatorname{nil}\, [M, \Omega A].$$

But we have seen already that in $\mathscr{C}$ both ΣA and ΩA are contractible for all A. Hence $[\Sigma A, M] = 0 = [M, \Omega A]$ for all $A \in \mathscr{C}$. This proves (3).

4. Numerical invariants in projective homotopy theory. This section deals with the model category $\mathscr{F}$. Thus the base ring $\wedge$ is a PID, and unless otherwise stated, by a module we mean a *finitely generated* module. The set $[A, B]$ for any A, B in $\mathscr{F}$ agrees with the set of projective homotopy classes of A in B. An object A of $\mathscr{F}$ is contractible if and only if A is free.

Remarks. 4.1. A module A dominates another module B if and only if there exist maps

$$B \xrightarrow{f} A, \quad A \xrightarrow{g} B \quad \text{such that } gf \simeq_p 1_B.$$

If B is dominated by a free module then B itself is free.

4.2. Let $N \in \mathscr{F}$ and $\epsilon : F \to N$ an epimorphism with F free. Define $\nu : N \oplus F \to N \oplus N = N \times N$ by $\nu(x, u) = (x, x + \epsilon(u))$ for all $x \in N, u \in F$. Then ν is clearly an epimorphism and $\nu j_N(x) = \nu(x, 0) = (x, x) = \Delta(x)$. Also $j_N : N \to N \oplus F$ is a p-homotopy equivalence. Thus Δ factors as

$$N \xrightarrow{j_N} N \oplus F \xrightarrow{\nu} N \times N$$

with j_N a *w.e.* and ν a fibration. It follows that $N \oplus F$ along with the maps

$$N \xrightarrow{j_N} N \oplus F, \quad \nu : N \oplus F \to N \times N$$

yields a path object for N in Quillen's set up [**6**, Chapter I]. Hence a candidate for ΩN in Quillen's set up is the kernel of $\nu : N \oplus F \to N \times N$. Now $(x, u) \in$

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Ker $\nu \Leftrightarrow (x, x + \epsilon(u)) = (0, 0) \Leftrightarrow x = 0, \epsilon(u) = 0$. It follows that $\lambda : \text{Ker } \epsilon \to \text{Ker } \nu$ defined by $\lambda(u) = (0, u)$ is an isomorphism of Ker ϵ onto Ker ν. But Ker ϵ is actually a candidate for ΩN in Hilton's set up of p-homotopy theory [**4**]. Since a submodule of a free module over a PID is free we see that ΩN is free. Since ΩN is determined unique up to p-homotopy type it follows that any candidate for ΩN in $\mathscr{F}$ is projective and hence free.

4.3. Let $N \in \mathscr{F}$ and $T(N)$ denote the torsion submodule of N. Then $N/T(N) = F$ is a free finitely generated module. Let $\eta : N \to F$ denote the canonical quotient map. Let $q : N \oplus N \to F \oplus F \oplus N$ be defined by $q(x, y) = (\eta(x), \eta(y), x + y)$. Then clearly $\delta_N \circ q : N \oplus N \to N$ is the same as $\nabla : N \oplus N \to N$ which carries (x, y) to $x + y$. The map $\delta_N : F \oplus F \oplus N \to N$ is a p-homotopy equivalence, since $F \oplus F$ is free. If R is any free module and $\alpha : N \oplus N \to R$ any map it is clear that $\alpha(T(N) \oplus T(N)) = 0$. Thus α induces a map $\bar{\alpha} : F \oplus F \to R$ satisfying $\bar{\alpha} \circ (\eta \oplus \eta) = \alpha$. If $\beta : F \oplus F \oplus N \to R$ is given by $\beta(u, v, x) = \bar{\alpha}(u, v)$ for all $(u, v) \in F \oplus F$ then clearly $\beta \circ q = \alpha$. Thus $q : N \oplus N \to F \oplus F \oplus N$ is a cofibration. It follows that $F \oplus F \oplus N$ together with the maps $q : N \oplus N \to F \oplus F \oplus N$, $\delta_N : F \oplus F \oplus N \to N$ yields a cylinder object for N in $\mathscr{F}$ in the sense of Quillen [**6**]. A candidate for ΣN is the quotient $F \oplus F \oplus N/q(N \oplus N)$.

Consider the map $\lambda : F \oplus F \oplus N \to F$ given by $\lambda(u, v, x) = u + v - \eta(x)$. We have $\lambda \circ q(x, y) = \lambda(\eta(x), \eta(y), x + y) = \eta(x) + \eta(y) - \eta(x + y) = 0$. Hence $q(N \oplus N) \subset \text{Ker } \lambda$. Suppose $(u, v, x) \in \text{Ker } \lambda$. Then $u + v = \eta(x)$. Choose a $z \in N$ satisfying $\eta(z) = u$. Then $v = \eta(x) - u = \eta(x) - \eta(z) = \eta(x - z)$. The element $(z, x - z) \in N \oplus N$ has the property $q(z, x - z) = (\eta(z), \eta(x - z), x) = (u, v, x)$. Hence Ker $\lambda \subset q(N \oplus N)$. Thus Ker $\lambda = q(N \oplus N)$. Moreover for any $u \in F$ we have $\lambda(u, 0, 0) = u$. Hence $\lambda : F \oplus F \oplus N \to F$ is onto. It follows that λ induces an isomorphism

$$\bar{\lambda} : \frac{F \oplus F \oplus N}{q(N \oplus N)} \simeq F.$$

Thus F is a candidate for ΣN. Since ΣN is determined unique up to homotopy type (which agrees with p-homotopy type here) we see that any ΣN is contractible.

The main result of this section is the following analogue of Theorem 3.4.

THEOREM 4.4. *Let* $M \in \mathscr{F}$, *Then*

(1) W-Cat $M = 0 =$ Ind Cat $M =$ Cocat M *whenever* M *is free;*
(2) W-Cat $M = 1$, Ind Cat $M = \infty =$ Cocat M *whenever* M *is not free;*
(3) Nil $M = 0 =$ Conil M *for all* $M \in \mathscr{F}$.

The following is the analogue of Lemma 3.5.

LEMMA 4.5. *Let* $M \in \mathscr{F}$. *If* Ind Cat $M < \infty$ (respectively Cocat $M < \infty$) *then* M *is free and hence* Ind Cat $M = 0$ (respectively Cocat $M = 0$).

Proof. For any integer $k \geqq 0$ let (S_k') and (T_k') be the statements given below.

(S_k'): In dCat $M \leqq k \Rightarrow M$ is free

(T_k'): Cocat $M \leqq k \Rightarrow M$ is free.

Both these statements will be proved by induction on k.

Clearly the statement (S_0') is true by the very definition of Inductive Category. Let $k \geqq 1$ and (S_l') be true for $1 \leqq k - 1$. If Ind Cat $M \leqq k$ there exists a cofibration

$$A \xrightarrow{q} Y$$

in $\mathscr{F}$ with Ind Cat $Y \leqq k - 1$ and the cofibre C of q dominating M. By (S_{k-1}') we see that Y is free. Let $T(A)$ denote the torsion submodule of A. Since $A \in \mathscr{F}$ it follows that $F = A/T(A)$ is free. Let $\eta : A \rightarrow F$ be the canonical quotient map.

Since Y is free it follows that $q(T(A)) = 0$. Hence q induces a map $\bar{q} : F \rightarrow Y$ satisfying $q = \bar{q} \circ \eta$. The cofibre C of q is the quotient $Y/q(A)$. Since $\eta : A \rightarrow F$ is onto we get $q(A) = \bar{q}(F)$. Hence $C = Y/\bar{q}(F)$.

Now $q : A \rightarrow Y$ is a cofibration in $\mathscr{F}$. Since F is free it follows that there exists a map $Y \xrightarrow{\beta} F$ such that $\beta \circ q = \eta$. The map

$$F \xrightarrow{\beta q} F$$

satisfies $(1_F - \beta\bar{q}) \circ \eta = \eta - \beta\bar{q}\eta = \eta - \beta q = \eta - \eta = 0$. Since $\eta : \mathrm{A} \rightarrow F$ is an epimorphism we get $1_F - \beta\bar{q}$. Hence the exact sequence

$$0 \rightarrow F \xrightarrow{\bar{q}} Y \rightarrow Y/\bar{q}(F) \rightarrow 0$$

splits. Thus $C = Y/\bar{q}(F)$ is a direct summand of Y. Hence C is projective. $\wedge$ being a PID it is free. Since C dominates M by 4.1 we see that M itself is free. This proves (S_k').

Clearly (T_0') is true by the very definition of Cocat. Let $k \geqq 1$ and let (T_l') be true for all $l \leqq k - 1$. Suppose Cocat $M \leqq k$. Then there exists a fibration

$$E \xrightarrow{\nu} B$$

with Cocat $E \leqq k - 1$ and the fibre H of ν dominating M. From (T_{k-1}') we see that E is free. H is a submodule of the finitely generated free module E over $\wedge$ (a PID). Hence H is a finitely generated free module. Since H dominates M by 4.1 we see that M is free. This proves (T_k').

Proof of Theorem 4.4. The proof is exactly similar to the proof of Theorem 3.4. In place of Lemma 3.5 we have only to use Lemma 4.5.

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