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A NOTE ON NILPOTENT-BY-ČERNIKOV GROUPS

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Abstract. In this note we prove that a locally graded group G in which all proper subgroups are (nilpotent of class not exceeding n)-by-Černikov, is itself (nilpotent of class not exceeding n)-by-Černikov.

As a preparatory result that is used for the proof of the former statement in the case of a periodic group, we also prove that a group G, containing a nilpotent of class n subgroup of finite index, also contains a characteristic subgroup of finite index that is nilpotent of class not exceeding n.

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1. Introduction. In various papers, by several authors (see, in particular, [**1**], [**8**] and [**9**]), groups whose proper subgroups satisfy the condition of being nilpotent-by-Černikov have been studied. It had been proved in [**8**] that the locally graded groups with this property are themselves nilpotent-by-Černikov, if they are not locally finite p-groups.

This case was studied by Asar in [**1**] where he proved, among other things, that the locally finite p-groups do not constitute an exception to the former statement.

Also, Otal and Peña proved, in Theorem 2 of [**9**], that a locally graded, periodic group whose proper subgroups are all abelian-by-Černikov, is itself abelian-by-Černikov. This property was proved for non-periodic groups by Napolitani and Pegoraro in [**8**].

It was now natural to ask if this were true also of locally graded groups whose proper subgroups are all (nilpotent of class not exceeding n)-by-Černikov, for any natural number n. Here we give a positive answer to this question.

Černikov groups are a particular case of groups with finite (Prüfer) rank (see [**12**, 1.4] for the basic definitions). In [**4**], the authors investigate a situation which is certainly related to the one that is examined here. They deal with groups G in which every proper subgroup H contains a normal nilpotent (or nilpotent of bounded class) subgroup K such that H/K is a group of finite rank. Some of the methods that we use here can be extended to that case, in such a way that it is possible to improve some of those results.

For example, Theorem 5 of [**4**] can be extended, by replacing "abelian" with "nilpotent of class not exceeding n", to get:

Let G be a group whose proper subgroups are all (nilpotent of class not exceeding n)-by-(finite rank). If G is locally nilpotent or locally finite with no infinite simple images, then G is also (nilpotent of class not exceeding n)-by-(finite rank).

In the present paper we prefer, however, to restrict our attention to the nilpotent-by-Černikov case, since the proofs for the other case are complicated and the results are less general than those obtained here. In fact, the statement of our main Theorem (see Theorem 2) is as follows:

MAIN THEOREM. *Let G be a locally graded group and suppose that for every proper subgroup H of G, H is (nilpotent of class not exceeding n)-by-Černikov, where n is a fixed natural number. Then also G is (nilpotent of class not exceeding n)-by-Černikov.*

Among other results that are used for the proof of the main Theorem is the following Lemma that generalizes Passman's Lemma [**10**, Chapter 12, Lemma 1.2]. We prove it in Section 2.

LEMMA 3. *Let G be a group and H be a subgroup of G, such that H is nilpotent of class not exceeding n, for some natural number n and let $[G : H] < \infty$. Then G contains a characteristic subgroup Z, such that Z is nilpotent of class not exceeding n and $[G : Z] < \infty$.*

In Section 2 we prove some preparatory results, and give the proof of the main Theorem, in the case of G periodic.

In Section 3 we complete the proof of the main Theorem, by examining the case of G non-periodic

We will mainly follow the notation of [**11**].

The notation $\mathcal{N}$ and $\mathcal{N}_n$ will indicate, respectively, the class of nilpotent groups and of groups with nilpotent class not exceeding n. The class of nilpotent-by-finite groups will be indicated by $\mathcal{NF}$. The notation $\mathcal{C}$ will indicate the class of Černikov groups; $\mathcal{N}_n\mathcal{F}$, $\mathcal{NC}$ and $\mathcal{N}_n\mathcal{C}$ will have the obvious meaning. Moreover, for any group G, $\pi(G)$ will indicate the set of all primes that divide the orders of elements of G.

2. Some preparatory results and the case of periodic G.

The first Lemma of this section is a generalization of a similar result that can be found in the proof of Lemma 4.7 of [**5**]. Here we prove it, with the same method, in a slightly different context.

LEMMA 1. *Let A be a periodic abelian group and let $T \leq Aut\ A$ be a radicable group. Then T acts nilpotently on A (i.e. there is a natural number k such that $[A,_k T] = 1$) if and only if $T = 1$.*

Proof. A radicable group is a group in which every element is an n-th power, for each natural number $n \geq 1$. Let k be a natural number, minimal such that $[A,_k T] = 1$. By way of contradiction we may suppose that $k > 1$. Let now $x \in [A,_{(k-2)} T]$, ($x \in A$ if $k = 2$). Hence $x \in A$, which is periodic and so there exists a natural s, such that $x^s = 1$. Let also $t \in T$. T is radicable and so there is $u \in T$ such that $t = u^s$. We get: $[x, t] = [x, u^s] = [x, u]^s = [x^s, u] = 1$, since $[x, u] \in [A,_{(k-1)} T]$ and so, by our hypotheses, is centralized by both x and u. Now, x and t were arbitrarily chosen and this gives that $[A,_{(k-1)} T] = 1$, a contradiction to our choice of k. Thus $k = 1$ and $T = 1$. □

The next Proposition is a direct consequence of the previous Lemma and also generalizes a result contained in the proof of Lemma 4.7 of [**5**].

PROPOSITION 1. *Let H be a periodic nilpotent group, A a normal subgroup of H, $A \in \mathcal{N}_n$ ($n \geq 1$) and such that H/A is divisible abelian. Then also $H \in \mathcal{N}_n$.*

Proof. If $n = 1$, then $A \leq C_H(A)$ and $T = H/C_H(A)$ is a divisible subgroup of Aut(A). By Lemma 1 we get that $A \leq Z_1(H)$ and so (by, e.g., [**7**, 5.3.5]) H is abelian. We now use induction on n. Consider $T = H/C_H(\gamma_n(A))$. Again, since $A \leq C_H(\gamma_n(A))$,

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T is a divisible subgroup of $\mathrm{Aut}(\gamma_n(A))$. Hence, by Lemma 1, we get that $\gamma_n(A) \leq Z_1(H)$. By the induction hypothesis, $H/Z_1(H)$ is nilpotent of class not exceeding $n-1$. Then $H \in \mathcal{N}_n$ and the proof is complete. □

LEMMA 2. *Let G be a group, H a subgroup of G,* $H \in \mathcal{N}_n$. *Let* $Z = Z_n(G)$ *be the n-th term of the ascending central series of G. Then* $HZ \in \mathcal{N}_n$.

Proof. For $n = 0$, $H = Z = 1$ and the statement is true. Suppose $n \geq 1$. By induction on i, with $1 \leq i \leq n$, it is easy to prove that $\gamma_i(HZ) \leq \gamma_i(H)Z_{n-i+1}(G)$. From this, with $i = n$, we get that $\gamma_n(HZ) \leq \gamma_n(H)Z_1(G)$. Thus $\gamma_{n+1}(HZ) \leq \gamma_{n+1}(H) = 1$ and the thesis follows. □

As it was said in Section 1, the next Lemma is a generalization of Passman's Lemma, [**10**, Chapter 12, Lemma 1.2].

LEMMA 3. *Let G be a group and H be a subgroup of G, such that* $H \in \mathcal{N}_n$ *and* $[G : H] < \infty$. *Then G contains a characteristic subgroup Z, such that* $Z \in \mathcal{N}_n$ *and* $[G : Z] < \infty$.

Proof. We use induction on n. For $n = 0$, G is finite and the statement is obviously true. If $n = 1$ the Lemma is true by ([**10**, Chapter 12, Lemma 1.2]).

Let now $n > 1$ and suppose that the Lemma is true for $n-1$. Let K be the characteristic closure of H in G. Hence $K = < H^\alpha : \alpha \in \mathrm{Aut}\, G >$. Since H has finite index in K, there is a finite number of automorphisms, $\alpha_1, \alpha_2, \dots \alpha_r$ of G, such that $K = < H^{\alpha_i} : i = 1, 2 \dots r >$. Set now $L = HZ_1(K)$, where $Z_1(K)$ is the centre of K. It is easily seen that $Z_1(K) = \bigcap_{i=1}^r Z_1(L^{\alpha_i})$ and that $K = < L^{\alpha_i} : i = 1, 2 \dots r >$. Moreover $L \in \mathcal{N}_n$ and has finite index in G. Set now $M = \bigcap_{i=1}^r L^{\alpha_i}$. Then M has finite index in G and $\gamma_n(M) \leq \gamma_n(L^{\alpha_i}) \leq Z_1(L^{\alpha_i})$, $\forall\, i = 1, 2 \dots r$. Thus $\gamma_n(M) \leq Z_1(K) \leq M$ and, since K is characteristic in G, $Z_1(K)$ is also a characteristic subgroup of G. Now $M/Z_1(K) \in \mathcal{N}_{n-1}$ and has finite index in $G/Z_1(K)$. By the induction hypothesis, $G/Z_1(K)$ contains a characteristic subgroup, say $T/Z_1(K)$, such that $T/Z_1(K) \in \mathcal{N}_{n-1}$ and $[G : T]$ is finite. It follows that $Z = T \cap K$ is characteristic in G, $[G : Z] < \infty$ and $Z \in \mathcal{N}_n$. This completes the induction and the Lemma is proved. □

It is now possible to prove our main Theorem, in the case of G periodic.

THEOREM 1. *Let G be a locally graded, periodic group and suppose that for every proper subgroup H of G,* $H \in \mathcal{N}_n\mathcal{C}$, *where n is a fixed natural number. Then also* $G \in \mathcal{N}_n\mathcal{C}$.

Proof. The Theorem is true if G contains a proper subgroup H of finite index, that we may choose normal in G: namely, if $K \in \mathcal{N}_n$ is a normal subgroup of H, then K has only a finite number of distinct conjugates in G. If H/K is Černikov, then also H/K_G, and hence G/K_G, are Černikov (K_G is the core of K in G) . Thus $G \in \mathcal{N}_n\mathcal{C}$, as desired.

We may therefore suppose that $G/\gamma_2(G)$ is divisible abelian. By [**8**, Theorem A] and [**1**, Theorem 1.3], there is a nilpotent subgroup N of G such that G/N is Černikov. If $G = N$, then G is abelian by [**11**, 5.2.5], since G is nilpotent and periodic. Hence we may suppose that N is a proper subgroup of G. Thus N has a normal subgroup K, $K \in \mathcal{N}_n$, such that N/K is a Černikov group. If $n = 0$, then $G \in \mathcal{C} = \mathcal{N}_0\mathcal{C}$ and the Theorem holds. Let now $n \geq 1$ and let D/K be the maximal divisible abelian subgroup of N/K. Then D has finite index in N and, by Proposition 1, $D \in \mathcal{N}_n$. It follows, by Lemma 3, that N contains a characteristic subgroup Z, such that $Z \in \mathcal{N}_n$ and N/Z is finite. Then Z is normal in G and G/Z is Černikov. Hence $G \in \mathcal{N}_n\mathcal{C}$ and Theorem 1 is proved. □

3. The case of non-periodic G. We will complete here the proof of the main Theorem, by examining the case of non-periodic G. The statement of the next Theorem will be general; in the proof we will refer to Theorem 1 for the periodic case.

THEOREM 2. *Let G be a locally graded group and suppose that for every proper subgroup H of G, $H \in \mathcal{N}_n\mathcal{C}$, where n is a fixed natural number. Then also $G \in \mathcal{N}_n\mathcal{C}$.*

Proof. By Theorem 1, we may assume that G is not periodic. Moreover, the proof of Theorem 1 shows that we may also assume that G does not contain proper subgroups of finite index. As above, by [**8**, Theorem A], there is a nilpotent subgroup N of G such that G/N is Černikov.

The proof that we give, will follow the steps of the proof of Theorem C of [**8**], with a few necessary adjustments. The notation will also be similar.

Since G/N is Černikov and therefore periodic, we have that N is non-periodic and nilpotent. Thus, by [**11**, 5.2.6], $N/\gamma_2(N)$ is non-periodic. Let $B/\gamma_2(N)$ be the torsion subgroup of $N/\gamma_2(N)$. Hence N/B is non-trivial, abelian and torsion free.

We may also suppose that N is a proper subgroup of G, since, if $G = N$, then G/B is non-trivial, divisible abelian and torsion free and so, even in this case, G would contain a proper normal (nilpotent) subgroup K with $G/K \in \mathcal{C}$ and we may now replace N by K, if necessary.

We will start by supposing that $N \in \mathcal{N}_{n+1}$.

Also, since N is a proper subgroup of G, there is a subgroup A, normal in N, such that $A \in \mathcal{N}_n$ and $N/A \in \mathcal{C}$.

Now G/N is a periodic abelian (in fact divisible abelian and Černikov) group, acting on the non-trivial, abelian and torsion free group N/B. Hence we can apply Lemma 2.3 of [**2**] and get that for each pair of primes p_1 and p_2, there exists a G-invariant subgroup M of N, such that $B < M$ and N/M is an abelian $\{p_1, p_2\}$-group contaning elements of orders p_1 and p_2. Moreover, if $\sigma = \pi(G/N)$ and $\pi = \pi(N/A)$, then $\sigma \cup \pi$ is a finite set (since both, G/N and N/A are Černikov groups) and we will chose p_1 and p_2 such that $p_1 \neq p_2$ and $\{p_1, p_2\} \cap (\sigma \cup \pi) = \emptyset$.

Notice also that N is the π-isolator of A in N and so, by Theorem 4.6 of [**6**], we get, (since $\gamma_{n+1}(A) = 1$), that $\gamma_{n+1}(N)$ is a π-group.

Now let H_i/M be the p_i-subgroup of N/M, for $i = 1, 2$. N/M is an abelian $\{p_1, p_2\}$-group and so $N/M = H_1/M \times H_2/M$. Moreover, by Theorem 3 of [**3**], since G/M is locally finite (it is metabelian and periodic) and $\pi(N/M) \cap \pi(G/N) = \emptyset$, there exists a subgroup C of G such that C/M is a complement of N/M in G/M.

Now we observe that, for $i = 1, 2$, CH_i is a proper subgroup of G and that $CH_i/H_i \simeq C/C \cap H_i = C/M \simeq G/N \in \mathcal{C}$. Thus it is easy to see that, for $i = 1, 2$ there exists $T_i \leq H_i$ such that T_i is normal in CH_i, $T_i \in \mathcal{N}_n$ and $CH_i/T_i \in \mathcal{C}$. Also, by our hypothesis on N and the choice of M, $\gamma_2(N) \leq M \cap Z_n(N)$). Thus, by Lemma 2, we may take $T_i \geq \gamma_2(N)$. It follows that $T_i \lhd NC = G$, for $i = 1, 2$.

It is now easy to prove that the groups $N/(T_1T_2)$ (and so, also $G/(T_1T_2)$) and $M/(T_1 \cap T_2)$ are Černikov groups.

Consider the periodic abelian group $(T_1T_2)/(T_1 \cap T_2)$. Let $P/(T_1 \cap T_2)$ be the $\{p_1, p_2\}$-component and $Q/(T_1 \cap T_2)$ be the $\{p_1, p_2\}'$-component, in $(T_1T_2)/(T_1 \cap T_2)$. Then $Q \leq M$ and so $Q/(T_1 \cap T_2) \in \mathcal{C}$. Consequently, $(T_1T_2)/P \in \mathcal{C}$ and, since $P \lhd G$, we deduce that $G/P \in \mathcal{C}$.

To conclude that $G \in \mathcal{N}_n\mathcal{C}$ we now prove that $P \in \mathcal{N}_n$. Since $P \leq N$, and $\gamma_{n+1}(N)$ is a π-group, we have that $\gamma_{n+1}(P)$ is a π-group. But also P is the $\{p_1, p_2\}$-isolator of

$T_1 \cap T_2$ in T_1T_2, and $T_1 \cap T_2$ is nilpotent of class not exceeding n. Hence, again as we did for N and A, by Theorem 4.6 of **[6]**, we get that $\gamma_{n+1}(P)$ is a $\{p_1, p_2\}$-group.

By the choice of $\{p_1, p_2\}$, we must have that $\gamma_{n+1}(P) = 1$. Thus $P \in \mathcal{N}_n$ as claimed.

To conclude the proof of Theorem 2, suppose now that $N \in \mathcal{N}_{n+r}$, with $r \geq 1$. We use induction on r, since the Theorem is true for $r = 1$. Set $l = n + r$.

By the induction hypothesis, the soluble and therefore locally graded quotient group $G/\gamma_l(N) \in \mathcal{N}_n\mathcal{C}$. Then it contains a normal subgroup $K/\gamma_l(N) \in \mathcal{N}_n$, such that $G/K \in \mathcal{C}$. Hence $\gamma_{n+1}(N \cap K) \leq \gamma_l(N) \cap K \leq Z_1(N) \cap K \leq Z_1(N \cap K)$. Thus $N \cap K \in \mathcal{N}_{n+1}$ and $G/(N \cap K) \in \mathcal{C}$. It follows, by the first part of this proof, that $G \in \mathcal{N}_n\mathcal{C}$. Hence the induction is complete and the Theorem is proved. □

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