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L^P-BOUNDS FOR PSEUDO-DIFFERENTIAL OPERATORS ON COMPACT LIE GROUPS

JULIO DELGADO AND MICHAEL RUZHANSKY

Department of Mathematics, Imperial College London,
180 Queen's Gate, London SW7 2AZ, UK (j.delgado@imperial.ac.uk;
m.ruzhansky@imperial.ac.uk)

Abstract   Given a compact Lie group G, in this paper we establish L^p-bounds for pseudo-differential operators in $L^p(G)$. The criteria here are given in terms of the concept of matrix symbols defined on the noncommutative analogue of the phase space $G \times \widehat{G}$, where $\widehat{G}$ is the unitary dual of G. We obtain two different types of L^p bounds: first for finite regularity symbols and second for smooth symbols. The conditions for smooth symbols are formulated using $\mathscr{S}^m_{\rho,\delta}(G)$ classes which are a suitable extension of the well-known (ρ, δ) ones on the Euclidean space. The results herein extend classical L^p bounds established by C. Fefferman on $\mathbb{R}^n$. While Fefferman's results have immediate consequences on general manifolds for $\rho > \max\{\delta, 1-\delta\}$, our results do not require the condition $\rho > 1-\delta$. Moreover, one of our results also does not require $\rho > \delta$. Examples are given for the case of $\mathrm{SU}(2) \cong \mathbb{S}^3$ and vector fields/sub-Laplacian operators when operators in the classes $\mathscr{S}^m_{0,0}$ and $\mathscr{S}^m_{\frac{1}{2},0}$ naturally appear, and where conditions $\rho > \delta$ and $\rho > 1-\delta$ fail, respectively.

Keywords: compact Lie groups; pseudo-differential operators; L^p bounds

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Contents

1. Introduction

In this work, we study the L^p boundedness of pseudo-differential operators on compact Lie groups. The investigation of the behaviour of pseudo-differential operators of Hörmander's class $S_{\rho,\delta}^m$ in L^p is a fundamental problem in the theory of pseudo-differential operators. The fact that the class $S_{1,\delta}^0$ begets bounded operators on L^p for every $1 < p < \infty$ is well known (e.g. [34, Ch. 13]). The boundedness on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ fails for general operators with symbols in $S_{\rho,\delta}^0(\mathbb{R}^n \times \mathbb{R}^n)$ with $\rho < 1$. Furthermore, when $m > 0$ is small, for operators with symbols in $S_{\rho,\delta}^{-m}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\rho < 1$ one can only get $L^p(\mathbb{R}^n)$ boundedness for finite intervals centred at $p = 2$, which is a consequence of C. Fefferman's estimates (cf. [7]) and the work on multipliers of Hirschman (e.g. [16]) and Wainger (cf. [35]). The obstruction for the boundedness on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ of operators in $\mathrm{Op}S_{\rho,\delta}^0(\mathbb{R}^n \times \mathbb{R}^n)$ with $\rho < 1$ is explained in a more general setting by the works of Beals [2] and [3]. The C. Fefferman's results were extended to symbols with finite regularity by Li and Wang in [19]. A version of L^p Fefferman type bounds in the setting of $S(m, g)$ classes has been established in [6].

The L^p boundedness on compact groups for invariant operators (Fourier multipliers) with symbols of finite regularity has been studied in [26, 28]. The case of the circle has been considered in [20].

The (ρ, δ) classes $\mathscr{S}_{\rho,\delta}^m(G)$ on compact Lie groups with $0 \leqslant \delta \leqslant \rho \leqslant 1$ ($\delta < 1$) have been introduced in [23] and then in [24] motivated by the study of sharp Gårding inequalities, and subsequently developed in [9, 26–29].

In this paper, we first establish L^p bounds for finite regularity symbols by applying multiplier results of [26] and [28]. Secondly, we extend Fefferman's bounds to compact Lie groups obtaining some improvement with respect to the range of (ρ, δ) from the point of view of pseudo-differential operators on compact manifolds. Our analysis will be based on the global quantisation developed in [23] and [25] as a noncommutative analogue of the Kohn–Nirenberg quantisation of operators on $\mathbb{R}^n$. The classes $\mathscr{S}_{\rho,\delta}^m(G)$ on a compact Lie group G extend the corresponding Hörmander ones when G is viewed as a manifold. The advantage here is that we will not impose the usual restriction $1 - \delta \leqslant \rho$ when dealing with those classes on manifolds. Thus, here we allow $\rho \leqslant \frac{1}{2}$ and $\rho = \delta$.

In order to illustrate our main results we first recall the $L^p(\mathbb{R}^n)$ bounds obtained by Fefferman [7]. In the following theorem, we denote by $\sigma(x, D)$ the pseudo-differential operator with symbol $\sigma(x, \xi)$, i.e.

$$\sigma(x, D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \sigma(x, \xi) \widehat{f}(\xi) \, d\xi.$$

Theorem A. (a) Let $0 \leqslant \delta < \rho < 1$ and $\nu < n(1 - \rho)/2$. Let $\sigma = \sigma(x, \xi) \in S_{\rho,\delta}^{-\nu}(\mathbb{R}^n)$. Then $\sigma(x, D)$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for

$$\left| \frac{1}{2} - \frac{1}{p} \right| \leqslant \frac{\nu}{n(1 - \rho)}.$$

(b) If $\left|\frac{1}{2} - \frac{1}{p}\right| > \frac{\nu}{n(1-\rho)}$, then the operator $\sigma(x, D)$ associated to the symbol

$$\sigma(x, \xi) = \sigma_{\rho,\nu}(\xi) = e^{i|\xi|^{1-\rho}}\langle\xi\rangle^{-\nu} \in S_{\rho,0}^{-\nu}, \tag{1.1}$$

is not bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

(c) Let $\sigma = \sigma(x, \xi) \in S_{\rho,\delta}^{-n(1-\rho)/2}(\mathbb{R}^n)$. Then $\sigma(x, D)$ does not have to be bounded on $L^1(\mathbb{R}^n)$ but is bounded from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

The part (a) can be deduced from (c) by complex interpolation. The part (b) corresponds to the classical counter-example due to Hardy–Littlewood–Hirschman–Wainger (cf. [16, 35, 36]). The complex interpolation and the duality $(H^1)' = BMO$ obtained by Stein and Fefferman in [8] reduce the proof of (c) to the estimation of $L^\infty - BMO$ bounds. We note that the conditions on the symbol in Theorem A restrict the choice of the parameter ρ depending on the order of the symbol. Part (b) shows that part (a) is sharp with respect to the size of the interval around $p = 2$. Moreover, the sharpness of the choice of the value ρ is explained by an estimate due to Hörmander [17]. In this paper, one of our main results will give an analogue of Theorem A on compact Lie groups.

In § 2, we recall basic elements of the theory of pseudo-differential operators on compact Lie groups. In § 3, we establish our main results on L^p boundedness, we consider two types of conditions, the first ones imposing finite regularity on the symbol and the second ones for C^∞-smooth symbols.

To give a taste of our results we state two of our main theorems. Here, we rely on the global noncommutative analogue of the Kohn–Nirenberg quantisation (3.7) on a compact Lie group G developed in [23, 25] providing a one-to-one correspondence between matrix symbols σ on the noncommutative phase space $G \times \widehat{G}$ and the corresponding operators $A \equiv \sigma(x, D)$ given by

$$Af(x) \equiv \sigma(x, D)f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}(\xi(x)\sigma(x, \xi)\widehat{f}(\xi)). \tag{1.2}$$

We refer to § 3 for the precise definitions of the appearing objects.

The following limited regularity result corresponds to Theorem 4.8. The notation $\mathbb{D}_\xi$ will indicate a suitable difference operator with respect to the discrete unitary dual.

Theorem 1.1. *Let G be a compact Lie group of dimension n, and let $0 \leqslant \delta, \rho \leqslant 1$. Denote by κ the smallest even integer larger than $\frac{n}{2}$. Let $1 < p < \infty$ and $\ell > \frac{n}{p}$ with $\ell \in \mathbb{N}$. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a linear continuous operator such that its matrix symbol σ satisfies*

$$\|\partial_x^\beta \mathbb{D}_\xi^\alpha \sigma(x, \xi)\|_{op} \leqslant C_{\alpha,\beta}\langle\xi\rangle^{-m_0-\rho|\alpha|+\delta|\beta|} \tag{1.3}$$

with

$$m_0 \geqslant \kappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right| + \delta\left(\left[\frac{n}{p}\right] + 1\right),$$

for all multi-indices α, β with $|\alpha| \leqslant \kappa$, $|\beta| \leqslant \ell$ and for all $x \in G$ and $[\xi] \in \widehat{G}$. Then the operator A is bounded from $L^p(G)$ to $L^p(G)$.

Here $[\frac{n}{p}]$ denotes the integer part of $\frac{n}{p}$.

For smooth symbols we prove the following theorem which corresponds to Theorem 4.15. In particular, we will not impose the well-known restrictions $\rho > \frac{1}{2}$ or $\rho \geqslant 1 - \delta$ when dealing with pseudo-differential operators on closed manifolds as an advantage of the global calculus employed here.

Theorem 1.2. *Let G be a compact Lie group of dimension n. Let $0 \leqslant \delta < \rho < 1$ and*

$$0 \leqslant \nu < \frac{n(1-\rho)}{2}.$$

Let $\sigma \in \mathscr{S}_{\rho,\delta}^{-\nu}(G)$. Then $\sigma(x, D)$ extends to a bounded operator from $L^p(G)$ to $L^p(G)$ for

$$\left| \frac{1}{p} - \frac{1}{2} \right| \leqslant \frac{\nu}{n(1-\rho)}.$$

Let us compare the statement of Theorem 1.2 on compact Lie groups with Theorem A on $\mathbb{R}^n$. First we can observe that Theorem A readily yields the corresponding L^p-boundedness result on a general compact manifold M for pseudo-differential operators with symbols in class $S_{\rho,\delta}^{-\nu}$. However, for these classes to be invariantly defined on M one needs the condition $\rho \geqslant 1 - \delta$ (see e.g. [30]). Together with condition $\rho > \delta$ this implies, in particular, that $\rho > \frac{1}{2}$. Therefore, Theorem A implies the statement of Theorem 1.2 under the additional assumption that $\rho \geqslant 1 - \delta$ (and hence also $\rho > \frac{1}{2}$); see also Remark 3.3 for the relation between operators in these symbol classes. Thus, the main point of Theorem 1.2 is to establish the L^p-boundedness without this restriction. This is possible due to the global symbolic calculus available thanks to G being a group. We point out that the condition $\delta \leqslant \rho$ has been also removed for the L^2-boundedness on $\mathbb{R}^n$ by Hounie [18].

2. Motivation and applications

Let us give several examples of one type of applications and relevance of the obtained results. Let $G = \mathrm{SU}(2) \simeq \mathbb{S}^3$ be equipped with the usual matrix multiplication of $\mathrm{SU}(2)$, or with the quaternionic product on $\mathbb{S}^3$. Let X, Y, Z be three left-invariant vector fields, orthonormal with respect to the Killing form. Then we have the following properties, established in [29]:

(i) Let $\mathcal{L}_{sub} = X^2 + Y^2$ be the sub-Laplacian (hypoelliptic by Hörmander's sum of squares theorem). Then its parametrix $\mathcal{L}_{sub}^\sharp$ has symbol in the symbol class $\mathscr{S}_{\frac{1}{2},0}^{-1}(G)$.

(ii) Let $\mathcal{H} = X^2 + Y^2 - Z$, it is also hypoelliptic by Hörmander's sum of squares theorem. Then its parametrix $H^\sharp$ has symbol in the symbol class $\mathscr{S}_{\frac{1}{2},0}^{-1}(G)$.

(iii) The operator $Z + c$ is globally hypoelliptic if and only if $ic \notin \frac{1}{2}\mathbb{Z}$. In this case its inverse $(Z + c)^{-1}$ exists and has symbol in the symbol class $\mathscr{S}_{0,0}^0(G)$.

We note that especially in the case (iii), the class $\mathscr{S}_{0,0}^0(G)$ is not invariantly defined in local coordinates while our global definition makes sense. For examples (i) and (ii), the class $\mathscr{S}_{\frac{1}{2},0}^{-1}(G)$ in local coordinates gives the Hörmander class $\mathscr{S}_{\frac{1}{2},\frac{1}{2}}^{-1}(\mathbb{R}^3)$. Consequently,

Theorem A cannot be applied since the condition $\rho > \delta$ is not satisfied in this case. Nevertheless, the results obtained in this paper apply, for example Theorem 1.1 works for the class $\mathscr{S}_{0,0}^{0}(G)$, and both Theorems 1.1 and 1.2 for the class $\mathscr{S}_{\frac{1}{2},0}^{-1}(G)$. Thus, this can be used to derive a priori estimates in L^p Sobolev spaces $W^{p,s}$, for example

$$\|f\|_{L^p(\mathbb{S}^3)} \leqslant C_p \|(Z+c)f\|_{W^{p,2|\frac{1}{p}-\frac{1}{2}|}(\mathbb{S}^3)}, \quad 1 < p < \infty,$$

or

$$\|f\|_{L^p(\mathbb{S}^3)} \leqslant C_p \|\mathcal{L}u\|_{W^{p,|\frac{1}{p}-\frac{1}{2}|-1}(\mathbb{S}^3)}, \quad 1 < p < \infty,$$

and

$$\|f\|_{L^p(\mathbb{S}^3)} \leqslant C_p \|\mathcal{H}u\|_{W^{p,|\frac{1}{p}-\frac{1}{2}|-1}(\mathbb{S}^3)}, \quad 1 < p < \infty.$$

The above examples show that similar to the introduction of the classes $S_{\rho,\delta}^{m}$ by Hörmander in the analysis of hypoelliptic operators on $\mathbb{R}^n$, the classes $\mathscr{S}_{\rho,\delta}^{m}(G)$ also appear in the analysis of (already) Fourier multipliers on Lie groups. Moreover, if a Lie group is acting on a homogeneous manifolds G/K, the Fourier analysis on G gives rise to the Fourier analysis on G/K in terms of class I representations of G.

We refer to [29] and [28] for other examples of the appearance of the globally defined classes $\mathscr{S}_{\rho,\delta}^{-\nu}(G)$ in the context of compact Lie groups, also for noninvariant operators, but let us give one explicit example here.

Let $f \in C^\infty(G)$ be a smooth function on G and let $\mathcal{L}$ be the Laplacian on G. Consider the (Schrödinger type) evolution problem

$$\begin{cases} i\partial_t u + f(x)(1-\mathcal{L})^{\delta/2}u = 0, \\ \qquad\qquad\qquad\quad u(0) = I. \end{cases} \tag{2.1}$$

Then, modulo lower-order terms, the main term of its solution operator can be seen as a pseudo-differential operator with symbol

$$\sigma_t(x,\xi) = e^{itf(x)\langle\xi\rangle^\delta},$$

where $\langle\xi\rangle$ stands for the eigenvalue of the elliptic operator $(1-\mathcal{L})^{1/2}$ corresponding to the representation ξ. One can check, for example using the functional calculus from [27], that $\sigma \in \mathscr{S}_{\rho,\delta}^{0}(G)$, with $\rho = 1 - \delta$. In particular, we may have $\rho \leqslant \delta$, depending on the range of δ in (2.1).

This example can be extended further if we take $\mathcal{L}$ in (2.1) to be a sub-Laplacian, since the matrix symbol of the sub-Laplacian can be also effectively controlled, see e.g. [13], with further dependence on indices, as already in the case of $\mathbb{S}^3$ in (i) above.

There are other examples of problems that can be effectively treated by the global calculus rather than by localisations of the classical Hörmander calculus. For example, let $\mathcal{L}_{sub}$ be a sub-Laplacian on G, i.e. a sum of squares of left-invariant vector fields satisfying Hörmander's commutator condition of order r. If we consider the Cauchy problem for the corresponding wave equation

$$\partial_t^2 u - a(t)\mathcal{L}_{sub}u = 0, \tag{2.2}$$

even with smooth function $a > 0$, it is weakly hyperbolic and its local analysis, while involving the microlocal structure of the sub-Laplacian, is rather complicated.

However, the problem (2.2) can be effectively analysed using the global techniques of pseudo-differential operators on groups. Thus, such results have been obtained in [13] with sharp regularity estimates (depending on the Hörmander commutation order r) for the solutions of the Cauchy problem for (2.2) allowing $a \geqslant 0$ to be also of Hölder regularity. The x-dependent pseudo-differential operators would appear if we allow a to also depend on x.

There is a variety of other problems in analysis that require the control of lower-order terms of the operator which are not provided by the classical theory of pseudo-differential operators using localisations but can be controlled using the global theory of pseudo-differential operators on groups or on homogeneous spaces. For example, using such techniques, estimates for the essential spectrum of operators on compact homogeneous manifolds have been obtained in [5], spaces of Gevrey functions and ultradistributions have been described in [4] relating them with the representation theory of the group acting on the space, Besov and other function spaces have been related to the representation theory of groups in [21]. It should be noted that many of the developed techniques work not only on groups but also on compact homogeneous manifolds G/K via class I representations of the compact Lie group G, thus covering the cases of real, complex or quaternionic spheres, projective spaces, and many other settings.

Furthermore, many techniques can be extended to noncompact situations, notably those of nilpotent Lie groups; see [12]. Nilpotent Lie groups, in turn, have a wide range of applications to various problems involving differential operators and equations on general manifolds due to the celebrated lifting techniques of Rothschild and Stein [22]. We refer to [12] for further explanations and examples of this (nilpotent) setting.

3. Preliminaries

In this section, we recall some basic facts about the theory of pseudo-differential operators on compact Lie groups and we refer to [23] and [25] for a comprehensive account of such topics.

Given a compact Lie group G, we equip it with the normalised Haar measure $\mu \equiv dx$ on the Borel σ-algebra associated to the topology of the smooth manifold G. The Lie algebra of G will be denoted by $\mathfrak{g}$. We also denote by $\widehat{G}$ the set of equivalence classes of continuous irreducible unitary representations of G and by $\mathrm{Rep}(G)$ the set of all such representations. Since G is compact, the set $\widehat{G}$ is discrete. For $[\xi] \in \widehat{G}$, by choosing a basis in the representation space of ξ, we can view ξ as a matrix-valued function $\xi : G \to \mathbb{C}^{d_\xi \times d_\xi}$, where d_ξ is the dimension of the representation space of ξ. By the Peter–Weyl theorem the collection

$$\left\{ \sqrt{d_\xi}\, \xi_{ij} \,:\, 1 \leqslant i, j \leqslant d_\xi, \ [\xi] \in \widehat{G} \right\}$$

is an orthonormal basis of $L^2(G)$. If $f \in L^1(G)$ we define its global Fourier transform at ξ by

$$\mathcal{F}_G f(\xi) \equiv \widehat{f}(\xi) := \int_G f(x)\xi(x)^* \, dx. \tag{3.1}$$

Thus, if ξ is a matrix representation, we have $\widehat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. The Fourier inversion formula is a consequence of the Peter–Weyl theorem, so that

$$f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}(\xi(x)\widehat{f}(\xi)). \tag{3.2}$$

Given a sequence of matrices $a(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$, we can define

$$(\mathcal{F}_G^{-1} a)(x) := \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}(\xi(x)a(\xi)), \tag{3.3}$$

where the series can be interpreted in a distributional sense or absolutely depending on the growth of (the Hilbert–Schmidt norms of) $a(\xi)$. For a further discussion we refer the reader to [23].

For each $[\xi] \in \widehat{G}$, the matrix elements of ξ are the eigenfunctions for the Laplacian $\mathcal{L}_G$ (or the Casimir element of the universal enveloping algebra), with the same eigenvalue which we denote by $-\lambda_{[\xi]}^2$, so that

$$-\mathcal{L}_G \xi_{ij}(x) = \lambda_{[\xi]}^2 \xi_{ij}(x) \quad \text{for all } 1 \leqslant i, j \leqslant d_\xi. \tag{3.4}$$

The weight for measuring the decay or growth of Fourier coefficients in this setting is $\langle \xi \rangle := (1 + \lambda_{[\xi]}^2)^{\frac{1}{2}}$, the eigenvalues of the elliptic first-order pseudo-differential operator $(I - \mathcal{L}_G)^{\frac{1}{2}}$. The Parseval identity takes the form

$$\|f\|_{L^2(G)} = \left(\sum_{[\xi] \in \widehat{G}} d_\xi \|\widehat{f}(\xi)\|_{\mathrm{HS}}^2 \right)^{1/2} \quad \text{where } \|\widehat{f}(\xi)\|_{\mathrm{HS}}^2 = \operatorname{Tr}(\widehat{f}(\xi)\widehat{f}(\xi)^*), \tag{3.5}$$

which gives the norm on $\ell^2(\widehat{G})$.

For a linear continuous operator A from $C^\infty(G)$ to $\mathcal{D}'(G)$ we define its *matrix-valued symbol* $\sigma(x, \xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ by

$$\sigma(x, \xi) := \xi(x)^*(A\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi}, \tag{3.6}$$

where $A\xi(x) \in \mathbb{C}^{d_\xi \times d_\xi}$ is understood as

$$(A\xi(x))_{ij} = (A\xi_{ij})(x),$$

i.e. by applying A to each component of the matrix $\xi(x)$. Then one has [23, 25] the global quantisation

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}(\xi(x)\sigma(x, \xi)\widehat{f}(\xi)) \equiv \sigma(x, D)f(x), \tag{3.7}$$

in the sense of distributions, and the sum is independent of the choice of a representation ξ from each equivalence class $[\xi] \in \widehat{G}$. If A is a linear continuous operator from $C^\infty(G)$ to $C^\infty(G)$, the series (3.7) is absolutely convergent and can be interpreted in the pointwise sense. The symbol σ can be interpreted as a matrix-valued function on $G \times \widehat{G}$. We refer to [23, 25] for the consistent development of this quantisation and the corresponding

symbolic calculus. If the operator A is left-invariant then its symbol σ does not depend on x. We often call such operators simply invariant.

The following inequality will be useful (e.g. [23, Theorem 12.6.1]): For $A, B \in \mathbb{C}^{n \times n}$, we have

$$\|AB\|_{\mathrm{HS}} \leqslant \|A\|_{op} \|B\|_{\mathrm{HS}}, \tag{3.8}$$

where $\|\cdot\|_{op}$ denotes the operator norm of the matrix A.

Our criteria will be formulated in terms of norms of the matrix-valued symbols. In order to justify their appearance, we recall that if $A \in \Psi^m_{\rho,\delta}(G)$ on a compact Lie group G is a pseudo-differential operator in Hörmander's class $\Psi^m_{\rho,\delta}(G)$, i.e. if all of its localisations to $\mathbb{R}^n$ are pseudo-differential operators with symbols in the class $S^m_{\rho,\delta}(\mathbb{R}^n)$, then the matrix symbol of A satisfies

$$\|\sigma(x, \xi)\|_{op} \leqslant C \langle \xi \rangle^m \quad \text{for all } x \in G, \ [\xi] \in \widehat{G}.$$

Here $\|\cdot\|_{op}$ denotes the operator norm of the matrix multiplication by the matrix $\sigma(x, \xi)$. For this fact, see e.g. [23, Lemma 10.9.1] or [25] in the $(1, 0)$ case. For the complete characterisation of Hörmander classes $\Psi^m_{1,0}(G)$ in terms of matrix-valued symbols, see also [29]. In particular, this motivates the usage of the operator norms of the matrix-valued symbols.

We say that Q_ξ is a *difference operator* of order k if it is given by

$$Q_\xi \widehat{f}(\xi) = \widehat{q_Q f}(\xi)$$

for a function $q = q_Q \in C^\infty(G)$ vanishing of order k at the identity $e \in G$, i.e.,

$$(P_x q_Q)(e) = 0$$

for all left-invariant differential operators $P_x \in \mathrm{Diff}^{k-1}(G)$ of order $k-1$. We denote the set of all difference operators of order k by $\mathrm{diff}^k(\widehat{G})$. For a given function $q \in C^\infty(G)$ it be will convenient to denote the associated difference operator, acting on Fourier coefficients, by

$$\Delta_q \widehat{f}(\xi) := \widehat{qf}(\xi). \tag{3.9}$$

Definition 3.1. A collection of k first-order difference operators

$$\Delta_1, \ldots, \Delta_k \in \mathrm{diff}^1(\widehat{G})$$

is called *admissible*, if the corresponding functions $q_1, \ldots, q_k \in C^\infty(G)$ satisfy

$$\nabla q_j(e) \neq 0, \ j = 1, \ldots, k,$$

and

$$\mathrm{rank}(\nabla q_1(e), \ldots, \nabla q_k(e)) = \dim G.$$

In particular, the group unit element e is an isolated common zero of the family $\{q_j\}_{j=1}^k$. An admissible collection is called *strongly admissible* if

$$\bigcap_{j=1}^k \{x \in G : q_j(x) = 0\} = \{e\}.$$

For a given admissible selection of difference operators on a compact Lie group G we use the multi-index notation

$$\Delta_\xi^\alpha := \Delta_1^{\alpha_1} \cdots \Delta_k^{\alpha_k} \quad \text{and} \quad q^\alpha(x) := q_1(x)^{\alpha_1} \cdots q_k(x)^{\alpha_k}.$$

Definition 3.2. Let $\{Y_j\}_{j=1}^{\dim G}$ be a basis for the Lie algebra of G, and let ∂_j be the left-invariant first-order differential operators corresponding to Y_j. For $\alpha \in \mathbb{N}_0^n$, we denote

$$\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}.$$

We use the notation ∂_x^α for ∂^α.

Let $\Delta_1, \ldots, \Delta_k \in \mathrm{diff}^1(\widehat{G})$ be a strongly admissible collection of first-order difference operators.

Let $0 \leqslant \delta \leqslant \rho \leqslant 1$ ($\delta < 1$). We say that a matrix-valued symbol $\sigma(x, \xi)$ belongs to $\mathscr{S}_{\rho,\delta}^m(G)$ if it is smooth in x, and for all multi-indices α, β there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\|\Delta_\xi^\alpha \partial_x^\beta \sigma(x, \xi)\|_{op} \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

holds uniformly in x and $\xi \in \mathrm{Rep}(G)$. If $\delta < \rho$, this class is independent of a strongly admissible collection $\Delta_1, \ldots, \Delta_k \in \mathrm{diff}^1(\widehat{G})$ of difference operators. Given a non-negative integer l we associate a seminorm $\|\sigma\|_{l;\mathscr{S}_{\rho,\delta}^m}$ defined by

$$\|\sigma\|_{l;\mathscr{S}_{\rho,\delta}^m} := \sup_{|\alpha|+|\beta|\leqslant l, (x,\xi)} \frac{\|\Delta_\xi^\alpha \partial_x^\beta \sigma(x, \xi)\|_{op}}{\langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}}.$$

Remark 3.3. If the group G is viewed as a manifold, the localised Hörmander class of operators is denoted by $\Psi_{\rho,\delta}^m(G, loc)$. In [29] for $(\rho, \delta) = (1, 0)$, and then in [9, Corollary 8.13] for more general ρ and δ, it has been shown that the class of operators $\Psi_{\rho,\delta}^m$ generated by the symbol class $\mathscr{S}_{\rho,\delta}^m(G)$, for $0 \leqslant \delta \leqslant \rho \leqslant 1$ ($\delta < 1$) coincides with the Hörmander class of operators $\Psi_{\rho,\delta}^m(G, loc)$ for $0 \leqslant \delta < \rho \leqslant 1$ and $1 - \delta \leqslant \rho$.

There is also a particular family of difference operators associated to representations that we need and that we now describe following [26, 28]. Such difference operators play an important role in the Mikhlin multiplier theorem proved in the above papers.

For a fixed irreducible representation ξ_0 let us define the (matrix-valued) difference operator

$$\xi_0 \mathbb{D} = (\xi_0 \mathbb{D}_{ij})_{i, j=1,\ldots,d_{\xi_0}}$$

corresponding to the matrix elements of the matrix-valued function $\xi_0(x) - \mathrm{I}$. In other words, if we set

$$q_{ij}(x) := \xi_0(x)_{ij} - \delta_{ij}$$

with δ_{ij} the Kronecker delta, and use the definition in (3.9), then

$$\xi_0 \mathbb{D}_{ij} = \Delta_{q_{ij}}.$$

If the representation is fixed, we may omit the index ξ_0 for brevity. For a sequence of difference operators of this type,

$$\mathbb{D}_1 = {}_{\xi_1}\mathbb{D}_{i_1 j_1}, \mathbb{D}_2 = {}_{\xi_2}\mathbb{D}_{i_2 j_2}, \ldots, \mathbb{D}_k = {}_{\xi_k}\mathbb{D}_{i_k j_k},$$

with $[\xi_m] \in \widehat{G}$, $1 \leqslant i_m, j_m \leqslant d_{\xi_m}$, $1 \leqslant m \leqslant k$, we define

$$\mathbb{D}^\alpha = \mathbb{D}_1^{\alpha_1} \cdots \mathbb{D}_k^{\alpha_k}.$$

Let us now fix a particular collection Δ_0 of representations: Let $\widetilde{\Delta_0}$ be the collection of the irreducible components of the adjoint representation, so that

$$\mathrm{Ad} = (\dim Z(G))1 \oplus \bigoplus_{\xi \in \widetilde{\Delta_0}} \xi,$$

where ξ are irreducible representations and 1 is the trivial one-dimensional representation. In the case when the centre $Z(G)$ of the group is nontrivial, we extend the collection $\widetilde{\Delta_0}$ to some collection Δ_0 by adding to $\widetilde{\Delta_0}$ a family of irreducible representations such that their direct sum is nontrivial on $Z(G)$, and such that the function

$$\rho^2(x) = \sum_{[\xi] \in \Delta_0} \left(d_\xi - \mathrm{trace}\xi(x) \right) \geqslant 0$$

(which vanishes only in $x = e$) would define the square of some distance function on G near the identity element. Such an extension is always possible, and we denote by Δ_0 any such extension; in the case of the trivial centre we do not have to take an extension and we set $\Delta_0 = \widetilde{\Delta_0}$. We denote further by $\mathbb{A}$ the second-order difference operator associated to $\rho^2(x)$,

$$\mathbb{A} = \mathscr{F}\rho^2(x)\mathscr{F}^{-1}.$$

In the sequel, when we write $\mathbb{D}^\alpha$, we can always assume that it is composed only of ${}_{\xi_m}\mathbb{D}_{i_m j_m}$ with $[\xi_m] \in \Delta_0$.

Such difference operators $\mathbb{D}^\alpha$ enjoy a number of additional algebraic properties compared to arbitrary difference operators, for example they satisfy the Leibniz formula, and lead to the distance function $\rho(x)$ that gives rise to the Calderon–Zygmund theory in the spirit of Coifman and Weiss; see [28] for the details.

4. L^p-boundedness

In this section, we establish the main results on the boundedness of operators on $L^p(G)$. We observe that from Theorem A by Fefferman and the equivalence of classes described in Remark 3.3, one can extend the Fefferman bounds to compact Lie groups for symbols in $\mathscr{S}_{\rho,\delta}^m(G)$ as an immediate consequence, but assuming that $\delta < \rho$ and $1 - \delta \leqslant \rho$. In particular, this type of argument leads to the restriction $\rho > \frac{1}{2}$. So the case of interest to us will be the one allowing $\rho \leqslant \frac{1}{2}$ and $\rho = \delta$. Moreover, we also obtain some L^p bounds for symbols with finite regularity, which cannot be deduced from the aforementioned equivalence of classes for smooth symbols. In the latter case we even allow $\delta > \rho$.

4.1. Symbols of finite regularity

In order to deduce some consequences the following lemma proved in [4] will be of importance to us.

Lemma 4.1. *Let G be a compact Lie group. Then we have*

$$\sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{-s} < \infty$$

if and only if $s > \dim G$.

In the next lemma we deduce a sufficient condition for the $L^\infty(G)$-boundedness.

Lemma 4.2. *Let G be a compact Lie group. Let σ be the symbol of a linear continuous operator $A : C^\infty(G) \to \mathcal{D}'(G)$ such that*

$$\operatorname*{ess\,sup}_{x\in G} \|\mathcal{F}_G^{-1}\sigma(x,\cdot)\|_{L^1(G)} < \infty. \tag{4.1}$$

Then A extends to a bounded operator from $L^\infty(G)$ to $L^\infty(G)$, and

$$\|Af\|_{L^\infty} \leqslant \operatorname*{ess\,sup}_{x\in G} \|\mathcal{F}_G^{-1}\sigma(x,\cdot)\|_{L^1(G)}\|f\|_{L^\infty}, \quad \forall f \in L^\infty(G).$$

Proof. We first observe that

$$
\begin{aligned}
Af(x) &= \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}(\xi(x)\sigma(x,\xi)\widehat{f}(\xi)) \\
&= \int_G \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}(\xi(x)\sigma(x,\xi)\xi(y)^*)f(y)\,dy \\
&= \int_G \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}(\xi(y)^*\xi(x)\sigma(x,\xi))f(y)\,dy \\
&= \int_G \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}(\xi(y^{-1}x)\sigma(x,\xi))f(y)\,dy \\
&= \int_G \mathcal{F}_G^{-1}\sigma(x,\cdot)(y^{-1}x)f(y)\,dy \\
&= (f * \mathcal{F}_G^{-1}\sigma(x,\cdot))(x).
\end{aligned}
$$

Hence

$$|Af(x)| \leqslant \|\mathcal{F}_G^{-1}\sigma(x,\cdot)\|_{L^1(G)}\|f\|_{L^\infty(G)} \leqslant C\|f\|_{L^\infty(G)},$$

for almost every $x \in G$.

Therefore,

$$\|Af\|_{L^\infty(G)} \leqslant \operatorname*{ess\,sup}_{x\in G} \|\mathcal{F}_G^{-1}\sigma(x,\cdot)\|_{L^1(G)}\|f\|_{L^\infty(G)},$$

completing the proof. □

The next statement gives a sufficient condition for an operator to be Hilbert–Schmidt on $L^2(G)$ and bounded on $L^p(G)$ for $2 \leqslant p \leqslant \infty$. It also shows an example of when the condition (4.1) is satisfied; see also Corollary 4.5. It will also imply further results on the L^p-boundedness. This is a particular feature of the compact situation.

Proposition 4.3. *Let G be a compact Lie group. Let σ be the symbol of a linear continuous operator $A : C^\infty(G) \to \mathcal{D}'(G)$ such that*

$$\operatorname*{ess\,sup}_{x \in G} \sum_{[\xi] \in \widehat{G}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 < \infty. \tag{4.2}$$

Then (4.1) holds and A extends to a Hilbert–Schmidt bounded operator from $L^2(G)$ to $L^2(G)$, and to a bounded operator from $L^p(G)$ to $L^p(G)$ for all $2 \leqslant p \leqslant \infty$.

The Hilbert–Schmidt norm of A is given by

$$\|A\|_{\mathrm{HS}}^2 = \int_G \sum_{[\xi] \in \widehat{G}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 \, dx. \tag{4.3}$$

Proof. We observe that the Cauchy–Schwarz inequality and the Parseval identity (3.5) imply

$$\int_G |\mathcal{F}_G^{-1} \sigma(x, \cdot)(y)| \, dy \leqslant \left(\int_G |\mathcal{F}_G^{-1} \sigma(x, \cdot)(y)|^2 \, dy \right)^{\frac{1}{2}}$$

$$= \left(\sum_{[\xi] \in \widehat{G}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 \right)^{\frac{1}{2}}$$

$$\leqslant \left(\operatorname*{ess\,sup}_{x \in G} \sum_{[\xi] \in \widehat{G}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 \right)^{\frac{1}{2}} < \infty. \tag{4.4}$$

Hence by Lemma 4.2 the operator A is bounded from $L^\infty(G)$ to $L^\infty(G)$.

Now, from the proof of Lemma 4.2 we see that the kernel K_A of A is given by

$$K_A(x, y) = \mathcal{F}_G^{-1} \sigma(x, \cdot)(y^{-1}x).$$

We observe that, by integrating (4.4) over G we get

$$\int_G \int_G |K(x, y)|^2 \, dy \, dx = \int_G \int_G |\mathcal{F}_G^{-1} \sigma(x, \cdot)(y^{-1}x)|^2 \, dy \, dx$$

$$= \int_G \int_G |\mathcal{F}_G^{-1} \sigma(x, \cdot)(z)|^2 \, dz \, dx$$

$$= \int_G \sum_{[\xi] \in \widehat{G}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 \, dx < \infty.$$

Hence A is a Hilbert–Schmidt operator on $L^2(G)$, (4.3) holds and in particular A is bounded on $L^2(G)$. By interpolating between $p = 2$ and $p = +\infty$, we conclude the proof. $\square$

Remark 4.4. (i) The $\operatorname{ess\,sup}_{x \in G}$ in the condition (4.2) can be removed if the symbol $\sigma(x, \xi)$ is continuous on G for all $[\xi] \in \widehat{G}$.

(ii) It is not hard to see that an analogous condition to (4.2) does not hold in the case of noncompact groups. Indeed, for $G = \mathbb{R}^n$ consider a symbol σ of the form

$$\sigma(x, \xi) = \beta(x)\alpha(\xi)$$

with $\beta \in L^\infty(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$, $\alpha \in L^2(\mathbb{R}^n) \setminus \{0\}$. Then,

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\sigma(x, \xi)|^2 \, d\xi < \infty,$$

but $\sigma \notin L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and σ does not beget a Hilbert–Schmidt operator on $L^2(\mathbb{R}^n)$.

As a consequence we obtain the next corollary with a condition in terms of the size of the symbol measured with the operator norm.

Corollary 4.5. *Let G be a compact Lie group and let m be a real number such that $m > \frac{\dim G}{2}$. Let σ be the symbol of a linear continuous operator $A : C^\infty(G) \to \mathcal{D}'(G)$ such that*

$$\|\sigma_A(x, \xi)\|_{op} \leqslant C\langle\xi\rangle^{-m}, \quad (x, \xi) \in G \times \widehat{G}. \tag{4.5}$$

Then A extends to a Hilbert–Schmidt bounded operator from $L^2(G)$ to $L^2(G)$, and to a bounded operator from $L^p(G)$ to $L^p(G)$ for all $2 \leqslant p \leqslant \infty$.

Proof. By applying (3.8) to the decomposition $\sigma(x, \xi) = \sigma(x, \xi)I_{d_\xi}$, where I_{d_ξ} is the identity matrix in $\mathbb{C}^{d_\xi \times d_\xi}$, Lemma 4.1 and the assumption on the symbol, we obtain for every $x \in G$:

$$\sum_{[\xi] \in \widehat{G}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 \leqslant \sum_{[\xi] \in \widehat{G}} d_\xi^2 \|\sigma(x, \xi)\|_{op}^2$$

$$\leqslant C \sum_{[\xi] \in \widehat{G}} d_\xi^2 \langle\xi\rangle^{-2m} < \infty.$$

Then, $\sup_{x \in G} \sum_{[\xi] \in \widehat{G}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 < \infty$, and an application of Proposition 4.3 concludes the proof. $\square$

Remark 4.6. (i) The condition (4.5) on the order of the symbol, $m > \frac{\dim M}{2}$, is well known (cf. [30]) in the context of compact manifolds M as a sharp order to ensure that a pseudo-differential operator is Hilbert–Schmidt. Here, in contrast, we do not assume any smoothness on the symbol nor do we require it to satisfy inequalities for the derivatives.

(ii) We observe that in the case of the torus $\mathbb{T}^n$, if the symbol σ only depends on ξ and for $k > \frac{n}{2}$, $k \in \mathbb{N}$, satisfies the inequalities

$$|\Delta_\xi^\alpha \sigma(\xi)| \leqslant C\langle\xi\rangle^{-|\alpha|} \quad \text{for all } \xi \in \mathbb{Z}^n, \tag{4.6}$$

and all multi-indices α such that $|\alpha| \leqslant k$, then the operator A is bounded on $L^p(\mathbb{T}^n)$ for all $1 < p < \infty$. Here

$$\Delta_\xi^\alpha = \Delta_{\xi_1}^{\alpha_1} \cdots \Delta_{\xi_n}^{\alpha_n}$$

are the usual partial difference operators on the lattice $\mathbb{Z}^n$. This classical result has been extended to noninvariant operators on the torus, replacing (4.6) by

$$|\partial_x^\beta \Delta_\xi^\alpha \sigma(x,\xi)| \leqslant C\langle\xi\rangle^{-|\alpha|} \quad \text{for all } \xi \in \mathbb{Z}^n, \tag{4.7}$$

and all multi-indices α, β such that $|\alpha| \leqslant k$, $|\beta| \leqslant k$.

(iii) Moreover, recently in [28, Theorem 2.1] a version of the condition (4.6) has been obtained for compact Lie groups. Let κ be the smallest even integer larger than $\frac{\dim G}{2}$. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a left-invariant linear continuous operator. Among other things it was shown in [28] that if its matrix symbol σ satisfies

$$\|\mathbb{D}_\xi^\alpha \sigma(\xi)\|_{op} \leqslant C_\alpha \langle\xi\rangle^{-|\alpha|} \tag{4.8}$$

for all multi-indices α with $|\alpha| \leqslant \kappa$ and for all $[\xi] \in \widehat{G}$, then the operator A is of weak type $(1, 1)$ and L^p-bounded for all $1 < p < \infty$.

(iv) Further, a condition of type $(\rho, 0)$ has been also obtained in [28, Corollary 5.1]. Let $\rho \in [0, 1]$ and let κ be as above. If $A : C^\infty(G) \to \mathcal{D}'(G)$ is left-invariant and its matrix symbol σ satisfies

$$\|\mathbb{D}_\xi^\alpha \sigma(\xi)\|_{op} \leqslant C_\alpha \langle\xi\rangle^{-\rho|\alpha|} \tag{4.9}$$

for all multi-indices α with $|\alpha| \leqslant \kappa$ and for all $[\xi] \in \widehat{G}$, then the operator A is bounded from the Sobolev space $W^{p,r}(G)$ to $L^p(G)$ for $1 < p < \infty$ and

$$r \geqslant \kappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Here the Sobolev space $W^{p,r}(G)$ consists of all the distributions f such that $(I - \mathcal{L}_G)^{\frac{r}{2}} f \in L^p(G)$.

(v) Lemma 4.2, Proposition 4.3 and Corollary 4.5 admit suitable extensions to general compact topological groups and nilpotent groups; see e.g. [11].

As a consequence of [28, Corollary 5.1] (or Remark 4.6(iv)), we obtain:

Theorem 4.7. *Let $\rho \in [0, 1]$ and let κ be the smallest even integer larger than $\frac{\dim G}{2}$. If $A : C^\infty(G) \to \mathcal{D}'(G)$ is left-invariant and its matrix symbol σ satisfies*

$$\|\mathbb{D}_\xi^\alpha \sigma(\xi)\|_{op} \leqslant C_\alpha \langle\xi\rangle^{-r-\rho|\alpha|} \tag{4.10}$$

with

$$r = \kappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|,$$

for all multi-indices α with $|\alpha| \leqslant \kappa$ and for all $[\xi] \in \widehat{G}$, $1 < p < \infty$, then the operator A is bounded from $L^p(G)$ to $L^p(G)$.

Proof. We observe that the symbol

$$\Gamma(\xi) = \sigma(\xi)\langle\xi\rangle^r$$

satisfies

$$\|\mathbb{D}_\xi^\alpha \Gamma(\xi)\|_{op} \leqslant C_\alpha \langle\xi\rangle^{-\rho|\alpha|},$$

for all multi-indices α with $|\alpha| \leqslant \kappa$ and for all $[\xi] \in \widehat{G}$. Then, the left-invariant operator $\mathrm{Op}(\Gamma)$ corresponding to Γ is bounded from $W^{p,r}(G)$ into $L^p(G)$ with $r = \kappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$. But

$$\mathrm{Op}(\Gamma) = A(I - \mathcal{L}_G)^{\frac{r}{2}}$$

and $(I - \mathcal{L}_G)^{\frac{r}{2}}$ is an isomorphism between $W^{p,r}(G)$ and $L^p(G)$. Therefore, A is bounded from $L^p(G)$ into $L^p(G)$. $\qquad\square$

We note that if the condition (4.10) holds for $\rho = 1$, then $r = 0$, and A is bounded on $L^p(G)$ for every $1 < p < \infty$. Hence, Theorem 4.7 absorbs the condition (4.10). We now derive a (ρ, δ)-type condition following the main idea in the proof of Theorem 5.2 in [28] and using Theorem 4.7.

Theorem 4.8. *Let $0 \leqslant \delta, \rho \leqslant 1$ and $\dim G = n$. Denote by κ the smallest even integer larger than $\frac{n}{2}$. Let $1 < p < \infty$ and let $\ell > \frac{n}{p}$ be an integer. If $A : C^\infty(G) \to \mathcal{D}'(G)$ is a linear continuous operator such that its matrix symbol σ satisfies*

$$\|\partial_x^\beta \mathbb{D}_\xi^\alpha \sigma(x,\xi)\|_{op} \leqslant C_{\alpha,\beta} \langle\xi\rangle^{-m_0 - \rho|\alpha| + \delta|\beta|} \qquad (4.11)$$

for all x, with

$$m_0 \geqslant \kappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right| + \delta\left(\left[\frac{n}{p}\right] + 1\right),$$

for all multi-indices α, β with $|\alpha| \leqslant \kappa$, $|\beta| \leqslant \ell$ and for all $[\xi] \in \widehat{G}$, then the operator A is bounded from $L^p(G)$ to $L^p(G)$.

Proof. We first write

$$Af(x) = (f * r_A(x))(x),$$

where

$$r_A(x)(y) = R_A(x, y)$$

denotes the right-convolution kernel of A. Let

$$A_y f(x) := (f * r_A(y))(x),$$

so that $A_x f(x) = Af(x)$.

Now we see that

$$\|Af\|_{L^p(G)}^p = \int_G |A_x f(x)|^p \, dx \leqslant \int_G \sup_{y \in G} |A_y f(x)|^p \, dx.$$

By applying the Sobolev embedding theorem, we obtain

$$\sup_{y \in G} |A_y f(x)|^p \leqslant C \sum_{|\gamma| \leqslant \ell} \int_G |\partial_y^\gamma A_y f(x)|^p \, dy.$$

Now, by Fubini Theorem we have

$$
\begin{aligned}
\|Af\|_{L^p(G)}^p &\leqslant C \sum_{|\gamma|\leqslant \ell} \int_G \int_G |\partial_y^\gamma A_y f(x)|^p \, dx \, dy \\
&\leqslant C \sum_{|\gamma|\leqslant \ell} \sup_{y\in G} \int_G |\partial_y^\gamma A_y f(x)|^p \, dx \\
&= C \sum_{|\gamma|\leqslant \ell} \sup_{y\in G} \|\partial_y^\gamma A_y f\|_{L^p(G)}^p \\
&\leqslant C \sum_{|\gamma|\leqslant \ell} \sup_{y\in G} \|f \mapsto f * \partial_y^\gamma r_A(y)\|_{\mathcal{L}(L^p(G))}^p \|f\|_{L^p(G)}^p.
\end{aligned}
$$

Thus, the operator A will be bounded on $L^p(G)$ provided that the left-invariant operators

$$
f \mapsto f * \partial_y^\gamma r_A(y)
$$

are uniformly bounded on $L^p(G)$ with respect to $y \in G$, $|\gamma| \leqslant \ell$. We shall now estimate for each $y \in G$, the conditions under which such operators are bounded on $L^p(G)$ according to Theorem 4.7. The symbol of the left-invariant operator $D_{y,\gamma} : f \mapsto f * \partial_y^\alpha r_A(y)$ is given by

$$
\sigma_{D_{y,\gamma}}(\xi) = \partial_y^\gamma \sigma(y, \xi).
$$

From (4.11) we have

$$
\|\mathbb{D}_\xi^\alpha \sigma_{D_{y,\gamma}}(\xi)\|_{op} \leqslant C_{\alpha,\beta} \langle \xi \rangle^{-m_0 + \delta|\gamma| - \rho|\alpha|}, \tag{4.12}
$$

for all multi-indices α, β with $|\alpha| \leqslant \kappa$, $|\gamma| \leqslant \ell$ and for all $[\xi] \in \widehat{G}$.

Hence $D_{y,\gamma}$ is bounded on $L^p(G)$ provided that

$$
-m_0 + \delta|\gamma| \leqslant -r = -\kappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|
$$

and $|\gamma| \leqslant \ell$. But this follows from the condition

$$
m_0 \geqslant \kappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right| + \delta\left(\left[\frac{n}{p}\right] + 1\right),
$$

since $\ell \geqslant [\frac{n}{p}] + 1$ and (4.12) holds for $|\gamma| \leqslant [\frac{n}{p}] + 1$. Then, the operators $D_{y,\gamma}$ are uniformly bounded on $L^p(G)$ with respect to $y \in G$, $|\gamma| \leqslant \ell$, which concludes the proof. $\qquad\square$

Remark 4.9. (i) If in Theorem 4.8 the operator A is left-invariant, then the symbol σ depends only on ξ and the conditions of the theorem recover those of Theorem 4.7.
(ii) In Theorem 4.8 the usual condition $\delta \leqslant \rho$ is not imposed, and in particular $\delta > \rho$ is allowed.
(iii) For relatively large values of p, for instance if $p > n$, the Theorem 4.8 only requires smoothness of the first order with respect to x for the symbol σ, i.e. reduces to conditions on first-order derivatives only.

(iv) By (ii) and (iii), in the extreme situation $\rho = 0, \delta = 1$, and for $p > n$, the condition (4.11) with $m_0 = \kappa(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right| + \delta([\frac{n}{p}] + 1)$ takes the form

$$\| \partial_x^\beta \mathbb{D}_\xi^\alpha \sigma(x, \xi) \|_{op} \leqslant C_{\alpha,\beta} \langle \xi \rangle^{-\kappa \left| \frac{1}{p} - \frac{1}{2} \right| + (|\beta| - 1)} \tag{4.13}$$

for all multi-indices α, β with $|\alpha| \leqslant \kappa$, $|\beta| \leqslant 1$ and for all $[\xi] \in \widehat{G}$.

We note that in such situation, the required regularity ($= 1$) is independent of the dimension $\dim G = n$, which is in contrast to the situation in the Euclidean setting. In particular, that is the case of the finite regularity improved version of Fefferman's bounds obtained by Li and Wang in [19].

For values near $p = 2$ the situation is opposite and the condition in Theorem 4.8 does not improve C. Fefferman's type conditions that can be obtained for C^∞-smooth symbols. We obtain sharper conditions on the symbol but requiring C^∞-smoothness. In particular, the usual restriction $\rho \geqslant \frac{1}{2}$ for (ρ, δ) classes on manifolds will not be imposed here as an advantageous consequence of the global symbolic calculus on compact Lie groups at our disposal.

Remark 4.10. Recently, the Mikhlin multiplier theorem obtained by the second author and Wirth in [26, 28] has been reobtained by Fischer in [10] using different collections of difference operators. If the integer part of $n/2$ is odd, the orders of required difference operators coincide, while if it is even, the order is improved by one. However, at the same time, the collections of difference operators one has to work with are different: the difference operators in [10] come from fundamental representations of the group, while our collection $\mathbb{D}^\alpha$ comes from the finite decomposition of the adjoint representation into irreducible components. However, since they are related, it is probable that the evenness of κ can be removed for our collection of difference operators $\mathbb{D}^\alpha$.

In any case, if in (4.11) one replaces the collection $\mathbb{D}^\alpha$ of difference operators by the collection of difference operators associated to fundamental representations of the group, a simple modification of the proof yields the statement of Theorem 4.8 with κ being the smallest integer larger than $\frac{n}{2}$ without requiring its evenness, giving an improvement of the order by one for half of the dimensions.

4.2. C^∞-smooth symbols

We turn now to a different perspective by looking for conditions for C^∞-smooth symbols. We employ the geodesic distance on the group G and it will be denoted by d. For the distances from the unit element e we write $|y| = d(y, e)$. The corresponding *BMO* space with respect to this distance will be denoted by $BMO(G)$. The following lemma will be useful to obtain $L^\infty - BMO(G)$ bounds by applying partitions of unity.

Here and in the sequel it will be also useful to introduce the number

$$a := 1 - \rho$$

that we use everywhere without special notice. Before formulating the following lemma we record asymptotic properties that will be of use on several occasions: asymptotically

as $\lambda \to \infty$ we have

$$\sum_{\langle \xi \rangle \leqslant \lambda} d_\xi^2 \langle \xi \rangle^{\alpha n} \asymp \lambda^{(\alpha+1)n} \quad \text{for } \alpha > -1, \tag{4.14}$$

and

$$\sum_{\langle \xi \rangle \geqslant \lambda} d_\xi^2 \langle \xi \rangle^{\alpha n} \asymp \lambda^{(\alpha+1)n} \quad \text{for } \alpha < -1. \tag{4.15}$$

We refer to [1] for their proof.

Lemma 4.11. *Let G be a compact Lie group of dimension n and let $0 < a < 1$.*
Let $0 \leqslant \delta < 1 - a$. Let $\sigma \in \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}(G)$ be supported in

$$\{(x, \xi) \in G \times \widehat{G} : R \leqslant \langle \xi \rangle \leqslant 3R\},$$

for some $R > 0$. Then $\sigma(x, D)$ extends to a bounded operator from $L^\infty(G)$ to $L^\infty(G)$,
and for $l \geqslant \frac{n}{2}$ we have

$$\|\sigma(x, D)f\|_{L^\infty(G)} \leqslant C \|\sigma\|_{l, \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \|f\|_{L^\infty(G)},$$

with C independent of σ, f and R.

Proof. Let $\sigma \in \mathscr{S}_{\rho,\delta}^{-m}(G)$ be supported in

$$\{(x, \xi) \in G \times \widehat{G} : R \leqslant \langle \xi \rangle \leqslant 3R\}, \tag{4.16}$$

for some fixed $R > 0$. In order to prove Lemma (4.11) we apply Lemma 4.2. We split G
into the form

$$G = \{y \in G : |y| \leqslant b\} \cup \{y \in G : |y| > b\},$$

where $b = R^{a-1}$.

By applying Cauchy–Schwarz inequality, Parseval identity and the inequality (3.8) to
the decomposition

$$\sigma(x, \xi) = \sigma(x, \xi) I_{d_\xi}$$

we obtain

$$\int_{|y| \leqslant b} |\mathcal{F}_G^{-1}\sigma(x, \cdot)(y)| \, dy \leqslant (\mu(\{|y| \leqslant b\}))^{\frac{1}{2}} \left(\int_G |\mathcal{F}_G^{-1}\sigma(x, \cdot)(y)|^2 \, dy \right)^{\frac{1}{2}}$$

$$= \left(\mu(\{|y| \leqslant R^{a-1}\}) \right)^{\frac{1}{2}} \left(\sum_{[\xi] \in \widehat{G}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 \right)^{\frac{1}{2}}$$

$$\leqslant C R^{\frac{n(a-1)}{2}} \left(\sum_{\{R \leqslant \langle \xi \rangle \leqslant 3R\}} d_\xi \|\sigma(x, \xi)\|_{\mathrm{HS}}^2 \right)^{\frac{1}{2}}$$

$$\leqslant C R^{\frac{n(a-1)}{2}} \left(\sum_{\{R \leqslant \langle \xi \rangle \leqslant 3R\}} d_\xi^2 \|\sigma(x, \xi)\|_{op}^2 \right)^{\frac{1}{2}}$$

$$\leqslant C \|\sigma\|_{0, \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} R^{\frac{n(a-1)}{2}} \left(\sum_{\{R \leqslant \langle \xi \rangle \leqslant 3R\}} d_\xi^2 \langle \xi \rangle^{-na} \right)^{\frac{1}{2}}$$

$$\leqslant C \|\sigma\|_{0, \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} R^{\frac{n(a-1)}{2}} R^{\frac{n(1-a)}{2}} \tag{4.17}$$

$$\leqslant C \|\sigma\|_{0, \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} < \infty, \tag{4.18}$$

with C independent of R and σ. For the inequality (4.17) we have applied the estimate (4.14).

We now consider the integral $\int_{|y| \geqslant b} |\mathcal{F}_G^{-1}\sigma(x, \cdot)(y)| \, dy$. To analyse it we take the difference operator Δ_q associated to q that vanishes at e of order l and e is its isolated zero, i.e., there exist constants $C_1, C_2 > 0$ such that

$$C_1 |y|^l \leqslant |q(y)| \leqslant C_2 |y|^l.$$

We first note that $|q(y)| \leqslant C|y|^l$, for small $|y|$, e.g. $|q(y)| \leqslant Cd^l$, for $|y| \leqslant d$ for some suitable d. We have, using the boundedness of r,

$$\int_{|y| \geqslant b} |\mathcal{F}_G^{-1}\sigma(x, \cdot)(y)| \, dy = \int_{|y| \geqslant b} \frac{|q(y)(\mathcal{F}_G^{-1}\sigma(x, \cdot))(y)|}{|q(y)|} \, dy$$

$$\leqslant \left(\int_{\{|y| \geqslant b\}} |q(y)|^{-2} \, dy \right)^{\frac{1}{2}} \left(\int_G |q(y)(\mathcal{F}_G^{-1}\sigma(x, \cdot))(y)|^2 \, dy \right)^{\frac{1}{2}}$$

$$\leqslant C \left(\int_{\{|y| \geqslant b\}} |y|^{-2l} \, dy \right)^{\frac{1}{2}} \left(\sum_{\{R \leqslant \langle \xi \rangle \leqslant 3R\}} d_\xi \|\Delta_q \sigma(x, \xi)\|_{\mathrm{HS}}^2 \right)^{\frac{1}{2}}$$

$$\leqslant C (b^{n-2l})^{\frac{1}{2}} \left(\sum_{\{R \leqslant \langle \xi \rangle \leqslant 3R\}} d_\xi^2 \|\Delta_q \sigma(x, \xi)\|_{op}^2 \right)^{\frac{1}{2}}$$

$$\leqslant C \|\sigma\|_{l, \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} R^{(a-1)(\frac{n}{2}-l)} \left(\sum_{\{R \leqslant \langle \xi \rangle\}} d_\xi^2 \langle \xi \rangle^{-na-2l(1-a)} \right)^{\frac{1}{2}}$$

$$\leqslant C \|\sigma\|_{l, \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} R^{(a-1)(\frac{n}{2}-l)} R^{(1-a)(\frac{n}{2}-l)} \tag{4.19}$$

$$\leqslant C \|\sigma\|_{l, \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} < \infty,$$

with C independent of R and σ. For the estimation of the integral $\int_{\{|y| \geqslant b\}} |y|^{-2l} \, dy$ we note that the essential case is b small, and so the bound can be reduced to a local estimation. $\qquad \square$

We now establish a $L^\infty(G) - BMO(G)$ estimate which will have as a consequence the main results for smooth symbols. The space $BMO(G)$ correspond to the system of balls $B(x, r)$ determined by the geodesic distance d.

Theorem 4.12. *Let G be a compact Lie group of dimension n and let $0 < \rho < 1$. Let $0 \leqslant \delta < \rho$ and $\sigma \in \mathscr{S}_{\rho,\delta}^{-\frac{n(1-\rho)}{2}}(G)$. Then $\sigma(x, D)$ extends to a bounded operator from $L^\infty(G)$ to $BMO(G)$ and moreover, for $l > \frac{n}{2}$ we have*

$$\|\sigma(x, D)f\|_{BMO(G)} \leqslant C\|\sigma\|_{l,\mathscr{S}_{\rho,\delta}^{-\frac{n(1-\rho)}{2}}}\|f\|_{L^\infty(G)},$$

with C independent of σ and f.

Proof. Here and everywhere we write $a := 1 - \rho$. Let us fix $f \in L^\infty(G)$ and $B = B(x_0, r) \subset G$. We show that there exist an integer k and a constant $C > 0$ independent of f and B such that

$$\frac{1}{\mu(B(x_0, r))} \int_B |\sigma(x, D)f(x) - g_B| \, dx \leqslant C\|\sigma\|_{k,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}}\|f\|_{L^\infty}, \qquad (4.20)$$

where we have written $g = \sigma(x, D)f$ and $g_B = \frac{1}{\mu(B(x_0,r))}\int_B g \, dx$.

We also write

$$R_0 = \sup\{R : \exists x \in G \text{ such that } B(x, R) \subset G\}.$$

We split $\sigma(x, \xi)$ into two symbols,

$$\sigma = \sigma^0 + \sigma^1,$$

with σ^0 supported in $\langle \xi \rangle \leqslant 2R_0 r^{-1}$ and σ^1 supported in $\langle \xi \rangle \geqslant \frac{1}{2}R_0 r^{-1}$, satisfying the following estimates

$$\|\sigma^0\|_{l,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}}, \|\sigma^1\|_{l,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \leqslant C_l \|\sigma\|_{l,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \quad \text{for every } l \geqslant 1. \qquad (4.21)$$

In order to establish the existence of the above splitting (4.21) one can consider a function $0 \leqslant \gamma \in C^\infty(\mathbb{R})$ which equals to 1 if $|t| \leqslant \frac{1}{2}$ and with $\operatorname{supp} \gamma = \{|t| \leqslant 1\}$. We set

$$\tilde{\gamma}(\xi) = \gamma(r\langle \xi \rangle)$$

and

$$\sigma^0(x, \xi) = \sigma(x, \xi)\tilde{\gamma}(\xi).$$

Moreover, the seminorms of σ^0 can be controlled by those of σ by

$$\|\sigma^0\|_{l,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \leqslant C_l \|\sigma\|_{l,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \quad \text{for all } l \geqslant 1. \qquad (4.22)$$

Now, by taking $\sigma^1 = \sigma - \sigma^0$, the estimate (4.22) is still valid for σ^1.

Now we note that for a left-invariant vector field X on G we have

$$X(\xi(x)\sigma(x, \xi)) = X(\xi(x))\sigma(x, \xi) + \xi(x)X\sigma(x, \xi)$$
$$= \xi(x)\sigma_X(\xi)\sigma(x, \xi) + \xi(x)X\sigma(x, \xi),$$

where we have used the fact that

$$\xi(x)\sigma_X(\xi) = (X\xi)(x)$$

from (3.6) applied to $A = X$.

Hence

$$XAf(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}\left(\xi(x)(\sigma_X(\xi)\sigma(x,\xi) + X\sigma(x,\xi))\widehat{f}(\xi)\right).$$

In particular, with $X = X_k = \partial_{x_k}$ being a left-invariant vector field we obtain

$$\partial_{x_k}\sigma^0(x,D)f(x) = \sigma'(x,D)f(x), \tag{4.23}$$

where

$$\sigma'(x,\xi) = \sigma_{X_k}(\xi)\sigma(x,\xi) + X_k\sigma(x,\xi).$$

By using a suitable partition of unity we write

$$\sigma'(x,\xi) = \sum_{j=1}^{\infty}\rho_j(x,\xi),$$

with ρ_j supported in $\langle\xi\rangle \sim 2^{-j}r^{-1}$, and such that

$$\|\rho_j\|_{l,\mathscr{S}^{-\frac{na}{2}}_{1-a,\delta}} \leqslant C2^{-j}r^{-1}\|\sigma\|_{l,\mathscr{S}^{-\frac{na}{2}}_{1-a,\delta}}. \tag{4.24}$$

In order to construct such partition of unity, we consider $\eta : \mathbb{R} \to \mathbb{R}$ defined by

$$\eta(t) = \begin{cases} 0 & \text{if } |t| \leqslant 1, \\ 1 & \text{if } |t| \geqslant 2. \end{cases}$$

We put $\rho(t) = \eta(t) - \eta(2^{-1}t)$. Then

$$\operatorname{supp}\rho = \{1 \leqslant |t| \leqslant 4\}.$$

One can see that

$$1 = \eta(t) + \sum_{j=1}^{\infty}\rho(2^j t) \quad \text{for all } t \in \mathbb{R}.$$

Indeed,

$$\eta(t) + \sum_{j=1}^{\ell}\rho(2^j t) = \eta(t) + \eta(2t) - \eta(t) + \eta(2^2 t) - \eta(2t) + \cdots + \eta(2^\ell t) - \eta(2^{\ell-1})$$

$$= \eta(2^\ell t) \to 1 \quad \text{as } \ell \to \infty.$$

In particular, we can write $t = r\langle\xi\rangle$ and then

$$1 = \eta(r\langle\xi\rangle) + \sum_{j=1}^{\infty}\rho(r2^j\langle\xi\rangle).$$

The support of η is $\{|t| > 1\}$ and if $r\langle\xi\rangle \leqslant 1$, then

$$\eta(r\langle\xi\rangle) \equiv 0$$

and hence

$$1 = \sum_{j=1}^{\infty}\rho(r2^j\langle\xi\rangle)$$

for $\{\langle\xi\rangle \leqslant r^{-1}\}$.

Now, since supp $\sigma' = $ supp σ^0 in view of (4.23), we have

$$\sigma'(x,\xi) = \sum_{j=1}^{\infty} \rho(r2^j\langle\xi\rangle)\cdot\sigma'(x,\xi).$$

We set

$$\rho_j(x,\xi) = \rho(r2^j\langle\xi\rangle)\cdot\sigma'(x,\xi) = \alpha_j(\xi)\sigma'(x,\xi).$$

The estimate (4.24) for the seminorms of ρ_j now follows from the relation between σ and σ' as well as from the fact that $\rho_j(x,\xi) = \alpha_j(\xi)\sigma'(x,\xi)$.

We now apply Lemma 4.11 to the pieces ρ_j obtaining

$$\begin{aligned}
\|\partial_{x_k}\sigma^0(x,D)f\|_{L^\infty} &\leqslant \sum_{j=0}^{\infty} \|\rho_j(x,D)f\|_{L^\infty} \\
&\leqslant Cr^{-1}\sum_{j=0}^{\infty} 2^{-j}\|\sigma\|_{l,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}}\|f\|_{L^\infty} \\
&\leqslant Cr^{-1}\|\sigma\|_{l,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}}\|f\|_{L^\infty}.
\end{aligned}$$

An application of the Mean Value Theorem gives us

$$|\sigma^0(x,D)f(x) - g_B| \leqslant C\|\sigma\|_{l;\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}}\|f\|_{L^\infty}.$$

Hence

$$\frac{1}{|B(x_0,r)|}\int_B |\sigma^0(x,D)f(x) - g_B|\,dx \leqslant C\|\sigma\|_{l,\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}}\|f\|_{L^\infty}, \tag{4.25}$$

which gives (4.20) for σ^0.

We now consider the term σ^1, we recall that we fixed a ball $B(x_0,r) \subset G$. We now also fix a cut-off function ϕ over G, with $0 \leqslant \phi \leqslant 10$, $\phi \geqslant 1$ on $B(x_0,r)$ and such that its Fourier transform $\widehat{\phi}$ verifies $\operatorname{supp}(\widehat{\phi}) \subset \{\langle\xi\rangle \leqslant (C^{-1}r)^{\frac{1}{1-a}}\}$. Let us write

$$\phi(x)\cdot\sigma^1(x,D)f(x) = \sigma^1(x,D)(\phi f)(x) + \left[\phi,\sigma^1(x,D)\right]f(x) = I + II. \tag{4.26}$$

For the estimation of I we begin by factorising it in a suitable way. Let L be the following power of the Laplacian on G, $L = (1 - \mathcal{L}_G)^{\frac{na}{2}}$. By [27, Theorem 4.2] we have $L \in \operatorname{Op}\mathscr{S}_{1,0}^{na}(G)$ and it is known that L^{-1} is a positive operator (cf. [14, 15, 33]), i.e., $L^{-1}(g) \geqslant 0$ if $g \geqslant 0$. We write

$$\sigma^1(x,D)(\phi f) = \left(\sigma^1(x,D)\circ L\right)\left(L^{-1}(\phi f)\right). \tag{4.27}$$

Since $\operatorname{Op}\mathscr{S}_{1,0}^{\frac{na}{2}}(G) \subset \operatorname{Op}\mathscr{S}_{1-a,\delta}^{\frac{na}{2}}(G)$, we note that $\sigma^1(x,D)\circ L$ is a pseudo-differential operator in $\operatorname{Op}\mathscr{S}_{1-a,\delta}^0$. We also have

$$L^{-1}: H^{-\frac{na}{2}} \longrightarrow L^2. \tag{4.28}$$

By the L^2 boundedness for operators of order 0 (cf. [9, Proposition 8.1]), applied to the operator $\sigma^1(x, D) \circ L$, we deduce the existence of a constant C and an integer l_0 such that

$$\|\sigma^1(x, D)(\phi f)\|_{L^2}^2 \leqslant C \|\sigma_A^1\|_{l_0; \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}}^2 \cdot \|L^{-1}(\phi f)\|_{L^2}^2. \tag{4.29}$$

On the other hand, by (4.28) we have

$$\|L^{-1}(\phi f)\|_{L^2}^2 \leqslant C \|\phi f\|_{H^{-\frac{na}{2}}}^2.$$

We also observe that $\|\widehat{\phi}(\xi)\|_{\mathrm{HS}}^2 \leqslant C' d_\xi$. Indeed, by the definition of the Fourier transform on compact groups, since $\xi(x)$ is unitary and $0 \leqslant \phi \leqslant 10$ we have

$$\|\widehat{\phi}(\xi)\|_{\mathrm{HS}} \leqslant \int_G \|\xi^*(x)\|_{\mathrm{HS}} |\phi(x)| \, dx \leqslant 10\sqrt{d_\xi},$$

where we have used the identities

$$\|\xi^*(x)\|_{\mathrm{HS}}^2 = \mathrm{Tr}(\xi^*(x)\xi(x)) = \mathrm{Tr}(I_{d_\xi}) = d_\xi.$$

Since L is a positive operator, we obtain

$$\begin{aligned}
\|L^{-1}(\phi f)\|_{L^2}^2 &\leqslant \|f\|_{L^\infty}^2 \|L^{-1}(\phi)\|_{L^2}^2 \\
&\leqslant C_1 \|f\|_{L^\infty}^2 \|\phi\|_{H^{-\frac{na}{2}}}^2 \\
&\leqslant C_1 \|f\|_{L^\infty}^2 (C^{-1} r)^n \\
&\leqslant C \|f\|_{L^\infty}^2 |B(x_0, r)|.
\end{aligned}$$

For the estimation of $\|\phi\|_{H^{-\frac{na}{2}}}^2$ we have used the following inequalities:

$$\begin{aligned}
\|\phi\|_{H^{-\frac{na}{2}}}^2 &\leqslant \sum_{\langle\xi\rangle \leqslant C r^{\frac{1}{1-a}}} d_\xi \langle\xi\rangle^{-na} \|\widehat{\phi}(\xi)\|_{\mathrm{HS}}^2 \\
&\leqslant C' \sum_{\langle\xi\rangle \leqslant C r^{\frac{1}{1-a}}} d_\xi^2 \langle\xi\rangle^{-na} \\
&\leqslant C_2 r^{\frac{1}{1-a}(1-a)n} = C_2 r^n.
\end{aligned}$$

For the last inequality we have applied the estimate (4.14).

Thus

$$\|\sigma^1(x, D)(\phi f)\|_{L^2}^2 \leqslant C \|\sigma^1\|_{l_0; \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}}^2 \|f\|_{L^\infty}^2 |B(x_0, r)|. \tag{4.30}$$

By the Cauchy–Schwarz inequality we get

$$\begin{aligned}
\frac{1}{|B(x_0, r)|} \int_B |\sigma^1(x, D)(\phi f)(x)| \, dx &\leqslant \left(\frac{1}{|B|} \int_B |\sigma^1(x, D)(\phi f)(x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\leqslant C \|\sigma^1\|_{l_0; \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \|f\|_{L^\infty}.
\end{aligned} \tag{4.31}$$

This proves the desired estimated for I.

For the estimation of II, we begin by writing $\big[\phi, \sigma^1(x, D)\big]f(x)$ in the form

$$\theta(x, D)f(x),$$

where $\theta(x, \xi)$ is a suitable symbol. To calculate $\theta(x, \xi)$ we write $B := \sigma^1(x, D)$, consider the convolution kernel k_x of B and observe that

$$
\begin{aligned}
\big[\phi, \sigma^1(x, D)\big]f(x) &= \phi(x)Bf(x) - B(\phi f)(x) \\
&= \phi(x)(f * k_x)(x) - ((\phi f) * k_x)(x) \\
&= \int_G \phi(x)f(xy^{-1})k_x(y)\,dy - \int_G \phi(xy^{-1})f(xy^{-1})k_x(y)\,dy \\
&= \int_G f(xy^{-1})\big(\phi(x)k_x(y) - \phi(xy^{-1})k_x(y)\big)\,dy.
\end{aligned}
$$

Hence, $\theta(x, \xi)$ is given by

$$\theta(x, \xi) = \int_G \big(\phi(x)k_x(y) - \phi(xy^{-1})k_x(y)\big)\xi(y)^*\,dy.$$

On the other hand, by using Taylor expansions on compact Lie groups (see e.g. [23] or [29]), we can write

$$\phi(xy^{-1}) = \phi(x) + \sum_{|\alpha|=1} \psi_\alpha(x, y)q_\alpha(y),$$

where $\psi_\alpha \in C^\infty(G \times G)$, $q_\alpha \in C^\infty(G)$, $q_\alpha(e) = 0$. Hence

$$
\begin{aligned}
\theta(x, \xi) &= \sum_{|\alpha|=1} \int_G \psi_\alpha(x, y)q_\alpha(y)k_x(y)\xi(y)^*\,dy \\
&= \sum_{|\alpha|=1} \Delta_{\psi_\alpha(x,\cdot)}\Delta_{q_\alpha}\sigma(x, \xi).
\end{aligned}
$$

Thus

$$\|\theta(x, \xi)\|_{op} \leqslant C\langle\xi\rangle^{-\frac{na}{2}-(1-a)}. \tag{4.32}$$

For the last inequality we have used the following estimate:

$$\sup_{[\xi]\in\widehat{G}} \|\Delta_{\psi_\alpha(x,\cdot)}\tau(\xi)\|_{op} \leqslant C\|\psi_\alpha(x, \cdot)\|_{C^k} \sup_{[\xi]\in\widehat{G}} \|\tau(\xi)\|_{op},$$

for $k > \frac{n}{2}$. We now write

$$\theta(x, \xi) = \sum_{j=0}^{\infty} \theta_j(x, \xi),$$

with $\theta_j(x, \xi)$ supported in $\langle\xi\rangle \sim 2^j r^{-1}$.

By (4.32) and from the inequalities for the support of θ_j one has

$$\langle\xi\rangle^{-(1-a)} \leqslant C2^{-j(1-a)}$$

and

$$\|\theta_j\|_{l;\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \leqslant C_l 2^{-j(1-a)}\|\sigma\|_{l;\mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}},$$

for all l.

According to Lemma 4.11 applied to the symbols θ_j and $l > \frac{n}{2}$ we have

$$
\begin{aligned}
\big\|\big[\phi, \sigma^1(x, D)f\big]\big\|_{L^\infty} &\leqslant \sum_{j=0}^\infty \|\theta_j(x, D)f\|_{L^\infty} \\
&\leqslant \sum_{j=0}^\infty C 2^{-j(1-a)} \|\sigma\|_{l; \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \|f\|_{L^\infty} \\
&\leqslant C \|\sigma\|_{l; \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \|f\|_{L^\infty}. \tag{4.33}
\end{aligned}
$$

Since $\phi \geqslant 1$ on $B(x_0, \delta)$, by using (4.31) and (4.33) into (4.26) we have

$$
\begin{aligned}
\frac{1}{|B(x_0, \delta)|} \int_B |\sigma^1(x, D)f(x)|\, dx &\leqslant \frac{1}{|B(x_0, \delta)|} \int_B |\phi(x) \cdot \sigma^1(x, D)f(x)|\, dx \\
&\leqslant C \|\sigma\|_{l; \mathscr{S}_{1-a,\delta}^{-\frac{na}{2}}} \|f\|_{L^\infty},
\end{aligned}
$$

which concludes the proof. $\qquad\square$

We now establish a theorem for symbols in $\mathscr{S}_{\rho,\delta}^m(G)$. We recall that the L^2 boundedness holds for operators with symbols in $\mathscr{S}_{\rho,\delta}^0(G)$ (cf. [9, Proposition 8.1]). As a consequence of real interpolation between L^2 boundedness and the previous $L^\infty - BMO$ boundedness we have:

Theorem 4.13. *Let G be a compact Lie group of dimension n and let $0 < \rho < 1$. Let $0 \leqslant \delta < \rho$ and $\sigma \in \mathscr{S}_{\rho,\delta}^{-\frac{n(1-\rho)}{2}}(G)$. Then $\sigma(x, D)$ extends to a bounded operator from $L^p(G)$ to $L^p(G)$ for $1 < p < \infty$.*

Proof. We write $A = \sigma(x, D)$. The symbol σ satisfies the condition of Theorem 4.12. Hence A is bounded from $L^\infty(G)$ to $BMO(G)$. Moreover, A is bounded from $L^2(G)$ to $L^2(G)$. This implies the boundedness of A from $L^p(G)$ to $L^p(G)$ for $2 \leqslant p \leqslant \infty$. On the other hand, since $\sigma_{A^*} \in \mathscr{S}_{\rho,\delta}^0(G)$ then $A^*: L^p(G) \to L^p(G)$ is bounded for $2 \leqslant p \leqslant \infty$ and hence by duality we get that $A: L^p(G) \to L^p(G)$ is bounded also for $1 \leqslant p \leqslant 2$. $\quad\square$

Remark 4.14. The index $\frac{n(1-\rho)}{2}$ in Theorem 4.13 cannot be improved, i.e., if one takes instead an index $\nu_0 < \frac{n(1-\rho)}{2}$, one only gets L^p boundedness for some finite interval around $p = 2$ and not for any p outside that interval. This situation will be explained in more detail by the next theorem and Remark 4.16.

We now apply the complex interpolation for an analytic family of operators (cf. [31, 32]) to obtain L^p bounds for orders ν with $0 \leqslant \nu \leqslant \frac{n(1-\rho)}{2}$:

Theorem 4.15. *Let G be a compact Lie group of dimension n and let $0 < \rho < 1$. Let $0 \leqslant \delta < \rho$ and $\sigma \in \mathscr{S}_{\rho,\delta}^{-\nu}$ with*

$$
0 \leqslant \nu < \frac{n(1-\rho)}{2}.
$$

Then $\sigma(x, D)$ extends to a bounded operator from $L^p(G)$ to $L^p(G)$ for

$$\left| \frac{1}{p} - \frac{1}{2} \right| \leqslant \frac{\nu}{n(1-\rho)}.$$

Proof. Let $\sigma \in \mathscr{S}_{1-a,\delta}^{-\nu}(G)$. We consider the family of operators $\{T_z\}_{0 \leqslant \text{Re } z \leqslant 1}$ defined by the symbols

$$\gamma_z(x, \xi) := e^{z^2} \sigma(x, \xi) \langle \xi \rangle^{\nu + \frac{na}{2}(z-1)},$$

where the operators T_z are defined via the global quantisation (3.7). One can verify that for every $z = t + is$ such that $0 \leqslant t \leqslant 1$, $s \in \mathbb{R}$ and $k \in \mathbb{N}$, we have

$$\|\gamma_z\|_{k, \mathscr{S}_{1-a,\delta}^0} \leqslant e^{t^2 - s^2} p(|z|) \|\sigma\|_{k, \mathscr{S}_{1-a,\delta}^{-\nu}},$$

where $p(\lambda)$ is a polynomial of degree k.

Since $0 \leqslant t \leqslant 1$, $-\infty < s < \infty$ and e^{s^2} dominates $|z|^k$ for $z \in \{0 \leqslant \text{Re } z \leqslant 1\}$, there exists a constant $C_k > 0$ independent of z such that

$$e^{t^2 - s^2} p(|z|) \leqslant C_k.$$

Hence

$$\|\gamma_z\|_{k, \mathscr{S}_{1-a,\delta}^0} \leqslant C_k \|\sigma\|_{k, \mathscr{S}_{1-a,\delta}^{-\nu}}, \tag{4.34}$$

with $C_k > 0$ independent of z. More precisely, C_k only depends on finitely many seminorms of σ.

By the L^2 boundedness for $\mathscr{S}_{\rho,\delta}^0$ classes there exist a constant $C > 0$ and an integer N such that

$$\|T_z f\|_{L^2(G)} \leqslant C \|\gamma_z\|_{N, \mathscr{S}_{1-a,\delta}^0} \|f\|_{L^2(G)}. \tag{4.35}$$

From (4.34) and (4.35) we obtain

$$\|T_z f\|_{L^2(G)} \leqslant C_1 \|\sigma\|_{N, \mathscr{S}_{1-a,\delta}^{-\nu}} \|f\|_{L^2(G)},$$

for a suitable constant $C_1 > 0$.

It is clear that the family $\{T_z\}_{0 \leqslant \text{Re } z \leqslant 1}$ is analytic in the strip

$$S = \{z = x + iy \in \mathbb{C} : 0 < x < 1\}$$

and continuous in $\overline{S}$. Thus, the family $\{T_z\}_{0 \leqslant \text{Re } z \leqslant 1}$ defines an analytic family of operators uniformly bounded on $\mathcal{L}(L^2(G), L^2(G))$. In order to apply the complex interpolation we observe that

$$\sup_{-\infty < s < \infty} \|T_{1+is} f\|_{L^2(G)} \leqslant C_1 \|\sigma\|_{N, \mathscr{S}_{1-a,\delta}^{-\nu}} \|f\|_{L^2(G)}, \quad f \in L^2(G),$$

where C_1 is independent of f.

On the other hand

$$T_{is} f(x) = \int_G \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}(\xi(y^{-1}x) e^{-s^2} \sigma(x, \xi) \langle \xi \rangle^{\nu - \frac{na}{2} + i \frac{na}{2} s}) f(y) \, dy.$$

Since

$$\gamma_{is}(x,\xi) = e^{-s^2}\sigma(x,\xi)\langle\xi\rangle^{\nu}\langle\xi\rangle^{-\frac{na}{2}}\langle\xi\rangle^{i\frac{na}{2}s},$$

and

$$\langle\xi\rangle^{i\frac{na}{2}s}I_{d_\xi} \in \mathscr{S}^0_{1-a,\delta}$$

(see [28]), we have

$$\gamma_{is} \in \mathscr{S}^{-\frac{na}{2}}_{1-a,\delta}.$$

Moreover,

$$\|\gamma_{is}\|_{k,\mathscr{S}^{-\frac{na}{2}}_{1-a,\delta}} \leqslant C\|\sigma\|_{k,\mathscr{S}^{-\nu}_{1-a,\delta}},$$

with $C > 0$ independent of s.

An application of Theorem 4.12 to the operator T_{is} gives

$$\|T_{is}f\|_{BMO(G)} \leqslant C\|\sigma\|_{k,\mathscr{S}^{-\nu}_{1-a,\delta}}\|f\|_{L^\infty(G)}.$$

The complex interpolation for an analytic family of operators gives us

$$\|T_t f\|_{L^p(G)} \leqslant C_p\|\sigma\|_{k,\mathscr{S}^{-\nu}_{1-a,\delta}}\|f\|_{L^p(G)},$$

where $p = \frac{2}{t}$, $0 < t \leqslant 1$. The corresponding symbol of the operator T_t is given by

$$\gamma_t(x,\xi) = e^{-t^2}\sigma(x,\xi)\langle\xi\rangle^{\nu+\frac{na}{2}(t-1)}.$$

Since $0 \leqslant \nu \leqslant \frac{na}{2}$, there exists t, $0 \leqslant t \leqslant 1$ such that

$$\nu = \frac{na}{2}(1-t).$$

Hence the operator $A = \sigma(x,D)$ is bounded from $L^p(G)$ into $L^p(G)$ for $p = \frac{2}{t}$ and

$$\|\sigma(x,D)f\|_{L^p(G)} \leqslant C_p\|\sigma\|_{k,\mathscr{S}^{-\nu}_{1-a,\delta}}\|f\|_{L^p(G)}.$$

We note that

$$\nu = \frac{na}{2}\left(1-\frac{2}{p}\right) = na\left(\frac{1}{2}-\frac{1}{p}\right).$$

By interpolation between $p = 2$ and $p = \frac{2}{t}$ we obtain the $L^p(G)$ boundedness for p verifying

$$\frac{1}{2} - \frac{1}{p} \leqslant \frac{\nu}{na}.$$

We can now apply a duality argument for the case $1 \leqslant p \leqslant 2$. Since the symbol of the operator $A^* = \sigma(x,D)^*$ also belongs to $\mathscr{S}^{-\nu}_{1-a,\delta}$ we have $A^* : L^p(G) \to L^p(G)$, for $\frac{1}{2} - \frac{1}{p} \leqslant \frac{\nu}{na}$. Then

$$A : L^p(G) \to L^p(G)$$

for $\frac{1}{p} - \frac{1}{2} \leqslant \frac{\nu}{na}$. Therefore, $A : L^p(G) \to L^p(G)$ is bounded for

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leqslant \frac{\nu}{na}. \qquad \square$$

Remark 4.16. The index $\frac{n(1-\rho)}{2}$ in Theorem 4.15 is sharp. Indeed, for $G = \mathbb{T}^1$, if $\nu_0 < \frac{1-\rho}{2}$ one only gets boundedness on finite intervals around $p = 2$. This is a consequence of the well-known classical multiplier theory on the torus (cf. [16]) and Wainger (cf. [35]).

Indeed, let $G = \mathbb{T}^1$ and $0 < \rho < 1$, $0 < v_0 < \frac{1-\rho}{2}$ and consider

$$\sigma(\xi) = \frac{e^{i\langle\xi\rangle^a}}{\langle\xi\rangle^{v_0}},$$

for $\xi \in \mathbb{Z}$. Then $\sigma \in \mathscr{S}_{\rho,0}^{-v_0}(\mathbb{T}^1)$ and the corresponding operator $\sigma(D)$ is bounded on $L^p(\mathbb{T}^1)$ for the interval

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{v_0}{a}$$

and is not bounded for p outside that interval. The L^p boundedness inside the interval with centre at $p = 2$ can also be obtained from the general result in Theorem 4.15.

References

1. R. Akylzhanov and M. Ruzhansky, Net spaces on lattices, Hardy–Littlewood type inequalities, and their converses, preprint, 2015, arXiv:1510.01251v1.
2. R. Beals, L^p and Holder estimates for pseudodifferential operators: necessary conditions, in *Harmonic Analysis in Euclidean Spaces (Proc. Symp. Pure Math., Williams Coll., Williamstown, MA, 1978)*, pp. 153–157 (American Mathematical Society, Providence, RI, 1979).
3. R. Beals, L^p and Holder estimates for pseudodifferential operators: sufficient conditions, *Ann. Inst. Fourier.* **29**(3) (1979), 239–260.
4. A. Dasgupta and M. Ruzhansky, Gevrey functions and ultradistributions on compact Lie groups and homogeneous spaces, *Bull. Sci. Math.* **138** (2014), 756–782.
5. A. Dasgupta and M. Ruzhansky, The Gohberg lemma, compactness, and essential spectrum of operators on compact Lie groups, *J. Anal. Math.* **128** (2016), 179–190.
6. J. Delgado, Estimations L^p pour une classe d'opérateurs pseudo-différentiels dans le cadre du calcul de Weyl–Hörmander, *J. Anal. Math.* **100** (2006), 337–374.
7. C. Fefferman, L^p bounds for pseudo-differential operators, *Israel J. Math.* **14** (1973), 413–417.
8. C. Fefferman and E. M. Stein., H^p-spaces of several variables, *Acta Math.* **129** (1972), 137–193.
9. V. Fischer, Intrinsic pseudo-differential calculi on any compact Lie group, *J. Funct. Anal.* **268** (2015), 3404–3477.
10. V. Fischer, Hörmander condition for Fourier multipliers on compact Lie groups, preprint, 2016, arXiv:1610.06348.
11. V. Fischer and M. Ruzhansky, Fourier multipliers on graded Lie groups, preprint, 2014, arXiv:1411.6950.
12. V. Fischer and M. Ruzhansky, *Quantization on nilpotent Lie groups*, Progress in Mathematics, Volume 314 (Birkhäuser, Basel, 2016).
13. C. Garetto and M. Ruzhansky, Wave equation for sums of squares on compact Lie groups, *J. Differential Equations* **258**(12) (2015), 4324–4347.
14. A. Grigor'yan, Estimates of heat kernels on Riemannian manifolds, in *Spectral theory and geometry. ICMS Instructional Conference, Edinburgh, 1998*,(ed. B. Davies and Yu. Safarov), London Mathematical Society Lecture Note Series, Volume 273, pp. 140–225 (Cambridge University Press, Cambridge, 1999).

15. A. GRIGOR'YAN, J. HU AND K.-S. LAU, Heat kernels on metric measure spaces, in *Geometry and Analysis on Fractals*, Springer Proceedings in Mathematics and Statistics, Volume 88, pp. 147–207 (Springer, Heidelberg, 2014).

16. I. I. HIRSCHMAN, Multiplier transformations I, *Duke Math. J.* **26** (1956), 222–242.

17. L. HÖRMANDER, Pseudo-differential operators and hypoelliptic equations, in *Proc. Symposium on Singular Integrals*, pp. 138–183 (American Mathematical Society, vol. 10, Providence, RI, 1967).

18. J. HOUNIE, On the L^2 continuity of pseudo-differential operators, *Communications in Partial Differential Equations* **11**(7) (1986), 765–778.

19. CH. LI AND R. WANG, On the L^p-boundedness of several classes of pseudo-differential operators, *Chin. Ann. Math. Ser. B* **5**(2) (1984), 193–213.

20. S. MOLAHAJLOO AND M. W. WONG, Pseudodifferential operators on $\mathbb{S}^1$, in *New Developments in Pseudo-differential operators*, Operator Theory: Advances and Applications, Volume 189, pp. 297–306 (Birkhäuser, Basel, 2009).

21. E. NURSULTANOV, M. RUZHANSKY AND S. TIKHONOV, Nikolskii inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **XVI**(5) (2016), 981–1017.

22. L. ROTHSCHILD AND E. M. STEIN, Hypoelliptic differential operators and nilpotent groups, *Acta Math.* **137**(3–4) (1976), 247–320.

23. M. RUZHANSKY AND V. TURUNEN, *Pseudo-differential Operators and Symmetries. Background Analysis and Advanced Topics*, Pseudo-Differential Operators. Theory and Applications, Volume 2 (Birkhäuser Verlag, Basel, 2010).

24. M. RUZHANSKY AND V. TURUNEN, Sharp Gårding inequality on compact Lie groups, *J. Funct. Anal.* **260**(10) (2011), 2881–2901.

25. M. RUZHANSKY AND V. TURUNEN, Global quantization of pseudo-differential operators on compact Lie groups, SU(2), 3-sphere, and homogeneous spaces, *Int. Math. Res. Not. IMRN* **2013**(11) (2013), 2439–2496.

26. M. RUZHANSKY AND J. WIRTH, On multipliers on compact Lie groups, *Funct. Anal. Appl.* **47**(1) (2013), 87–91.

27. M. RUZHANSKY AND J. WIRTH, Global functional calculus for operators on compact Lie groups, *J. Funct. Anal.* **267** (2014), 144–172.

28. M. RUZHANSKY AND J. WIRTH, L^p Fourier multipliers on compact Lie groups, *Math. Z.* **280** (2015), 621–642.

29. M. RUZHANSKY, V. TURUNEN AND J. WIRTH, Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity, *J. Fourier Anal. Appl.* **20** (2014), 476–499.

30. M. A. SHUBIN, *Pseudodifferential Operators and Spectral Theory*, second edition (Springer, Berlin, 2001). Translated from the 1978 Russian original by Stig I. Andersson.

31. E. M. STEIN, Interpolation of linear operators, *Trans. Amer. Math. Soc.* **83** (1956), 482–492.

32. E. M. STEIN AND G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series, No. 32 (Princeton University Press, Princeton, NJ, 1971).

33. R. STRICHARTZ, Analysis of the laplacian on the complete Riemannian manifold, *J. Funct. Anal.* **52** (1983), 48–79.

34. M. E. TAYLOR, *Partial Differential Equations III. NonLinear Equations* (Springer, New York, 1996).

35. S. WAINGER, Special trigonometric series in k-dimensions, *Mem. Amer. Math. Soc. No.* **59** (1965), 102.

36. A. ZYGMUND, *Trigonometric Series I and II*, second edition (Cambridge University Press, Cambridge, New York, Melbourne, 1977).