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Extension Operators for Biholomorphic Mappings

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Abstract. Suppose that $D \subset \mathbb{C}$ is a simply connected subdomain containing the origin and $f(z_1)$ is a normalized convex (resp., starlike) function on D. Let

$$\Omega_N(D) = \Big\{ (z_1, w_1, \dots, w_k) \in \mathbb{C} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k} : \|w_1\|_{p_1}^{p_1} + \cdots + \|w_k\|_{p_k}^{p_k} < \frac{1}{\lambda_D(z_1)} \Big\},$$

where $p_j \geq 1$, $N = 1 + n_1 + \cdots + n_k$, $w_1 \in \mathbb{C}^{n_1}, \dots, w_k \in \mathbb{C}^{n_k}$ and λ_D is the density of the hyperbolic metric on D. In this paper, we prove that

$$\Phi_{N,1/p_1,\dots,1/p_k}(f)(z_1, w_1, \dots, w_k) = \big(f(z_1), (f'(z_1))^{1/p_1} w_1, \dots, (f'(z_1))^{1/p_k} w_k \big)$$

is a normalized convex (resp., starlike) mapping on $\Omega_N(D)$. If D is the unit disk, then our result reduces to Gong and Liu via a new method. Moreover, we give a new operator for convex mapping construction on an unbounded domain in $\mathbb{C}^2$. Using a geometric approach, we prove that $\Phi_{N,1/p_1,\dots,1/p_k}(f)$ is a spiral-like mapping of type α when f is a spiral-like function of type α on the unit disk.

1 Introduction

Let B_n be the unit ball of $\mathbb{C}^n$. In the case of complex plane $\mathbb{C}$, B_1 is always written by U. A biholomorphic mapping $f: B_n \to \mathbb{C}^n$ is said to be *normalized* if $f(0) = 0$ and $J_f(0) = I_n$, where I_n is the identity matrix and J_f is the Jacobian matrix of f. A normalized biholomorphic mapping $f: B_n \to \mathbb{C}^n$ is said to be convex (resp. starlike) if $f(B_n)$ is convex (resp. starlike with resect to the origin); see [17]. Let $\mathcal{K}(B_n)$ and $\mathcal{S}^*(B_n)$ denote the class of normalized convex and starlike mappings on B_n, respectively. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $p \geq 1$, we denote $\|z\|_p = (\sum_{k=1}^n |z_k|^p)^{1/p}$ by the p-norm in $\mathbb{C}^n$.

In a very influential paper, Roper and Suffridge [14] introduced an extension operator. This operator is defined for a normalized locally biholomorphic function $f(z_1)$ on U by

$$F(z) = \Phi_{1/2}(f)(z) = \big(f(z_1), \sqrt{f'(z_1)} z_0 \big),$$

where $(z_1, z_0) \in B_n$ and the branch of the square root is chosen such that $\sqrt{f'(0)} = 1$.

It is well known that the Roper–Suffridge operator has the following two remarkable properties:

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- Property 1: if $f \in \mathcal{K}(U)$, then $F \in \mathcal{K}(B_n)$;
- Property 2: if $f \in \mathcal{S}^*(U)$, then $F \in \mathcal{S}^*(B_n)$.

Roper and Suffridge proved Property 1. Graham and Kohr [10] provided a simplified proof of Property 1 and proved Property 2. More properties have been explored by various authors; see *e.g.*, [2–5, 11–13, 15]. Using the Roper–Suffridge extension operator, a lot of convex mappings and starlike mappings on B_n can be easily constructed, which explains its popularity.

Generally, Graham and Kohr [10] proposed the following problem:

Consider the "egg" domain

$$\Omega_{2,p} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^p < 1\},$$

where $p \geq 1$. Does the operator

$$\Phi_{1/p}(f)(z) = \big(f(z_1), [f'(z_1)]^{1/p} z_2\big)$$

extend convex functions on U to convex mappings on $\Omega_{2,p}$?

In [6], Gong and Liu introduced the ϵ-starlike mappings on a domain in $\mathbb{C}^n$ as follows.

Definition 1.1 Suppose Ω is a domain in $\mathbb{C}^n$, and suppose $f\colon \Omega \to \mathbb{C}^n$ is a locally biholomorphic mapping and $0 \in f(\Omega)$. Given a positive number ϵ, $0 \leq \epsilon \leq 1$, f is said to be an *ϵ-starlike* mapping on Ω if $f(\Omega)$ is starlike with respect to every point in $\epsilon f(\Omega)$.

Note that, when $\epsilon = 0$ and $\epsilon = 1$, ϵ-starlike reduces to starlike and convex, respectively.

In [7], Gong and Liu proved the following result.

Theorem 1.2 *If $f(z_1)$ is a normalized biholomorphic ε-starlike function on the unit disk U, then*

$$\Phi_{N,1/p_1,\dots,1/p_k}(f)(z_1, w_1, \dots, w_k) = \big(f(z_1), (f'(z_1))^{1/p_1} w_1, \dots, (f'(z_1))^{1/p_k} w_k\big)$$

is a normalized biholomorphic ε-starlike mapping on the domain

$$\Omega_N = \Big\{(z_1, w_1, \dots, w_k) \in \mathbb{C}\times\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_k} : |z_1|^2 + \|w_1\|_{p_1}^{p_1} + \cdots + \|w_k\|_{p_k}^{p_k} < 1\Big\}, \quad p_j \geq 1,$$

where $N = 1 + n_1 + \cdots + n_k$, $w_1 \in \mathbb{C}^{n_1}, \dots, w_k \in \mathbb{C}^{n_k}$. The branch is chosen so that $(f'(z_1))^{1/p_j}|_{z_1=0} = 1$, $j = 1, \dots, k$.

When $n = 2$ and $\epsilon = 1$, Theorem 1.1 answered the above problem. A completely new solution to the problem was recently given by Wang and Liu [16].

The well-known Riemann mapping theorem states that any non-empty, open, simply connected, proper subset of $\mathbb{C}$ is conformally equivalent to the unit disk U. Naturally, we would like to ask:

> *How can one generalize Theorem 1.2 from the unit disk U to any given simply connected proper subdomain $D \subset \mathbb{C}$?*

The answer to the above question is the following result.

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Theorem 1.3 *Let $D \subsetneqq \mathbb{C}$ be a simply connected domain containing the origin, and let $\|\cdot\|_{p_j}$ be the Banach norms of $\mathbb{C}^{n_j}$, $j = 1, 2, \ldots, k$, where n_j are positive integers and $p_j \geq 1$. Suppose*

$$\Omega_N(D) = \Big\{ (z_1, w_1, \ldots, w_k) \in \mathbb{C} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k} : \|w_1\|_{p_1}^{p_1} + \cdots + \|w_k\|_{p_k}^{p_k} < \frac{1}{\lambda_D(z_1)} \Big\},$$

where λ_D is the hyperbolic metric on D, $N = 1 + n_1 + \cdots + n_k$, $w_1 \in \mathbb{C}^{n_1}, \ldots, w_k \in \mathbb{C}^{n_k}$.
If $f(z_1)$ is a normalized biholomorphic ε-starlike function on D, then

$$\Phi_{N,1/p_1,\ldots,1/p_k}(f)(z_1, w_1, \ldots, w_k) = \big(f(z_1), (f'(z_1))^{1/p_1} w_1, \ldots, (f'(z_1))^{1/p_k} w_k \big)$$

is a normalized biholomorphic ε-starlike mapping on the domain $\Omega_N(D)$, where we choose the branch so that $(f'(z_1))^{1/p_j}|_{z_1=0} = 1$, $j = 1, \ldots, k$.

Remark When D is the unit disk U, Theorem 1.3 reduces to Theorem 1.2. The Roper–Suffridge extension operator that we mentioned above starts from an ϵ-starlike function f of one complex variable on a simply connected domain $D \subset \mathbb{C}$; via the Roper–Suffridge extension operator, we can get an ϵ-starlike mapping $\Phi_{N,1/p_1,\ldots,1/p_k}(f)$ on the domain in $\Omega_N(D) \subset \mathbb{C}^N$. It is well known that the convex mapping is more delicate. Naturally, we ask the following question:

> Can we have a new convex construction in several complex variables other than use of the Roper–Suffridge extension operator?

Interestingly, we have the following new operator construction for convex mappings.

Theorem 1.4 *Assume $D \subsetneqq \mathbb{C}$ is a simply connected domain containing the origin. Let f be a biholomorphic convex function on D with $f'(0) = 1$, and let*

$$G_2 = \Big\{ (z_1, z_2) \in \mathbb{C}^2 : |e^{z_2}| < \frac{1}{\lambda_D(z_1)} \Big\}.$$

Suppose F is defined by

$$F(z_1, z_2) = \big(f(z_1), \log f'(z_1) + z_2 \big),$$

where $(z_1, z_2) \in G_2$ and the branch is chosen so that $\log 1 = 0$. Then $F(z_1, z_2)$ is a biholomorphic convex mapping on the domain G_2.

By using a geometric approach, as for spiral-like mappings of type α associated with the Roper–Suffridge extension operator $\Phi_{1/p}$, we can prove the following theorem.

Theorem 1.5 *Let*

$$\Omega_{n,p} = \Big\{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n} |z_j|^p < 1 \Big\}, \quad p \geq 1,$$

$$\Phi_{1/p}(f)(z) = \big(f(z_1), [f'(z_1)]^{1/p} z_0 \big), \quad z = (z_1, z_0) \in \Omega_{n,p}.$$

If f is a spiral-like function of type $\alpha(-\pi/2 < \alpha < -\pi/2)$ on the unit disk U, then $\Phi_1/p(f)$ is a spiral-like function of type α on $\Omega_{n,p}$. In particular, if $f \in \mathcal{S}^*(U)$, then $\Phi_{1/p}(f) \in \mathcal{S}^*(\Omega_{n,p})$.

2 Notation and Two Lemmas

2.1 Notation

Let us give the following notation:

- Let $G \subset \mathbb{C}^n$ be a bounded convex circular domain that contains the origin. A normalized biholomorphic mapping $f: G \to \mathbb{C}^n$ is said to be *spiral-like* of type $\alpha(-\pi/2 < \alpha < \pi/2)$ if $e^{-te^{-i\alpha}} f(G) \subset f(G)$ holds for all $t > 0$; see [9]. Of course, a domain $D \subset \mathbb{C}^n$ is said to be *spiral-like* of type α if $e^{-te^{-i\alpha}} z \in D$ holds for all $t > 0$ and $z \in D$.
- Let $D \subset \mathbb{C}$ be a domain containing the origin, and let f and g be two holomorphic functions on D. If there is a holomorphic function $\varphi: D \to D$ such that $\varphi(0) = 0$ and $f = g \circ \varphi$, then f is subordinate to g and is denoted by $f \prec g$ on D.
- Let D be a simply connected proper subdomain of $\mathbb{C}$, and let f be a conformal (biholomorphic) mapping of the unit disk U onto D. The hyperbolic metric of D is defined by

$$\lambda_D(f(z))|dz| = \frac{|dz|}{(1-|z|^2)|f'(z)|},\ z \in U.$$

It is not difficult to show that this value of $\lambda_D(f(z))$ is independent of the choice of conformal mapping f. Hence, convenient choice is available for us in this paper. For any fixed $z \in D$, if we choose the conformal mapping f satisfying $f(0) = z$ and $f'(0) > 0$, then

$$\lambda_D(z) = \frac{1}{f'(0)}. \tag{2.1}$$

The function $\lambda_D(z)$ is real analytic in D and the metric $\lambda_D(z)|dz|$ has constant (Gaussian) curvature -4. Recall that

$$\kappa(z) = -\Delta \log \lambda_D(z)/\lambda_G^2(z)$$

is the curvature of $\lambda_D(z)$.

It is not difficult to check the following elementary property on hyperbolic metric, for instance; see [1].

Conformal Invariance If f is a conformal mapping from the domain D onto Ω, then $\lambda_\Omega(f(z))|f'(z)| = \lambda_D(z)$, $\forall z \in D$.

2.2 Two Lemmas

The following lemma gives an interesting characterization for hyperbolic metric on ε-starlike domain, which plays an important role to prove our main theorem; see Wang and Liu [16].

Lemma 2.1 *Let $D \subset \mathbb{C}$ contain the origin. If D is an ε-starlike domain and $D \neq \mathbb{C}$, then given $z_1, z_2 \in D$,*

$$\frac{1}{\lambda_D((1-t)z_1 + \varepsilon t z_2)} \geq \frac{1-t}{\lambda_D(z_1)} + \frac{\varepsilon t}{\lambda_D(z_2)}.$$

In order to prove Theorem 1.5, we need the following lemma.

Lemma 2.2 *Let $-\pi/2 < \alpha < \pi/2$ and $D \subset \mathbb{C}$ be a spiral-like of type α domain. Then*

$$\frac{e^{-t\cos\alpha}}{\lambda_D(z)} \leq \frac{1}{\lambda_D(e^{-te^{-i\alpha}}z)}$$

holds for all $z \in D$ and $t > 0$.

Proof For any fixed $z \in D$, the Riemann mapping theorem shows that there is a conformal mapping $g: U \to D$ so that $g(U) = D$, $g(0) = z$. Because D is a spiral-like of type α domain, it means that $e^{-te^{-i\alpha}}z \in D$. Also, let h be a conformal mapping of U onto D so that $h(0) = e^{-te^{-i\alpha}}z$.

Noting that $e^{-te^{-i\alpha}}g: U \to D$, we obtain that $e^{-te^{-i\alpha}}g$ is subordinate to h. Thereby,

$$|e^{-te^{-i\alpha}}g'(0)| \leq |h'(0)|.$$

In terms of (2.1), we get

$$\lambda_D(g(0)) = \frac{\lambda_U(0)}{|g'(0)|} \quad \text{and} \quad \lambda_D(h(0)) = \frac{\lambda_U(0)}{|h'(0)|}.$$

Hence,

$$\frac{e^{-t\cos\alpha}}{\lambda_D(z)} \leq \frac{1}{\lambda_D(e^{-te^{-i\alpha}}z)}.$$

Consequently, we complete the proof of Lemma 2.2 ∎

3 ε-starlike

In this section, we prove that the Roper–Suffridge extension operator preserves ε-starlike property.

Proof of Theorem 1.3 It is easy to see that $\Phi_{N,1/p_1,\dots,1/p_k}(f)$ is a normalized biholomorphic mapping on $\Omega_N(D)$. Hence, we need to verify that the image of $\Phi_{N,1/p_1,\dots,1/p_k}(f)$ is an ε-starlike domain of $\mathbb{C}^N$. Write

$$(u_1, v_1, \dots, v_k) = \left(f(z_1), (f'(z_1))^{1/p_1}w_1, \dots, (f'(z_k))^{1/p_k}w_k\right).$$

Then

$$\begin{aligned} u_1 &= f(z_1),\\ v_1 &= (f'(z_1))^{1/p_1}w_1,\\ &\vdots\\ v_k &= (f'(z_1))^{1/p_k}w_k. \end{aligned}$$

That is,

$$\begin{aligned} u_1 &= f(z_1),\\ \|v_1\|_{p_1}^{p_1} &= |f'(z_1)|\|w_1\|_{p_1}^{p_1},\\ &\vdots\\ \|v_k\|_{p_k}^{p_k} &= |f'(z_1)|\|w_k\|_{p_k}^{p_k}. \end{aligned} \tag{3.1}$$

Let $\Omega_1 = f(D)$. Then

$$\lambda_{\Omega_1}(f(z_1))|f'(z_1)| = \lambda_D(z_1) \tag{3.2}$$

follows from Conformal Invariance on hyperbolic metric.

From the definition of $\Omega_N(D)$, we get

$$\|w_1\|_{p_1}^{p_1} + \cdots + \|w_k\|_{p_k}^{p_k} < \frac{1}{\lambda_D(z_1)}. \tag{3.3}$$

Putting (3.1), (3.2), and (3.3) together, it yields that the image of $\Phi_{N,1/p_1,\ldots,1/p_k}(f)$ obeys

$$\widetilde{\Omega}_N = \Big\{(u_1, v_1, \ldots, v_k) \in \mathbb{C} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k} : \|v_1\|_{p_1}^{p_1} + \cdots + \|v_k\|_{p_k}^{p_k} - \frac{1}{\lambda_{\Omega_1}(u_1)} < 0\Big\}.$$

It remains to prove that $\widetilde{\Omega}_N$ is an ε-starlike domain in $\mathbb{C}^N$. In fact, for all $t \in [0,1]$, $(u_1, v_1, \ldots, v_k) \in \widetilde{\Omega}_N$ and $(a_1, b_1, \ldots, b_k) \in \widetilde{\Omega}_N$. Since x^{p_j} is real convex function on $x \in [0, \infty)$ when $p_j \geq 1$, we have that

$$\begin{aligned}
&\|(1-t)v_1 + \varepsilon t b_1\|_{p_1}^{p_1} + \cdots + \|(1-t)v_k + \varepsilon t b_k\|_{p_k}^{p_k} \\
&\quad\leq \big((1-t)\|v_1\|_{p_1}^{p_1} + t\|\varepsilon b_1\|_{p_1}^{p_1}\big) + \cdots + \big((1-t)\|v_k\|_{p_k}^{p_k} + t\|\varepsilon b_k\|_{p_k}^{p_k}\big) \\
&\quad\leq (1-t)\|v_1\|_{p_1}^{p_1} + \varepsilon^{p_1} t\|b_1\|_{p_1}^{p_1} + \cdots + (1-t)\|v_k\|_{p_k}^{p_k} + t\varepsilon^{p_k}\|b_k\|_{p_k}^{p_k} \\
&\quad\leq (1-t)\|v_1\|_{p_1}^{p_1} + \varepsilon t\|b_1\|_{p_1}^{p_1} + \cdots + (1-t)\|v_k\|_{p_k}^{p_k} + \varepsilon t\|b_k\|_{p_k}^{p_k} \\
&\quad= (1-t)\big(\|v_1\|_{p_1}^{p_1} + \cdots + \|v_k\|_{k}^{p_k}\big) + \cdots + \varepsilon t\big(\|b_1\|_{p_1}^{p_1} + \cdots + \|b_k\|_{p_k}^{p_k}\big).
\end{aligned}$$

By using Lemma 2.1, we have

$$-\frac{1}{\lambda_{\Omega_1}((1-t)u_1 + \varepsilon t a_1)} \leq -\frac{1-t}{\lambda_{\Omega_1}(u_1)} - \frac{\varepsilon t}{\lambda_{\Omega_1}(a_1)}.$$

Therefore,

$$\begin{aligned}
&\|(1-t)v_1 + \varepsilon t b_1\|_{p_1}^{p_1} + \cdots + \|(1-t)v_k + \varepsilon t b_k\|_{p_k}^{p_k} - \frac{1}{\lambda_{\Omega_1}((1-t)u_1 + \varepsilon t a_1)} \\
&\quad\leq (1-t)\Big(\|v_1\|_{p_1}^{p_1} + \cdots + \|v_k\|_{p_k}^{p_k} - \frac{1}{\lambda_{\Omega_1}(u_1)}\Big) \\
&\qquad + \varepsilon t\Big(\|b_1\|_{p_1}^{p_1} + \cdots + \|b_k\|_{p_k}^{p_k} - \frac{1}{\lambda_{\Omega_1}(a_1)}\Big) \\
&\quad< 0.
\end{aligned}$$

Consequently,

$$(1-t)(u_1, v_1, \ldots, v_k) + \varepsilon t(a_1, b_1, \ldots, b_k) \in \widetilde{\Omega}_N,$$

which implies $\widetilde{\Omega}_N$ is an ε-starlike domain.

Thus, $\Phi_{N,1/p_1,\ldots,1/p_k}(f)$ is a biholomorphic ε-starlike mapping on $\Omega_N(D)$. ∎

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4 New Operator for Convex Mappings

Inspired by Theorem 1.3, we establish the new convex mappings of several complex variables by using convex functions of one complex variable, which is Theorem 1.4. Interestingly, it seems that there are no convex mappings construction on the non-Reinhardt domain.

Proof of Theorem 1.4 Let

$$u_1 = f(z_1),$$
$$u_2 = \log f'(z_1) + z_2.$$

Then

$$f(z_1) = u_1,$$
$$|f'(z_1)| = |e^{u_2 - z_2}|.$$

Set $\Omega = f(D)$. By Conformal Invariance of hyperbolic metric, we have

$$\lambda_\Omega\big(f(z_1)\big)|f'(z_1)| = \lambda_D(z_1).$$

Then

$$\lambda_D(u_1)|e^{u_2-z_2}| = \lambda_D(z_1).$$

Since

$$G_2 = \Big\{(z_1, z_2) \in \mathbb{C}^2 : |e^{z_2}| < \frac{1}{\lambda_D(z_1)}\Big\},$$

we get the image domain of F is

$$F(G_2) = \Big\{(u_1, u_2) \in \mathbb{C}^2 : |e^{u_2}| - \frac{1}{\lambda_D(u_1)} < 0\Big\}.$$

Note that $|e^{u_2}| = e^{\Re u_2}$ is a convex function. By using Lemma 2.1 for $\epsilon = 1$, we have that $-1/\lambda_D(u_1)$ is also convex. Hence, $F(G_2)$ is a convex domain, which shows that F is a convex mapping on the domain G_2. ■

5 Spiral-like

By using Lemma 2.2, we can prove the Roper–Suffridge operator $\Phi_{1/p}$ preserves spiral-like function of type α as follows.

Proof of Theorem 1.5 Without loss of generalization, we need only to prove the case of dimension $n = 2$, because the general case can be similarly obtained.

Let

$$(u_1, u_2) = \big(f(z_1), [f'(z_1)]^{1/p} z_2\big).$$

Then

$$u_1 = f(z_1),$$
$$u_2 = [f'(z_1)]^{1/p} z_2.$$

This yields that

$$z_1 = f^{-1}(u_1),$$
$$z_2 = \frac{u_2}{[f'[f^{-1}(u_1)]]^{1/p}}.$$

Let $G = f(U)$. In terms of $|z_1|^2 + |z_2|^p < 1$, we obtain the range of mapping $\Phi_\beta(f)$ is

$$\Phi_\beta(f)(\Omega_{2,p}) = \Big\{ (u_1, u_2) \in G \times \mathbb{C} : \frac{|u_2|^p}{|f'[f^{-1}(u_1)]|} < 1 - |f^{-1}(u_1)|^2 \Big\}.$$

The equation (2.1) implies that

$$\lambda_G(u_1) = \frac{1}{(1-|z_1|^2)|f'(z_1)|} = \frac{1}{(1-|f^{-1}(u_1)|^2)|f'[f^{-1}(u_1)]|}.$$

Hence, the range of mapping $\Phi_{1/p}(f)$ satisfies

$$\Phi_{\frac{1}{p}}(f)(\Omega_{2,p}) = \Big\{ (u_1, u_2) \in G \times \mathbb{C} : |u_2|^p - \frac{1}{\lambda_G(u_1)} < 0 \Big\},$$

where $G = f(U)$. Note that $\Phi_{1/p}(f)$ is spiral-like of type α on $\Omega_{2,p}$ if and only if $\Phi_{1/p}(f)(\Omega_{2,p})$ is a spiral-like of type α domain in $\mathbb{C}^2$. Hence, we need to prove

$$e^{-te^{-i\alpha}} \Phi_{1/p}(f)(\Omega_{2,p}) \subset \Phi_{1/p}(f)(\Omega_{2,p}).$$

In fact, for any $(u_1, u_2) \in \Phi_{1/p}(f)(\Omega_{2,p})$, we have

$$|u_2|^p - \frac{1}{\lambda_G(u_1)} < 0.$$

In terms of Lemma 2.2, we get

$$\begin{aligned} |e^{-te^{-i\alpha}} u_2|^p - \frac{1}{\lambda_G(e^{-te^{-i\alpha}} u_1)} &\le e^{-pt\cos\alpha} |u_2|^p - \frac{e^{-t\cos\alpha}}{\lambda_G(u_1)} \\ &\le e^{-t\cos\alpha}\Big(|u_2|^p - \frac{1}{\lambda_G(u_1)} \Big) \\ &\le 0. \end{aligned}$$

So $e^{-te^{-i\alpha}}(u_1, u_2) \in \Phi_{1/p}(f)(\Omega_{2,p})$. Namely, $\Phi_{1/p}(f)$ is spiral-like of type α. ∎

Remark If $\alpha = 0$ and $p = 2$, then $f \in \mathcal{S}^*(U)$. Theorem 1.5 shows that $\Phi_{1/2}(f) \in \mathcal{S}^*(B_n)$, which reduces to that of Graham and Kohr [10, Theorem 2.2]. Moreover, our proof is different.

6 Some Comments

In general, another well-known extension operator (see [8,18]) is defined by

$$\Psi_{\beta,\gamma}(f)(z) = \Big(f(z_1), [f'(z_1)]^\beta \Big[\frac{f(z_1)}{z_1}\Big]^\gamma z_0 \Big), \quad z = (z_1, z_0) \in B_n,$$

where $0 \le \beta \le 1/2$, $0 \le \gamma \le 1$, and $\beta + \gamma \le 1$. In [8, Corollary 2.2], they proved that if $f \in \mathcal{S}^*(U)$, then $\Psi_{\beta,\gamma}(f) \in \mathcal{S}^*(B_n)$. According to the idea of proving Theorem 1.3, it is not difficult to show that if $f \in \mathcal{S}^*(U)$, then $\Psi_{\beta,\gamma}(f) \in \mathcal{S}^*(\Omega_{n,p})$, where $0 \le \beta \le 1/p$, $0 \le \gamma \le 1$, and $\beta + \gamma \le 1$. Interestingly, it seems entirely new for the Roper–Suffridge operator to establish this connection between the unit ball B_n and Reinhardt domain $\Omega_{n,p}$.

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