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ON THE COMPACTIFICATION OF PRODUCTS

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Let $\{X_a, a \in A\}$ be a family of completely regular Hausdorff spaces, $\{\beta X_a\}$ the corresponding family of their Stone–Čech compactifications and $\Pi_a X_a$ the usual topological product. The following theorem was proved by Glicksberg [2] and subsequently by Frolík [1].

THEOREM. *If $\{X_a, a \in A\}$ is a family of infinite completely regular Hausdorff spaces then $\beta(\Pi_a X_a) = \Pi_a \beta X_a$ if and only if $\Pi_a X_a$ is pseudocompact.*

A topological space X is pseudocompact if each function in $C(X)$, the space of real valued continuous functions on X, is bounded. The difficult part of the proof is the sufficiency. Frolík's proof considerably simplifies that of Glicksberg. It is the purpose of this note to show that a very easy proof of the sufficiency may be obtained from a corollary to a theorem of Frolík.

All topological spaces under consideration will be assumed to be completely regular and Hausdorff, and where a topology is mentioned on the set $C(X)$ of real valued continuous functions on the pseudocompact space X it is invariably the metrizable topology given by the sup norm. The only facts about pseudocompact spaces that will be needed are: (a) if $\Pi_a X_a$ is pseudocompact then each coordinate space X_a is also, (b) a pseudocompact metric space is compact and (c) the continuous image of a pseudocompact space is pseudocompact.

LEMMA (Frolík). *Let $X \times Y$ be pseudocompact and let $f \in C(X \times Y)$. For each x in X define $F(x) = \mathrm{Sup}_{y \in Y} f(x, y)$. Then F is a continuous function on X.*

COROLLARY. *Let f be a continuous function on the pseudocompact space $X \times Y$ and let x_0 be a fixed point of X. Then the function $G(x) = \mathrm{Sup}_{y \in Y} |f(x_0, y) - f(x, y)|$ is a continuous function of x.*

Proof. $|f(x_0, y) - f(x, y)|$ is continuous on $X \times Y$.

THEOREM. *Let $X \times Y$ be pseudocompact. Then $\beta(X \times Y) = \beta X \times \beta Y$.*

Proof. Let $f \in C(X \times Y)$. For x fixed in X define $f_1(x)(y) = f(x, y)$. It is routine to show that f_1 is a bounded function on X into $C(Y)$. Further since

$$\begin{aligned} \|f_1(x_0) - f_1(x)\| &= \mathrm{Sup}_{y \in Y} |f_1(x_0)(y) - f_1(x)(y)| \\ &= \mathrm{Sup}_{y \in Y} |f(x_0, y) - f(x, y)| \end{aligned}$$

it follows from the corollary to the proceeding lemma that f_1 is continuous at each point x_0 of X. Now for each $x, f_1(x)$ is in $C(Y)$ so that we may extend $f_1(x)$ continuously to βY and consider f_1 to be a continuous mapping of X into $C(\beta Y)$. Since X is pseudocompact, $f_1(X)$ is a pseudocompact subset of the metric space $C(\beta Y)$ and therefore $f_1(X)$ is compact. By the Stone–Čech compactification theorem f_1 extends continuously to βX. Define f_2 on $\beta X \times \beta Y$ by the equation $f_2(x, y) = f_1(x)(y)$. It is easily verified that f_2 is a continuous extension of f to $\beta X \times \beta Y$. Since f was an arbitrary element of $C(X \times Y)$ it follows that $\beta(X \times Y) = \beta X \times \beta Y$.

The extension of the theorem to finite products is immediate. The extension to arbitrary products results from noting, as have Glicksberg and Frolík, that a continuous function on the pseudocompact product $\Pi_a X_a$ may be uniformly approximated by continuous functions depending on only a finite number of coordinates.

References

1. Z. Frolík, *The topological product of two pseudocompact spaces*, Czechoslovak Math. J. **10** (1960), 339–349.

2. I. Glicksberg, *Stone–Čech compactifications of products*, Trans. Amer. Math. Soc. **90** (1959), 369–382.

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