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SIGN-VARIATIONS OF SOLUTIONS OF NONLINEAR DISCRETE BOUNDARY VALUE PROBLEMS

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In this paper, we study two-point boundary value problems for the nonlinear second order difference equation

$$\Delta^2 u(i-1) + g\big(u(i)\big) = f(i), \quad i \in \{1, \dots, T+1\},$$
$$u(0) = u(T+2) = 0.$$

We establish the relationship between the number of sign-variation of f on $\{0, \dots, T+2\}$ and the one of the solution u of the above problem.

1. INTRODUCTION

In [**2**], Bellman considered the following two-point boundary value problem for linear second order ordinary differential equation in the form

$$u''(t) + q(t)u(t) = f(t), \qquad 0 \leqslant t \leqslant 1,$$
$$u(0) = u(1) = 0. \tag{1.1}$$

Assuming $q(t) \leqslant \pi^2$ and $q(t) \not\equiv \pi^2$, he proved, by the method of calculus of variation, that if $f(\cdot)$ has n simple zeros in $(0,1)$, the solution $u(\cdot)$ of (1.1) has at most n simple zeros in $(0,1)$. A version of this result was proved by Lazer and McKenna [**4**] and some important applications of it were also given in [**4**]. Bellman's idea was generalised by Boucherif and Slimini [**3**] to boundary value problems of nonlinear second order ordinary differential equations of the form

$$u''(t) + g\big(u(t)\big) = f(t), \qquad 0 \leqslant t \leqslant 1,$$
$$u(0) = u(1) = 0. \tag{1.2}$$

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A key condition they used is the following:

$$0 < \alpha \leqslant g'(s) \leqslant \beta < \pi^2, \quad \text{for some constants } \alpha,\ \beta \in (0, \infty).$$

Motivated by [**2, 3, 4**], we study boundary value problems for the second order difference equation of the form

$$\begin{gathered} \Delta^2 u(i-1) + c(i)u(i) = f(i), \quad i \in \{1, \dots, T+1\}, \\ u(0) = u(T+2) = 0. \end{gathered} \tag{1.3}$$

and

$$\begin{gathered} \Delta^2 u(i-1) + g\big(u(i)\big) = f(i), \quad i \in \{1, \dots, T+1\}, \\ u(0) = u(T+2) = 0. \end{gathered} \tag{1.4}$$

We conclude with some results similar to those of [**2, 3, 4**]. The methods we apply are rather similar to those in [**3, 4**]. However a great deal of additional effort has to be made due to the existence of *nodes* in the discrete cases.

2. The Preliminaries

Let T be an integer with $T \geqslant 3$. Let $\mathbb{T} := \{0, 1, \dots, T+2\}$. We denote the closure of an interval $I \subset \mathbb{R}$ by $\overline{I}$.

Lemma 2.1. ([5, Theorem 7.6]) *The Sturm-Liouville problem*

$$\begin{gathered} \Delta^2 u(i-1) + \lambda u(i) = 0, \quad i \in \{1, \dots, T+1\}, \\ u(0) = u(T+2) = 0 \end{gathered}$$

has a sequence of eigenvalues: $\lambda_1 < \lambda_2 < \dots < \lambda_{T+1}$.

Let

$$D^* = \Big\{ u \mid u = \big(u(0), u(1), \dots, u(T+2)\big),\ u(j) \in \mathbb{R} \text{ for } i \in \{0, \dots, T+2\} \Big\},$$

and

$$D = \big\{ u \in D^* \mid u(0) = u(T+2) = 0 \big\}. \tag{2.1}$$

Assume that

(H1) $c : \{1, \dots, T+1\} \to \mathbb{R}$ is a function satisfying

$$c(j) < \lambda_1, \quad \forall j \in \{1, \dots, T+1\}.$$

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Lemma 2.2. *Let $u \in D$. Then*

$$\sum_{k=0}^{T+1} \left|\Delta u(k)\right|^2 \geqslant \lambda_1 \sum_{k=0}^{T+1} \left|u(k)\right|^2.$$

Proof: From Kelley and Perterson [5, Theorem 7.7], we have that for $u \in D$,

$$\lambda_1 \leqslant \frac{\sum_{t=1}^{T+2} \left[\Delta u(t-1)\right]^2}{\sum_{t=1}^{T+1} u^2(t)} \leqslant \frac{\sum_{\tau=0}^{T+1} \left[\Delta u(\tau)\right]^2}{\sum_{t=1}^{T+1} u^2(t)} = \frac{\sum_{k=0}^{T+1} \left[\Delta u(k)\right]^2}{\sum_{k=0}^{T+1} u^2(k)}.$$

∎

Remark 2.1. It is worth remarking that by taking $n = T+2$, $p = q = 1, u_k = v_k = u(k)$, in Pachpatte [6, Theorem 2] we obtain

$$\sum_{k=0}^{T+1} \left|\Delta u(k)\right|^2 \geqslant \left(\frac{2}{T+2}\right)^2 \sum_{k=0}^{T+1} \left|u(k)\right|^2, \quad u \in D.$$

However the constant $\left(2/(T+2)\right)^2$ may be smaller than λ_1. This can be seen from the linear eigenvalue problem

$$\Delta^2 u(i-1) + \lambda u(i) = 0, \quad i \in \{1,\ 2,\ 3\},$$
$$u(0) = u(4) = 0.$$

From [5, Example 7.1], $\lambda_1 = 2 - \sqrt{2}$. Obviously $\lambda_1 > \left(\frac{2}{T+2}\right)^2$.

Lemma 2.3. *Let $u, w \in D$. Then*

$$\sum_{k=0}^{T+1} w(k)\Delta^2 u(k-1) = -\sum_{k=0}^{T+1} \Delta u(k)\Delta w(k).$$

Proof: Since $w(0) = w(T+2) = 0$, we have

$$\begin{aligned}
&\sum_{k=0}^{T+1} w(k)\Delta^2 u(k-1) \\
&\quad = \sum_{k=1}^{T+1} w(k)\Delta^2 u(k-1) \qquad \text{(by } w(0) = 0\text{)} \\
&\quad = \sum_{j=0}^{T} w(j+1)\Delta^2 u(j) \qquad \text{(by setting } j = k-1\text{)} \\
&\quad = \sum_{j=0}^{T} w(j+1)\left(\Delta u(j+1) - \Delta u(j)\right) \\
&\quad = \sum_{j=0}^{T} \Delta u(j+1)w(j+1) - \sum_{j=0}^{T} \Delta u(j)w(j+1)
\end{aligned}$$

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$$= \sum_{l=1}^{T+1} \Delta u(l)w(l) - \sum_{j=0}^{T} \Delta u(j)w(j+1) \qquad \text{(by setting } l = j+1\text{)}$$

$$= \Delta u(T+1)w(T+1) + \sum_{l=1}^{T} \Delta u(l)w(l) - \left[\Delta u(0)w(1) + \sum_{j=1}^{T} \Delta u(j)w(j+1)\right]$$

$$= \Delta u(T+1)\left[w(T+1) - w(T+2)\right] - \sum_{l=1}^{T} \Delta u(l)\Delta w(l) - \Delta u(0)\left[w(1) - w(0)\right]$$

$$= -\sum_{l=0}^{T+1} \Delta u(l)\Delta w(l).$$

□

Lemma 2.4. *Let $f : \{1, \dots, T+1\} \to \mathbb{R}$ be a function. Let (H1) be satisfied, and let u satisfy*

$$\Delta^2 u(i-1) + c(i)u(i) = f(i), \quad i \in \{1, \dots, T+1\},$$
$$u(0) = u(T+2) = 0. \tag{2.2}$$

Assume that there exist $i_0 \in \{0, \dots, T\}$ and an integer $p > 1$ with $i_0 + p \leqslant T+2$, such that

(i) *either*

$$u(i_0) \leqslant 0, \; u(i_0 + p) \leqslant 0, \; u(i_0 + j) > 0, \quad j \in \{1, \dots, p-1\},$$

or

$$u(i_0) \geqslant 0, \; u(i_0 + p) \geqslant 0, \; u(i_0 + j) < 0, \quad j \in \{1, \dots, p-1\};$$

(ii) *either $f(i_0 + j) > 0$ for all $j \in \{1, \dots, p-1\}$ or $f(i_0 + j) < 0$ for all $j \in \{1, \dots, p-1\}$.*

Then $u(i_0 + j)f(i_0 + j) < 0$ for all $j \in \{1, \dots, p-1\}$.

Proof: Notice that the set D is a Hilbert space under the inner product

$$\langle u, v \rangle := \sum_{k=0}^{T+2} u(k)v(k).$$

Clearly

$$\langle u, v \rangle = \sum_{k=1}^{T+1} u(k)v(k).$$

since $u(0) = u(T+2) = 0$. For $v \in D$, let

$$J(v) := \frac{1}{2}\sum_{k=0}^{T+1} \left(\Delta v(k)\right)^2 - \frac{1}{2}\sum_{k=1}^{T+1} c(k)v^2(k) + \sum_{k=1}^{T+1} f(k)v(k).$$

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If $v(k) = u(k) + w(k)$, then by (2.2), Lemma 2.2, Lemma 2.3, the fact $u, v, w \in D$, it follows that

$$\begin{aligned}
&J(v) - J(u)\\
&= \frac{1}{2}\sum_{k=0}^{T+1}\big(\Delta u(k) + \Delta w(k)\big)^2 - \frac{1}{2}\sum_{k=1}^{T+1} c(k)\big(u(k)+w(k)\big)^2\\
&\quad + \sum_{k=1}^{T+1} f(k)\big(u(k)+w(k)\big) - \frac{1}{2}\sum_{k=0}^{T+1}\big(\Delta u(k)\big)^2 + \frac{1}{2}\sum_{k=1}^{T+1} c(k)u^2(k) - \sum_{k=1}^{T+1} f(k)u(k)\\
&= \sum_{k=0}^{T+1}\big(\Delta u(k)\Delta w(k)\big) + \frac{1}{2}\sum_{k=0}^{T+1}\big(\Delta w(k)\big)^2\\
&\quad - \sum_{k=1}^{T+1} c(k)u(k)w(k) - \frac{1}{2}\sum_{k=1}^{T+1} c(k)\big(w(k)\big)^2 + \sum_{k=1}^{T+1} f(k)w(k)\\
(2.3)\quad &= \sum_{k=0}^{T+1}\big[-\Delta^2 u(k-1) - c(k)u(k) + f(k)\big]w(k)\\
&\quad + \frac{1}{2}\sum_{k=0}^{T+1}\big(\Delta w(k)\big)^2 - \frac{1}{2}\sum_{k=1}^{T+1} c(k)\big(w(k)\big)^2\\
&= \frac{1}{2}\sum_{k=0}^{T+1}\big(\Delta w(k)\big)^2 - \frac{1}{2}\sum_{k=1}^{T+1} c(k)\big(w(k)\big)^2\\
&\geqslant \frac{1}{2}\sum_{k=0}^{T+1}\big[\lambda_1 - c(k)\big]\big(w(k)\big)^2\\
&\geqslant 0
\end{aligned}$$

If, contrary to the assertion of the lemma,

$$u(i_0+j)f(i_0+j) > 0, \qquad j \in \{1, \ldots, p-1\}, \tag{2.4}$$

and we set

$$v^*(i) = \begin{cases} u(i), & i \in \{0, \ldots, T+2\} \setminus \{i_0+1, \ldots, i_0+p-1\},\\ -u(i), & i \in \{i_0+1, \ldots, i_0+p-1\},\end{cases}$$

then $v^* \in D$. It is easy to check that

$$\big|\Delta u(i_0)\big| \geqslant \big|\Delta v^*(i_0)\big|, \quad \big|\Delta u(i_0+p-1)\big| \geqslant \big|\Delta v^*(i_0+p-1)\big|,$$

and

$$\big|\Delta u(i_0+j)\big| = \big|\Delta v^*(i_0+j)\big|, \qquad j \in \{1, \ldots, p-2\},$$

which together with (2.4) implies $J(v^*) < J(u)$, contrary to (2.3). This contradiction shows that $u(i_0+j)f(i_0+j) < 0$ for all $j \in \{1, \ldots, p-1\}$. ∎

Lemma 2.5. *Let m, n be integers with $n < m$. Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ and $0 = \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} = 1$ be given. Denote*

$$I_i := (t_{i-1}, t_i), \quad i = 1, \dots, n+1; \qquad J_j := (\tau_{j-1}, \tau_j), \quad j = 1, \dots, m+1.$$

Let's dye each of these open intervals in blue or red, such that

(a) *any two adjacent open intervals in $\{I_i \mid i = 1, \dots, n+1\}$ have different colours;*

(b) *any two adjacent open intervals in $\{J_j \mid j = 1, \dots, m+1\}$ have different colours.*

Then there exist I_{i_0} and J_{j_0} such that

(i) *$J_{j_0} \subseteq I_{i_0}$;*

(ii) *I_{i_0} and J_{j_0} are in same colour.*

Proof: Without loss of the generality, we assume that I_1 is blue.

It is easy to see that in the case $n = 1$, the lemma holds.

Assume that in the case $n = k$, the result is true. We make a couple of fundamental observations.

Observation 1. If there exist I_{i^*}, J_{j^*} and J_{j^*+1} such that $(J_{j^*} \cup J_{j^*+1}) \subseteq I_{i^*}$, then we have done.

Observation 2. The results of Lemma 2.5 with $n \leqslant k$ are still true if we replace the interval $[0, 1]$ with a general interval $[\alpha, \beta]$.

Let's consider the case that $n = k + 1$.

So we may assume that for each $i \in \{1, \dots, k+2\}$ and $j \in \{1, \dots, m\}$ $(m > k+2)$,

$$(J_j \cup J_{j+1}) \not\subseteq I_i. \tag{2.5}$$

By Observation 1 and (2.5), we only need to consider the following three cases:

Case 1. J_1 is blue and $I_1 \subset J_1$;

Case 2. J_1 is red and $J_1 \subset I_1 \subset (J_1 \cup J_2)$;

Case 3. J_1 is red and $I_1 \subseteq J_1$.

In Case 1, there exists $r \in \{2, \dots, k+2\}$ such that $\tau_1 \in [t_r, t_{r+1})$ and

$$(\overline{J}_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m) \subseteq (\overline{I}_r \cup \overline{I}_{r+1} \cup \cdots \cup \overline{I}_{k+2}).$$

If

$$(\overline{J}_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m) \subset (\overline{I}_r \cup \overline{I}_{r+1} \cup \cdots \cup \overline{I}_{k+2}),$$

we take $I'_r := I_r \cap (J_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m)$ so that

$$(\overline{J}_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m) = (\overline{I}'_r \cup \overline{I}_{r+1} \cup \cdots \cup \overline{I}_{k+2}). \tag{2.6}$$

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Thus we can reduce the problem with $n = k+1$ to a new problem with $n = k+2-r$ $(\leqslant k)$. So by reduction and Observation 2, there exists $j_0 \in \{2, \ldots, m\}$, such that either

$$J_{j_0} \subseteq I'_r \ (\subset I_r) \text{ and they have same colour,} \tag{2.7}$$

or for some $i_0 \in \{r+1, \ldots, k+2\}$,

$$J_{j_0} \subseteq I_{i_0} \text{ and they have same colour.} \tag{2.8}$$

In Case 2, set $J'_2 := (J_1 \cup \overline{J}_2) \setminus \overline{I}_1$. Then $J'_2 \neq \emptyset$ and

$$\overline{I}_2 \cup \overline{I}_3 \cup \cdots \cup \overline{I}_{k+2} = \overline{J}'_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m, \tag{2.9}$$

and again we reduce the problem to a new problem with $n = k$. So by reduction and Observation 2, either there exists $j_0 \in \{3, \ldots, m\}$, such that for some $i_0 \in \{2, \ldots, k+2\}$,

$$J_{j_0} \subseteq I_{i_0} \text{ and they have same colour,} \tag{2.10}$$

or

$$J'_2 \subset I_2, \text{ and they have same colour.} \tag{2.11}$$

However (2.11) can not occur in Case 2 since J'_2 and I_2 have two different colours. Therefore, (2.10) holds.

In Case 3, there exists $l \in \{2, \ldots, k+2\}$ such that $\tau_1 \in [t_l, t_{l+1})$ and

$$(\overline{J}_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m) \subseteq (\overline{I}_l \cup \overline{I}_{l+1} \cup \cdots \cup \overline{I}_{k+2}). \tag{2.12}$$

If

$$(\overline{J}_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m) \subset (\overline{I}_l \cup \overline{I}_{l+1} \cup \cdots \cup \overline{I}_{k+2})$$

holds, we take $I'_l := I_l \cap (J_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m)$. Then $I'_l \neq \emptyset$ and

$$(\overline{J}_2 \cup \overline{J}_3 \cup \cdots \cup \overline{J}_m) = (\overline{I}'_l \cup \overline{I}_{l+1} \cup \cdots \cup I_{k+2}). \tag{2.13}$$

Thus we can reduce the problem with $n = k+1$ to the new problem with $n = k+2-l$. So by the reduction and Observation 2, there exists $j_0 \in \{2, \ldots, m\}$, such that either

$$J_{j_0} \subset I'_l \ (\subset I_l) \text{ and they have same colour} \tag{2.14}$$

or for some $i_0 \in \{l+1, \ldots, k+2\}$,

$$J_{j_0} \subseteq I_{i_0} \text{ and they have same colour.} \tag{2.15}$$

□

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3. The Main Results

Definition 3.1: A function $u \in D^*$ has a *zero* $j \in \{0, \dots, T+2\}$ if $u(j) = 0$. If

$$u(j) = 0, \quad \text{and } u(j-1)u(j+1) < 0$$

for some $j \in \{1, \dots, T+1\}$, then we say that j is a *simple zero* of u. If $u(k)u(k+1) < 0$ for some $k \in \{1, \dots, T+1\}$, then we say that u has a *node* at $k + 1/2$.

We say j is a *point of sign-variation* if it is a simple zero and or if it is a node. We shall denote by $\text{NSV}u(\mathbb{T})$ the number of the points of sign-variations of a function u on $\mathbb{T}$.

Remark 3.2. The point s given by the definition of node of u does not belong to the set $\{0, 1, \dots, T+2\}$. This idea of nodes can be found from Agarwal, Bohner and Wong [**1**].

Theorem 3.3. *Assume that* (H1) *is satisfied. If all zeros of f in $\{1, \dots, T+1\}$ are simple zeros, and if u is the unique solution of* (2.2) *and all zeros of u in $\{0, 1, \dots, T+2\}$ are simple zeros. Then $NSVu(\mathbb{T}) \leqslant NSVf(\mathbb{T})$.*

Proof: The case $NSVf(\mathbb{T}) = 0$ is trivial.

We now deal with the case $NSVf(\mathbb{T}) \geqslant 1$.

Let all of the points of sign-variation of u and f on $\mathbb{T}$ be given by

$$a_1 < a_2 < \dots < a_r, \quad \text{and} \quad b_1 < b_2 < \dots < b_l$$

respectively. Then

$$0 = a_0 < a_1 < a_2 < \dots < a_r < a_{r+1} = T + 2,$$

and

$$0 = b_0 < b_1 < b_2 < \dots < b_l < b_{l+1} = T + 2,$$

and

$$\text{NSV}f(\mathbb{T}) = r, \quad \text{NSV}u(\mathbb{T}) = l.$$

If $f(j) > 0$ for all $j \in (a_s, a_{s+1}) \cap \mathbb{T}$, then we dye the interval (a_s, a_{s+1}) in blue; if $f(j) < 0$ for all $j \in (a_s, a_{s+1}) \cap \mathbb{T}$, then we dye the interval (a_s, a_{s+1}) in red; If $u(j) > 0$ for all $j \in (b_\tau, b_{\tau+1}) \cap \mathbb{T}$, then we dye the interval $(b_\tau, b_{\tau+1})$ in blue; if $f(i) < 0$ for all $j \in (b_\tau, b_{\tau+1}) \cap \mathbb{T}$, then we dye the interval $(b_\tau, b_{\tau+1})$ in red.

Suppose on the contrary that $r > l$. Since all zeros of f and u in $\{1, \dots, T+1\}$ are simple zeros, we are in the position of applying Lemma 2.5 now. Hence there exist $j_0 \in \{1, \dots, r+1\}$ and $i_0 \in \{1, \dots, l+1\}$, such that

(i) $(a_{j_0-1}, a_{j_0}) \subseteq (b_{i_0-1}, b_{i_0})$;

(ii) $f(k)u(k) > 0$ for $k \in \{a_{j_0-1} + 1, \dots, a_{j_0-1} - 1\}$.

However this contradicts Lemma 2.4. Therefore $r \leqslant l$. ∎

4. The Nonlinear Problem

Assume the following

(H2) $g \in C^1(\mathbb{R}, \mathbb{R})$;

(H3) There exists two constants α_0 and β_0, such that

$$0 < \alpha \leqslant g'(s) \leqslant \beta < \lambda_1.$$

For $\psi \in D = \{u \in D^* \mid u(0) = u(T+2) = 0\}$, let

$$\|\psi\|_D := \sum_{j=0}^{T+2} |\psi(j)|^2.$$

Let

$$Y = \{\varphi \mid \varphi : \{1, \dots, T+1\} \to \mathbb{R}\}$$

and for $\varphi \in Y$, let $\|\varphi\|_Y := \sum_{j=1}^{T+1} |\varphi(j)|^2$. It is clear that the above are norms on D and Y, respectively, and that the finite dimensionality of these spaces makes them Banach spaces.

Theorem 4.1. *Assume that* (H2) *and* (H3). *Then*

$$\begin{aligned} &\Delta^2 u(j-1) + g(u(j)) = f(j), \quad j \in \{1, \dots, T+1\} \\ &u(0) = u(T+2) = 0, \end{aligned} \tag{4.1}$$

has a unique solution.

Proof: Define an operator $L : D \to Y$ by

$$(Lu)(j) = \Delta^2 u(j-1) + \frac{\beta+\alpha}{2}(u(j)), \quad j \in \{1, \dots, T+1\}. \tag{4.2}$$

Then it is easy to check that L is an bijection from Y onto D, and

$$\|L^{-1}\|_{Y \to D} = \frac{1}{\lambda_1 - (\beta+\alpha)/2}. \tag{4.3}$$

Now (4.1) is equivalent to the fixed point problem

$$u(j) = L^{-1}\Big[\frac{\beta+\alpha}{2}(u(j)) - g(u(j)) + f(j)\Big] := (Tu)(j), \quad j \in \{0, 1, \dots, T+2\}. \tag{4.4}$$

For every $u, v \in D$,

$$\begin{aligned} &|Tu(j) - Tv(j)| \\ &= \|L^{-1}\|_{Y \to D} \Big|\frac{\beta+\alpha}{2} - g'\big(\theta(j)u(j) + (1-\theta(j))v(j)\big)\Big| \, |u(j) - v(j)| \end{aligned} \tag{4.5}$$

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for some $\theta(j) \in (0,1)$. This together with (H3) and (4.3) implies that $T : D \to D$ is a contraction mapping. So by the Contraction Mapping Principle, T has unique fixed point in D, and accordingly (4.1) has unique solution. ∎

Theorem 4.2. *Assume that* (H2) *and* (H3). *Assume that all zeros of f in $\{1,\dots,T+1\}$ are simple zeros. Let u be the unique solution of* (4.1) *and assume that all zeros of u in $\{0,1,\dots,T+2\}$ are simple zeros. Then $NSVu(\mathbb{T}) \leqslant NSVf(\mathbb{T})$.*

Remark 4.1. Theorem 4.2 is a similar result to [**3**, Theorem 9]. However the main conditions are much weaker than those imposed on [**3**, Theorem 9] where the following restrictions are needed

(h.2) $g(0) = 0$;

(h.5) $G(u) = G(-u)$ where $G(u) = \int_0^u g(s)ds$.

Moreover the proof of Theorem 4.2 is much simple than the proof of [**3**, Theorem 9].

Proof of Theorem 4.2: Set

$$c^*(k) = \begin{cases} \dfrac{g(u(k))}{u(k)}, & \text{if } u(k) \neq 0, \\ \dfrac{1}{2}\lambda_1, & \text{if } u(k) = 0. \end{cases}$$

Then (4.1) can be rewritten as

$$\begin{aligned} \Delta^2 u(j-1) + c^*(j)u(j) = f(j), \quad j \in \{1,\dots,T+1\} \\ u(0) = u(T+2) = 0. \end{aligned} \tag{4.6}$$

Now the desired result is an immediate consequence of Theorem 3.3. ∎

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