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BANACH ALGEBRAS OF TOPOLOGICALLY BOUNDED INDEX

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We consider normed and Banach algebras satisfying a condition topologically analogous to bounded index for rings. We investigate stability properties, prove a topological version of a theorem of Jacobson, and find in many cases co-incidence with well-known finiteness properties.

INTRODUCTION

An element a of a ring R is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$, and R is *of bounded index* if there is some $N \in \mathbb{N}$ with $a^N = 0$ for all nilpotent $a \in R$. Such rings have attracted several authors, for example Jacobson [**6**] and Klein [**9**], and it seems natural to consider topological analogues of these rings in the context of Banach algebras. We pursue such an analogy here in the spirit of P.G. Dixon's work on topologically nilpotent Banach algebras.

For an element a of a normed algebra A we write $\sigma(a)$ for the *spectrum* of a and

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

which is, of course, the spectral radius when A is a Banach algebra. If $r(a) = 0$ we shall say that a is *topologically nilpotent*. We denote the set of such elements by $T(A)$.

DEFINITION 1.1: A normed algebra A is of *topologically bounded index* if

$$S_{T(A)}(n) = \sup\left\{\|a^n\|^{1/n} : a \in T(A),\ \|a\| \leqslant 1\right\} \to 0$$

as $n \to \infty$.

Radical normed algebras (that is those with $T(A) = A$) satisfying the above are called *uniformly topologically nil* (also *uniformly radical*) and have been investigated in [**4**] and [**5**].

In this article we determine some elementary properties of general normed algebras of topologically bounded index. We consider stability, prove a topological analogue of a theorem of Jacobson, and conclude by investigating several examples.

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For a normed algebra A the set $N(A)$ of nilpotent elements is contained in $T(A)$ and it is clear that normed algebras of bounded index with $T(A) = N(A)$ are of topologically bounded index. The following examples show that, in general, bounded index and topologically bounded index are quite different.

The commutative Banach algebra $C[0,1]$ of continuous functions $f : [0,1] \to \mathbb{C}$, with supremum norm and convolution product

$$f * g(s) := \int_0^s f(t)g(s-t)dt \qquad (f,\, g \in C[0,\, 1])$$

is known to be uniformly topologically nil, while Titchmarsh's theorem shows that it is not of bounded index. On the other hand consider for a suitable weight function $\omega\colon [0,\, \infty) \to (0,\, \infty)$, the algebra $L^1(\omega)$ of Lebesgue measurable functions $f : [0,\infty) \to \mathbb{C}$, with convolution product and weighted L^1 norm

$$\|f\|_\omega := \int_0^\infty |f(t)|\, \omega(t)dt \qquad \big(f \in L^1(\omega)\big).$$

This is a commutative radical Banach algebra if, for example, we take $\omega(t) = \exp\left(-te^{-t}\right)$ but has a bounded approximate identity. By [**11,** Proposition 2.4] and [**4,** Theorem 2.1] it is not uniformly topologically nil — particularly, it is not of topologically bounded index. Another application of Titchmarsh's theorem shows that $L^1(\omega)$ has no non-zero nilpotent elements and so is vacuously of bounded index.

We conclude the introduction with a lemma which will simplify several subsequent arguments.

Lemma 1.2. *The sequence* $\big(S_{T(A)}(n)\big)$ *converges to a limit in* $[0,1]$ *and*

$$\lim_{n\to\infty} S_{T(A)}(n) = \inf_{n\in\mathbb{N}} S_{T(A)}(n).$$

Proof: This follows by a well known argument from the observation that

$$S_{T(A)}(n+m)^{n+m} \leqslant S_{T(A)}(n)^n S_{T(A)}(m)^m \qquad (n, m \in \mathbb{N}).$$

(See for example [**2,** Proposition 2.8]). ∎

2. Stability

Topologically bounded index seems to have rather poor stability properties — perhaps because $T(A)$ need not, in general, even be a subspace. It is, however, easy to see that subalgebras and the unitisation of topologically bounded index normed algebras are of topologically bounded index.

Topologically bounded index is not in general preserved when we take a quotient, as is seen from the following example. Let A_0 denote the algebra over $\mathbb{C}$ with generators a_n $(n \in \mathbb{N})$ and relations $a_n a_m = 0$ $(i \neq j)$. Thus a typical element $x \in A_0$ is of the form

$$x = \sum_{n,m \in \mathbb{N}} \lambda_{n,m} a_n^m$$

where only finitely many of the $\lambda_{n,m} \in \mathbb{C}$ are non-zero. With the ℓ^1 norm,

$$\|x\| = \sum_{n,m \in \mathbb{N}} |\lambda_{n,m}|,$$

A_0 is a normed algebra and we denote by A the completion.

If $x \in A$ is non-zero then we can find n, m such that

$$\lambda_{n,1} = \lambda_{n,2} = \cdots = \lambda_{n,m-1} = 0 \neq \lambda_{n,m}$$

so that for each k the coefficient of a_n^{km} in x^k is exactly $\lambda_{n,m}^k$. Then $\|x^k\| \geqslant |\lambda_{n,m}^k|$ so $r(x) \geqslant |\lambda_{n,m}| > 0$ and consequently A is vacuously of topologically bounded index.

Now write $I = \operatorname{span}\{a_n^m : 1 \leqslant n \leqslant m\}$, which is a closed ideal. With $[x]_I$ denoting the equivalence class of x in the quotient A/I we have ${[a_n]_I}^{n+1} = 0$ for $n \in \mathbb{N}$ while

$$\|[a_n]_I^n\| = \|[a_n^n]_I\| = \|a_n^n\| = 1 = \|[a_n]_I\|.$$

It follows that $S_{A/I}(n) = 1$ $(n \in \mathbb{N})$ and A/I is not of topologically bounded index.

There is one case in which available structure theory may be used to show that uniformly topologically nil extensions are topologically bounded index.

A ring R is a PI-ring if there is some non-zero polynomial p, in say n indeterminates, such that

$$p(x_1, \ldots, x_n) = 0 \qquad (x_1, \ldots, x_n \in R).$$

We consider Banach algebras which are PI modulo the radical (that is $A/\operatorname{rad}(A)$ is PI), and make use of the generalised Gelfand transform developed in [**10**, Chapter IV] as follows.

Suppose that A is a unital Banach algebra which is PI modulo the radical. Then there exists $n \in \mathbb{N}$ such that $a \in A$ is invertible if and only if $\pi(a)$ is invertible for all representations $\pi : A \to M_n(\mathbb{C})$ with the property that

$$|\pi(a)_{i,j}| \leqslant \|a\| \qquad (a \in A,\ 1 \leqslant i, j \leqslant n). \tag{1}$$

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With this value of n fixed, the set of representations satisfying (1), which we denote Φ_A, can be given a compact Hausdorff topology. If $C(\Phi_A, M_n(\mathbb{C}))$ denotes the Banach algebra of continuous functions from Φ_A to $M_n(\mathbb{C})$ then *the generalised Gelfand transform* is the linear mapping

$$\begin{aligned} A &\longrightarrow C(\Phi_A,\, M_n(\mathbb{C})) \\ a &\longmapsto \widehat{a} \end{aligned}$$

where $\widehat{a}(\pi) = \pi(a)$ $(a \in A,\ \pi \in \Phi_A)$. It is known that:

1. the generalised Gelfand transform is a continuous homomorphism;
2. $a \in A$ is in the radical if and only if $\widehat{a} = 0$;
3. the spectrum of $a \in A$ is the union of the spectra of $\widehat{a}(\pi)$ $(\pi \in \Phi_A)$.

Proposition 2.1. *Let A be a Banach algebra which is PI modulo the radical. Then A is of topologically bounded index if and only if* $\mathrm{rad}\,(A)$ *is uniformly topologically nil.*

Proof: That a topologically bounded index Banach algebra has a uniformly topologically nil radical is obvious. Moreover a unitisation argument shows that we need only prove the converse in the case when A is unital.

Suppose that $\mathrm{rad}\,(A)$ is uniformly topologically nil and $a \in A$ is topologically nilpotent. Using the above notation we have

$$\sigma(a) = \bigcup \{\sigma(\widehat{a}(\pi)) : \pi \in \Phi_A\} = \{0\},$$

hence the matrix $\widehat{a}(\pi)$ is topologically nilpotent and, by Cayley-Hamilton, $(\widehat{a}(\pi))^n = 0$ $(\pi \in \Phi_A)$. Consequently $\widehat{(a^n)} = 0$ and so $a^n \in \mathrm{rad}\,(A)$. Thus whenever $a \in T(A)$ with $\|a\| \leqslant 1$ we find

$$\left\|a^{kn}\right\|^{1/kn} = \left\|(a^n)^k\right\|^{1/kn} \leqslant S_{\mathrm{rad}\,(A)}(k)^{1/n} \qquad (k \in \mathbb{N})$$

and taking the supremum over all such a we obtain

$$S_{T(A)}(kn) \leqslant S_{\mathrm{rad}\,(A)}(k)^{1/n} \qquad (k \in \mathbb{N}).$$

The proposition now follows from Lemma 1.2. ∎

Corollary 2.2. *A semisimple PI Banach algebra is of topologically bounded index.*

We now turn to the question of whether topologically bounded index extends to the closure. It is easy to see, by a continuity of norm argument, that for a normed algebra A we have $S_{T(A)}(n) = S_{\overline{T(A)}}(n)$ for each $n \in \mathbb{N}$. However, we may find that $\overline{T(A)}$ is properly contained in $T(\overline{A})$ as occurs in the following example of a topologically bounded index normed algebra whose completion is not of topologically bounded index.

Let (e_i) denote the natural basis of ℓ^1 and for each $m, n \in \mathbb{N}$ write $L_{m,n}$ for the weighted left shift operator on ℓ^1 given by

$$L_{m,n}e_i = \begin{cases} 0 & \text{if } i = 1, \\ e_{i-1} & \text{if } i = 2, \dots, n+1, \\ (1/m)e_{i-1} & \text{if } i = n+2, \dots . \end{cases}$$

Denote by FS_2 the free semigroup on symbols s, t and write $\ell^1(FS_2)$ for the ℓ^1 semigroup algebra of FS_2. We let $p_m = s^m t^m \in FS_2$ and define A_n to be the normed subalgebra of the Cartesian product $\ell^1(FS_2) \times \mathcal{B}(\ell^1)$ generated by the elements $(1/m\, p_m, L_{m,n})$ $(m \in \mathbb{N})$. The algebraic operations on A_n are pointwise and the norm is the supremum norm.

It is known that $\ell^1(FS_2)$ has no non-zero topologically nilpotent elements (see [**2**, Example 46.6]) and so if $(a, b) \in A_n$ is topologically nilpotent we have $a = 0$. Since A_n is generated by the $(1/m\, p_m, L_{m,n})$ there is some $M \in \mathbb{N}$ and a polynomial P such that

$$(a,\, b) = P\left((p_1,\, L_{1,n}), \dots (1/M\, p_M, L_{M,n})\right)$$

and so $P(p_1, \dots, 1/M\, p_M) = 0$. But this implies that P is trivial since distinct products of the p_i produce distinct elements of FS_2. Hence A_n contains no non-zero topologically nilpotent elements.

Now let $L_n \in \mathcal{B}(\ell^1)$ be defined by

$$L_n e_i = \begin{cases} 0 & \text{if } i = 1, \\ e_{i-1} & \text{if } i = 2, \dots, n+1, \\ 0 & \text{if } i = n+2, \dots \end{cases}$$

so that

$$(L_{m,n} - L_n)e_i = \begin{cases} 0 & \text{if } i = 1, \dots, n+1, \\ (1/m)e_{i-1} & \text{if } i = n+2, \dots . \end{cases}$$

We then have, for $\sum_{i=1}^{\infty} \lambda_i e_i \in \ell^1$

$$\left\| (L_{m,n} - L_n) \sum_{i=1}^{\infty} \lambda_i e_i \right\| = \frac{1}{m} \sum_{i=n+2}^{\infty} |\lambda_i| \leqslant \frac{1}{m} \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|$$

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so $\|L_{m,n} - L_n\| \leqslant 1/m$ for each $m \in \mathbb{N}$. It follows that $(0, L_n) \in \overline{A_n}$ and a short calculation shows that $\|(0, L_n)^n\| = \|(0, L_n)\| = 1$ while $(0, L_n)^{n+1} = 0$.

Our construction is almost complete. Let A denote the set of sequences whose n-th co-ordinate is in A_n. With pointwise algebraic operations and supremum norm A becomes a normed algebra and using the above we see that A contains no topologically nilpotent elements. However, considering the sequence in $\overline{A}$ with $(0, L_n)$ in the n-th co-ordinate and zero elsewhere we see that $S_{T(\overline{A})}(n) = 1$ for each $n \in \mathbb{N}$.

3. A topological Jacobson theorem

Many of the results on rings of bounded index do not seem to have analogues in the topological case. The following theorem of Jacobson [**6**] is a welcome exception.

Theorem. (Jacobson, [**6**]) *If A is a ring (with unit) of bounded index then for any $a, b \in A$ with $ab = 1$ we have $ba = 1$.*

Our topological version has a weaker (and topological) conclusion.

Proposition 3.2. *Let A be a unital normed algebra of topologically bounded index and suppose there are $a, b \in A$ with $ab = 1$. Then either;*

1. *$ba = 1$ or*
2. *$\|b^n a^n\|$ is not bounded.*

Proof: We proceed as in the proof of Jacobson's theorem. Suppose that $ab = 1 \neq ba$ and define matrix units by

$$e_{i,j} = b^{i-1}(1 - ba)a^{j-1} \qquad (i, j \in \mathbb{N}).$$

It is quickly confirmed that

$$e_{i,j}e_{k,l} = \delta_j^k e_{i,l} \qquad (i, j, k, l \in \mathbb{N})$$

and that the $e_{i,j}$ are linearly independent.

Now write, for each $n \in \mathbb{N}$

$$v_n = \sum_{i=1}^{n} e_{i,i+1}$$

so that

$$v_n^n b^n = 1 - ba \neq 0, \qquad v_n^{n+1} = 0 \qquad (n \in \mathbb{N}).$$

(This proves Jacobson's theorem since it shows that A contains nilpotents of arbitrarily large index so it is not of bounded index). For our topological version, we assume in addition that $\|b^n a^n\|$ is bounded. Since $v_n^n b^n = 1 - ba$ we have

$$\|1 - ba\| = \|v_n^n b^n\| \leqslant \|v_n^n\| \, \|b^n\| \qquad (n \in \mathbb{N})$$

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and so

$$\|v_n^n\|^{1/n} \geqslant \frac{\|1-ba\|^{1/n}}{\|b^n\|^{1/n}} \geqslant \frac{\|1-ba\|^{1/n}}{\|b\|} \qquad (n \in \mathbb{N}).$$

Thus, for sufficiently large n

$$(2) \qquad \|v_n^n\|^{1/n} \geqslant \frac{1}{2\,\|b\|}.$$

Now, by hypothesis, there is some $K > 0$ such that

$$\|b^n a^n\| \leqslant K \qquad (n \in \mathbb{N})$$

and since

$$v_n = \sum_{i=1}^{n} e_{i,i+1} = (1 - b^n a^n)a \qquad (n \in \mathbb{N})$$

we have

$$(3) \qquad \|v_n\| \leqslant (1 + \|b^n a^n\|)\,\|a\| \leqslant (1+K)\,\|a\| \qquad (n \in \mathbb{N}).$$

Combining (2) and (3) we find that, for sufficiently large n

$$S_{T(A)}(n) \geqslant \frac{\|v_n^n\|^{1/n}}{\|v_n\|} \geqslant \frac{1}{2\,\|a\|\,\|b\|\,(1+K)}$$

and so A is not of topologically bounded index. ∎

It would be interesting know if there is a unital Banach algebra A of topologically bounded index with $a, b \in A$ and $ab = 1 \neq ba$.

4. Examples

4.1 The ℓ^1-algebra of a semigroup. Let S be a semigroup and $\ell^1(S)$ the ℓ^1-semigroup algebra of S (see for example [**13,** Section 4.8.6]). We find some necessary conditions on S for $\ell^1(S)$ to be of topologically bounded index.

A *periodic* element $s \in S$ is one with $\langle s \rangle = \{s, s^2, \dots\}$ finite. For such an element there are unique $m = m(s)$, $k = k(s) \in \mathbb{N}$ with s, s^2, ..., s^{m+k-1} distinct, $s^{m+k} = s^m$ and $\langle s \rangle = \{s, s^2, \dots, s^{m+k-1}\}$ (see [**3**]). In this case we say that s has *index* m and *period* k.

Proposition 4.1. *If S is a semigroup such that $\ell^1(S)$ is topologically bounded index then*

$$\{m(s)/k(s) : s \in S \text{ is periodic}\}$$

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is bounded or empty.

PROOF: We shall write A for $\ell^1(S)$ and use the notation $[t]$ for the integer part of $t \in \mathbb{R}$. The result will follow from the fact that if some $s \in S$ is periodic with index m and period k then $S_{T(A)}([m/k]) = 1$ provided $2k \leqslant m$. For such s we write

$$x = \frac{1}{2}\left(s - s^k\right) \in A$$

so that $\|x\|_1 = 1$ and if we write $d = [m/k]$ we have

$$\left\|x^d\right\|_1 = \frac{1}{2^d}\left\|\sum_{r=0}^{d}(-1)^r \binom{d}{r} s^{(d-r)+rk}\right\|_1 = \frac{1}{2^d}\sum_{r=0}^{d}\binom{d}{r} = 1$$

since $s^d, s^{d-1+k}, \ldots, s^{dk}$ are distinct by choice of d. Moreover we find that

$$x^m = \frac{1}{2^m}\sum_{r=0}^{m}(-1)^r \binom{m}{r} s^{(m-r)+rk} = 0$$

since $s^{m-r+rk} = s^{m+(k-1)r} = \cdots = s^m$ for $r = 0, \ldots, m$. ∎

We also mention the following; The bicyclic semigroup is the semigroup with unit generated by symbols s, t, 1 subject to the relation $st = 1$. We see from the topological version of Jacobson's theorem that if a semigroup S has the bicyclic semigroup as a subsemigroup then $\ell^1(S)$ cannot be of topologically bounded index.

Semigroups S such that $\ell^1(S)$ *is* of topologically bounded index are easily found: it is known that $\ell^1(S)$ is a semisimple Banach algebras when S is a group, and PI if S is also Abelian-by-finite (see [**12,** Theorem 3, Chapter 18] for example).

4.2 VON NEUMANN ALGEBRAS. In the case of a von Neumann algebras we can characterise topologically bounded index in terms of the *type decomposition* of the algebra. We refer the reader to [**14,** Chapter 10] for a detailed treatment as we need only the following facts.

1. A von Neumann algebra A admits a decomposition

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_\infty \oplus A_C$$

into orthogonal von Neumann subalgebras $A_1, A_2, \ldots, A_\infty$ and A_C where A_i is of type I_i for $i = 1, \ldots, \infty$ and A_C is continuous [**14,** 4.17, E4.14 and E4.15].

2. A type I_n von Neumann algebra is isometrically isomorphic to the algebra $C(X, M_n(\mathbb{C}))$ of continuous functions from some compact Hausdorff space

X, to the $n \times n$ matrices over $\mathbb{C}$, with pointwise product and supremum norm [**8**, 6.6.5].

3. If A is a continuous or type I_∞ von Neumann algebra then for each $n \in \mathbb{N}$ there are projections $e_1, \dots, e_n \in A$ with $e_1 + \cdots + e_n = 1$ and which are equivalent and pairwise orthogonal. [**14**, 4.12] and [**8**, 6.5.6].

Proposition 4.2. *A von Neumann algebra A is of topologically bounded index if and only if its type decomposition is the direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ where A_i is type $I_{n(i)}$ for some $n(1), \dots, n(k) \in \mathbb{N}$.*

Proof: We first note that if a Banach algebra A is the direct sum of finitely many orthogonal closed subalgebras then an application of the open mapping theorem shows that A is of topologically bounded index if, and only if, all its summands are.

If B is a type I_n von Neumann algebra the we identify B with the algebra $C(X, M_n(\mathbb{C}))$ as mentioned above and suppose that $f : X \to M_n(\mathbb{C})$ is topologically nilpotent. Then we have $f(x) \in M_n(\mathbb{C})$ topologically nilpotent for each $x \in X$ and so $f(x)^n = 0$ by Cayley-Hamilton. Since the product is pointwise we have $f^n = 0$ and so B is of topologically bounded index. The above remark then shows that a direct sum of finitely many such von Neumann subalgebras is also of topologically bounded index.

To see the reverse implication we first note that each type I_n algebra contains an element v with

$$\left\|v^{n-1}\right\| = \|v\| = 1, \quad v^n = 0. \tag{4}$$

(Consider the function whose constant value is the matrix with ones on the first superdiagonal and zeros elsewhere.) This shows that a von Neumann algebra whose type decomposition contains type $\mathrm{I}_{n(i)}$ summands for an increasing sequence $n(i)$ is not topologically bounded index.

The remaining case is when the type decomposition of A contains continuous or type I_∞ summands. Since we may write $A = (A_1 \oplus \cdots) \oplus A_\infty \oplus A_C$ we see that it suffices to show that a continuous or type I_∞ algebra B is not topologically bounded index. To see this apply the third fact listed at the start of this section to B to obtain, for each n, projections $e_1, e_2, \dots, e_n$ which are orthogonal, equivalent and with sum 1. For $i = 1, 2, \dots, n-1$ denote by $v_{i,i+1}$ a partial isometry implementing the equivalence $e_{i+1} \sim e_i$. Then for $1 \leqslant i < j \leqslant n$ define

$$v_{i,j} = v_{i,i+1} v_{i+1,i+2} \cdots v_{j-1,j}.$$

One quickly confirms, using a routine Hilbert space orthogonality argument, that $v = v_{1,2} + v_{2,3} + \cdots + v_{n-1,n}$ satisfies the condition (4) and this completes the proof. ∎

We can combine the characterisation 4.2 with known results to see, for von Neumann algebras, coincidence of topologically bounded index with some well known finiteness properties. Namely subhomogeneity, satisfaction of a polynomial identity (Johnson [**7**, Proposition 6.1]) and injectivity in the sense of Varopoulos [**15**] (see Aristov [**1**]). Inspection of the proof of 4.2 also shows coincidence with bounded index.

4.3 ALGEBRAS OF OPERATORS ON A BANACH SPACE. For a Banach space $\mathfrak{X}$, it seems natural to imagine that $\mathcal{B}(\mathfrak{X})$, the algebra of bounded operators on $\mathfrak{X}$, is not of topologically bounded index. The following argument shows that this is true for many Banach spaces although we have not been able show full generality. We start with an old result due to Banach.

THEOREM 4.3. (Banach 1932) Let $\mathfrak{X}$ be a Banach space with a Schauder basis (e_i). Then the projections

$$P_n : \sum_{i=1}^{\infty} \alpha_i e_i \longmapsto \sum_{i=1}^{n} \alpha_i e_i$$

are bounded and $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$.

PROPOSITION 4.4. *Suppose that $\mathfrak{X}$ is a Banach space possessing a Schauder basis (e_i) and that the left and right unicellular shift operators, denoted L, R and defined by $L(e_n) = e_{n-1}$ and $R(e_n) = e_{n+1}$, relative to this basis are bounded. Then any subalgebra of $\mathcal{B}(\mathfrak{X})$ containing L and R is not of topologically bounded index.*

PROOF: Suppose the contrary and note that $LR = 1$ but $Re_1L = 0$ so $RL \neq 1$. Then by the topological Jacobson theorem we have that $\|R^nL^n\|$ is not bounded. But since $R^nL^n = 1 - P_n$ we have

$$\|R^nL^n\| \leqslant 1 + \|P_n\|$$

and so $\|P_n\|$ is not bounded, contrary to the above theorem of Banach. ∎

We conclude with a discussion of a property that may be considered to be to topologically bounded index what topological nilpotence (of a Banach algebra A — as in [**4**]) is to uniformly topologically nil. Namely the condition that

$$\sup\left\{\left(\frac{\|x_1\cdots x_n\|}{\|x_1\|\cdots\|x_n\|}\right)^{1/n} : x_1,\ldots,x_n \in S,\ S\subseteq T(A) \text{ is a semigroup}\right\} \to 0. \tag{5}$$

It is fairly easy to see that (5) implies topologically bounded index. Indeed we only know of one example (due to Dixon and Müller in [**5**]) of a Banach algebra which is of topologically bounded index but does not satisfy (5). Also (5) coincides with topological nilpotence for radical Banach algebras, and with topologically bounded index for commutative Banach algebras. The following suggests that it is better behaved.

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Proposition 4.5. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space. Then $\mathcal{B}(\mathfrak{X})$ does not satisfy (5).*

Proof: Let n be fixed and take Y to be an $n+1$ dimensional subspace of $\mathfrak{X}$. By a result of Auerbach we can find linearly independent vectors $e_1, \dots, e_{n+1} \in Y$ and $f_1, \dots, f_{n+1} \in Y^*$ all of unit norm and such that

$$f_i(e_j) = \delta_{i,j} \qquad (i, j = 1, \dots, n+1).$$

Using the Hahn-Banach theorem we may extend each f_i to a linear functional (also denoted by f_i) on $\mathfrak{X}$ with the same (unit) norm. Now define, for $i = 1, \dots, n$

$$\begin{aligned} T_i : \mathfrak{X} &\longrightarrow \mathfrak{X} \\ x &\longmapsto f_i(x) e_{i+1} \end{aligned}$$

Note that

$$\|T_i x\| = |f_i(x)| \, \|e_{i+1}\| \leqslant \|f_i\| \, \|x\| = \|x\|$$

so that $\|T_i\| \leqslant 1$ and that

$$T_i T_j x = f_i(f_j e_{j+1}) e_{i+1} = f_i(e_{j+1}) f_j(x) e_{i+1}$$

so $T_i T_j = 0$ for $i \neq j+1$. Thus the semigroup S generated by $\{T_1, \dots, T_n\}$ consists of nilpotent operators. Now we have $T_1, \dots, T_n \in S \subseteq T(\mathcal{B}(\mathfrak{X}))$ and

$$T_n \cdots T_1 e_1 = e_{n+1}$$

so $\|T_n \cdots T_1\| = 1$. ∎

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