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ON CONTINUOUS IMAGES OF MOORE SPACES

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In [**4–8**], the author has obtained several counterexamples to conjectures involving chain conditions, normality conditions, completeness, and the existence of point countable bases in Moore spaces. Each of these examples was obtained by constructing, by various means, a Moore space based on another space X_0. In this paper, the author unifies these construction techniques and states some of the relationships between the original spaces and the derived Moore spaces. It is hoped that this discussion will prove useful for obtaining other counterexamples. In addition, the author investigates the mapping properties of these constructions and obtains two new results of unexpected generality. The first result shows that a well known theorem of Stone in [**10**] can not be extended from metrizable spaces to Moore spaces and the second answers a question raised by Arhangel'skiĭ in [**1**].

A. H. Stone in [**10**] proved the following three results which have been the inspiration of a great deal of work: (all spaces are to be T_1)

(1) The image of a metrizable space under a closed continuous map is metrizable if it is first countable.

(2) The image of a metrizable space under a continuous map that is both open and closed is metrizable.

(3) The regular image of a locally separable metrizable space under an open continuous mapping such that point inverses are separable is locally separable and metrizable.

In [**11**], J. M. Worrell showed that results (1) and (2) hold true if "metrizable space" is replaced by "Moore space" in both the hypothesis and conclusion. In part I, the author shows that result (3) can not be extended to Moore spaces by establishing that each locally separable regular first countable T_1-space is the open countable-to-one continuous image of a locally separable Moore space.

In [**1**], A. Arhangel'skiĭ raised the following question: If X is a completely regular space and there exists a space Y with a countable base such that Y is the image of X under a continuous open map f such that $f^{-1}(y)$ is a compactum (i.e., compact metrizable space) for each $y \in Y$, must X have a countable network? In part II, the author shows that each uncountable separable metrizable space is the image of a completely regular, separable, non-metrizable Moore space under a continuous, open, countable-to-one map by which point inverses are compact. And since each separable, non-metrizable

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Moore space does not have a countable network and each compact Moore space is a compactum [**3**, pp. 9, 10, and 380], Arhangel'skiĭ's question is answered in the negative.

Preliminaries. By a development for a T_1-space S is meant a sequence $G_1, G_2, \ldots$ of open coverings of S such that (1) $G_{i+1} \subset G_i$ for each i, and (2) if $p \in S$ and p is contained in the open set D, then there exists an n such that each element of G_n containing p is contained in D. A regular T_1-space having a development is a Moore space. A Moore space is complete provided it has a complete development, i.e., a development $G_1, G_2, \ldots$ such that if for each i, M_i is a closed point set contained in an element of G_i and $M_{i+1} \subset M_i$, then $\bigcap_{i=1}^{\infty} M_i \neq \emptyset$ [**3**]. A Moore space is completable provided it can be embedded in a complete Moore space.

A space has the countable chain condition (CCC) [discrete countable chain condition (DCCC)] provided each [discrete] collection of mutually exclusive open sets is countable. A T_1-space S has a G_δ-diagonal [regular G_δ-diagonal] provided the diagonal in $S \times S$ is a G_δ-set [regular G_δ-set]. Each Moore space has a G_δ-diagonal and continuously semi-metrizable or submetrizable Moore spaces have regular G_δ-diagonals [**12**; **2**; **9**].

1. Moore spaces based on first countable T_2 and T_3-spaces. The author will now construct a Moore space X based on an arbitrary first countable T_2-space and a more complicated Moore space S based on an arbitrary first countable T_3-space. The interesting feature of these constructions is that the Moore spaces so constructed are almost always non-metrizable.

Construction of X. Let X_0 be a first countable T_2-space. For each $x \in X_0$, denote by $u_1(x), u_2(x), \ldots$ a non-increasing sequence of open sets in X_0 which forms a local base at x in X_0. Now denote by S_1 a copy of X_0 and for each i, denote by $S_{(1,i)}$ a unique copy of X_0 distinct from S_1 such that all copies are pairwise disjoint. Let $X = S_1 \cup (\bigcup_{i=1}^{\infty} S_{(1,i)})$ and for each $p \in X$, denote by x_p the element of X_0 which is identified with p. If $p \in X$ and j is a positive integer, define $g_j(p)$ as follows: (1) If $p \in S_{(1,i)}$ for some i, $g_j(p) = \{p\}$; (2) If $p \in S_1$, let

$$g_j(p) = \{p\} \cup \{q \in S_{(1,i)} | i \geqq j \quad \text{and} \quad x_q \in u_i(x_p) \text{ in } X_0\}.$$

It follows that $B = \{g_j(p) | p \in X \text{ and } j \text{ is a positive integer}\}$ is a basis for a topology on X and that $G_1, G_2, \ldots$, where for each n, $G_n = \{g_j(p) | p \in X$ and $j \geqq n\}$ is a development for the completely regular, complete Moore space X. Note that basic open sets are both open and closed and S_1 is a closed subset of S which has no limit point.

Properties of X. In [**7**], it was noted that if X_0 is the Michael line, Sorgenfrey line, the real line, or the space of countable ordinals with the order topology, then the associated Moore space X is not normal. In fact, the existence of two subsets H and K in X_0 such that K is uncountable and K is not the union of countably many subsets each of which has no limit point in H ensures that X will not be normal. Observe also that if X_0 does not have a G_δ-diagonal then X does not have a regular G_δ-diagonal.

The map from X onto X_0. Consider the natural map f from X onto X_0 such that if $p \in X$, then $f(p) = x_p$ in X_0. It is easily verified that f is continuous, countable-to-one, and point inverses are compact.

The following construction is an "infinite dimensional" expansion of the space X, i.e., countably infinitely copies of X_0 are constructed over each $S_{(1,i)}$, then countably infinitely many copies over each of these and continue the process countably infinitely many times. Finally, let S be their union and define a topology on S by constructing "cones" at each level extending up through the dimensions making each copy of X_0 have similar properties to that of S_1 in X. However, regularity of X_0 is required to ensure the regularity of S.

Construction of S. Let X_0 be a regular first countable T_1-space. For each $x \in X_0$, denote by $u_1(x), u_2(x), \ldots$ a sequence of open sets in X_0 which forms a local base at x such that for each i, $\overline{u_{i+1}(x)} \subset u_i(x)$. Now for each positive integer m, let $A_m = \{(n_1, n_2, \ldots, n_m) | n_1 = 1$ and for $1 \leqq i \leqq m$, n_i is a positive integer$\}$. Let $A = \bigcup_{m=1}^{\infty} A_m$. For each $a = (n_1, n_2, \ldots, n_m) \in A$, denote by S_a a unique copy of X_0 such that all copies are pairwise disjoint. And for each $x \in X_0$, denote by $(x_{n_1}, x_{n_2}, \ldots, x_{n_m})$ the element of S_a which is identified with x. Let $S = \bigcup\{S_a | a \in A\}$ and define a development for S as follows: For each positive integer j, $a = (n_1, n_2, \ldots, n_m) \in A$, and $p = (y_{n_1}, y_{n_2}, \ldots, y_{n_m}) \in S_a$, let

$$g_j(p) = \{p\} \cup \{(x_{n_1}, x_{n_2}, \ldots, x_{n_m}, x_{k_1}, x_{k_2}, \ldots, x_{k_c}) | x \in X_0, c \text{ is a positive integer, for } 1 \leqq i \leqq c, k_i \geqq j, \text{ and } x \in u_{k_1+j}(y) \text{ in } X_0\}.$$

It follows that $B = \{g_j(p) | p \in S$ and j is a positive integer$\}$ is a basis for a topology on S and that $G_1, G_2, \ldots$, where for each n, $G_n = \{g_j(p) | p \in S$ and $j \geqq n\}$, is a development for the Moore space S. Note that the space X previously constructed is a closed subset of S.

Properties of S. X_0 (i) is separable, (ii) is locally separable, (iii) has the countable chain condition, or (iv) has the discrete countable chain condition, if and only if S has the corresponding property. Thus, if X_0 is uncountable and separable, then S is a separable Moore space which contains an uncountable point set $S_{(1)}$ with no limit point and hence is not metrizable. Also, since the space X is a subspace of S, the properties listed for X hold also for S. However, although X is always complete, the space S is often non-completable. The author noted in [**7**], that if X_0 is the space of countable ordinals or the Michael line, then S is non-completable.

The map from S onto X_0. Consider the natural map f from S onto X_0 such that if $p = (x_{n1}, x_{n2}, \ldots, x_{nm}) \in S$, then $f(p) = x$ in X_0. It is easily verified that f is continuous, countable-to-one, and open.

Thus, the following theorem is established.

THEOREM 1. *Each locally separable regular first countable T_1-space X_0 is the open, continuous, countable-to-one image of a locally separable Moore space S.*

Thus, since there are many examples of regular, locally separable, first countable T_1-spaces, which are not Moore spaces, Stone's result (3) does not extend to Moore spaces. This is somewhat surprising since results (1) and (2) do extend to Moore spaces and it is easily seen that regular images of Moore spaces under open finite-to-one continuous maps are Moore spaces.

2. Moore spaces based on other Moore spaces.

Construction of Z. Let X_0 be a Moore space. Denote by $U_1, U_2, \ldots$ a development for X_0. For each $x \in X_0$, denote by $u_1(x), u_2(x), \ldots$ a sequence of open sets in X_0 such that for each i, $u_i(x) \in U_i$ and $\overline{u_{i+1}(x)} \subset u_i(x)$. Now, denote by T_1 a copy of X_0 and for each i, denote by $T_{(1,i)}$ a unique copy of X_0 distinct from T_1. And for each $x \in X_0$ and each positive integer i, denote by x_1 and $x_{(1,i)}$, the elements of T_1 and $T_{(1,i)}$ respectively which are identified with x. Let $Z = T_1 \cup (\bigcup_{i=1}^{\infty} T_{(1,i)})$ and define a development for Z as follows:

(1) If j is a positive integer and $p = x_{(1,i)} \in T_{(1,i)}$ for some i, let $g_j(p) = \{q = y_{(1,i)} \in T_{(1,i)} | y \in u_j(x) \text{ in } X_0\}$; (2) If j is a positive integer and $p = x_1 \in T_1$, let $g_j(p) = \{p\} \cup \{q = y_{(1,i)} \in T_{(1,i)} | i \geqq j \text{ and } y \in u_{i+j}(x) \text{ in } X_0\}$.

It follows that $B = \{g_j(p) | p \in Z \text{ and } j \text{ is a positive integer}\}$ is a basis for a topology on Z and that $G_1, G_2, \ldots$, where for each n, $G_n = \{g_j(p) | p \in Z$ and $j \geqq n\}$ is a development for the Moore space Z. Note that each $T_{(1,i)}$ is homeomorphic to X_0 and T_1 is a subset of Z which has no limit point.

Properties of Z. Again, the existence of two subsets H and K in X_0 such that K is uncountable and is not the union of countably many subsets each of which has no limit point in H ensures that Z will not be normal. And X_0 (i) is separable, (ii) is locally separable, (iii) has the countable chain condition, (iv) has the discrete countable chain condition, and (v) is complete if and only if Z has the corresponding property. Thus, as noted before, if X_0 is separable and uncountable, then Z is not metrizable.

Also, if X_0 is completely regular, then Z can be constructed so as to be completely regular by the following modification: For each $x \in X_0$, denote by $u_1(x), u_2(x), \ldots$, a sequence of open sets in X_0 such that for each i, $u_i(x) \in U_i$, $u_{i+1}(x) \subset u_i(x)$ and there exists a continuous map f_i^x from X_0 into $[0, 1]$ such that (i) $f_i^x(x) = 0$, (ii) $f_i^x(X_0 - u_i(x)) = 1$, and (iii) if $y \in u_{i+1}(x)$, then $f_i^x(y) < 1/i$.

The map from Z onto X_0. Consider the natural map f from Z onto X_0 such that (1) if $p = x_{(1,i)} \in T_{(1,i)}$ for some i, then $f(p) = x$ in X_0 and (2) if $p = x_1 \in T_1$, then $f(p) = x$ in X_0. It is easily verified that f is open, continuous, countable-to-one and point inverses are compact.

Thus, the following theorem is established.

THEOREM 2. *Each uncountable separable metrizable space X_0 is the image of a completely regular, separable, non-metrizable Moore space Z under a continuous, open, countable-to-one map by which point inverses are compact.*

Thus, since a non-metrizable Moore space cannot have a countable network, Arhangel'skiĭ's question in [**1**] is answered in the negative.

Added in proof. The author has recently learned that there exist earlier, although less general, solutions to Arhangel'shiĭ's question than the one given by Theorem 2. See T. Przymusiński, Colloq. Math. *24* (1972), 175–180, for references.

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