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TYPES OVER $C(K)$ SPACES

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Abstract

Let K be a compact Hausdorff space and $C(K)$ the Banach space of all real-valued continuous functions on K, with the sup norm. Types over $C(K)$ (in the sense of Krivine and Maurey) are represented here by pairs (l, u) of bounded real-valued functions on K, where l is lower semicontinuous and u is upper semicontinuous, $l \leq u$ and $l(x) = u(x)$ for every isolated point x of K. For each pair the corresponding type is defined by the equation $\tau(g) = \max\{\|l + g\|_\infty, \|u + g\|_\infty\}$ for all $g \in C(K)$, where $\|\cdot\|_\infty$ is the sup norm on bounded functions. The correspondence between types and pairs (l, u) is bijective.

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1. Statement of the Main Theorem

The concept of type over a Banach space E was first introduced by Krivine and Maurey [5] in the context of separable Banach spaces. The reader is referred to Garling's monograph [2] for more details. We consider types over general, not necessarily separable Banach spaces.

Let E be a Banach space. For every $x \in E$, we define a function $\tau_x : E \to \mathbb{R}$ by letting $\tau_x(y) = \|x + y\|$ for all $y \in E$.

DEFINITION 1.1. A function $\tau : E \to \mathbb{R}$ is a *type over* E if τ is in the closure (with respect to the topology of pointwise convergence) of the set $\{\tau_x : x \in E\}$.

Throughout we take K to be a compact Hausdorff topological space. We let $\ell_\infty(K)$ denote the Banach lattice of bounded real-valued functions on K equipped with the sup-norm. For $f, g \in \ell_\infty(K)$ the lattice ordering is defined pointwise.

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DEFINITION 1.2. An *sc pair* (*semicontinuous pair*) is a pair of functions (l, u) from $\ell_\infty(K)$ such that l is lower semicontinuous, u is upper semicontinuous, $l \leq u$, and $l(x) = u(x)$ for all isolated points x of K.

This paper is devoted to proving the following theorem:

THEOREM 1.3 (Characterization of types over $C(K)$).

(i) *Let τ be a type over $C(K)$. There exists an sc pair (l, u) such that*

$$\tau(g) = \max\{\|l+g\|, \|u+g\|\} \quad \textit{for all } g \in C(K). \tag{1.1}$$

(ii) *Let (l, u) be an sc pair. Then the function $\tau : C(K) \to \mathbb{R}$ defined by* (1.1) *is a type over $C(K)$.*

(iii) *The correspondence between types over $C(K)$ and sc pairs is bijective.*

A special case of (i) was observed by Haydon and Maurey [3]. The proof of Theorem 1.3 is provided in Section 4.

2. Preliminaries

The purpose of this section is to introduce the notation and concepts that will be used in the proof of Theorem 1.3.

A special type of nets and their convergence are used to generalize sequences.

DEFINITION 2.1. (i) Let I be a nonempty set which is partially ordered by $\leq$. In this paper, $(I, \leq)$ is a *net* if

(a) $(I, \leq)$ has no maximal element;
(b) for every element $\alpha \in I$, the set $\{\beta \in I : \beta \leq \alpha\}$ of *predecessors* of α is finite;
(c) for any $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. Such an element γ is called a *successor* of α (and β).

(ii) Let $(I, \leq)$ be a net. For every element $\alpha_0 \in I$, define the number of its predecessors by $|\alpha_0| = \operatorname{card}(\{\alpha \in I : \alpha \leq \alpha_0\})$.

(iii) Let $(I, \leq)$ and $(J, \leq)$ be nets. A function $k : I \to J$ is *order-preserving* if $\alpha \leq \beta \in I$ implies $k(\alpha) \leq k(\beta)$. A function $k : I \to J$ is *cofinal* if for every $\gamma \in J$ there exists $\alpha \in I$ such that $\gamma \leq k(\alpha)$.

(iv) A *subnet* of I is a cofinal order-preserving function $j : I \to I$.

(v) Let $(I, \leq)$ be a net and K be a topological space. We say that $(x_\alpha)_{\alpha \in I}$ is a *net in K indexed by I* if $x_\alpha \in K$ for all $\alpha \in I$. If K is a normed space then $(x_\alpha)_{\alpha \in I}$ is *bounded* if $\{\|x_\alpha\| : \alpha \in I\}$ is bounded in $\mathbb{R}$.

(vi) Let $(I, \leq)$ be a net, K a topological space and $(x_\alpha)_{\alpha\in I}$ a net in K indexed by I. Let $x \in K$. Then $\lim_{\alpha,I} x_\alpha = x$ if and only if for every neighbourhood U of x in K there exists $\alpha \in I$ such that $x_\beta \in U$ for all $\beta \geq \alpha$.

(vii) Let $(I, \leq)$ be a net and $(r_\alpha)_{\alpha\in I}$ a bounded net of real numbers. Then we define

$$\limsup_{\alpha,I} r_\alpha = \inf_{\alpha\in I} \sup\{r_\beta : \beta \in I \text{ and } \beta \geq \alpha\}$$

and

$$\liminf_{\alpha,I} r_\alpha = \sup_{\alpha\in I} \inf\{r_\beta : \beta \in I \text{ and } \beta \geq \alpha\}.$$

Throughout this paper, $(I, \leq)$ will denote a net in the sense of Definition 2.1.

The following proposition is immediate from Definitions 1.1 and 2.1; see [6] for more equivalent conditions and a detailed proof.

PROPOSITION 2.2. *Let E be a Banach space and $\tau : E \to \mathbb{R}$ a function. Then the following are equivalent*:

(i) *τ is a type over E.*

(ii) *For every finite subset $\alpha \subseteq E$ and every $\varepsilon > 0$, there exists an element $x = x(\alpha, \varepsilon) \in E$ such that $|\tau(y) - \|x + y\|| < \varepsilon$ for all $y \in \alpha$.*

(iii) *There exists a bounded net $(x_\alpha)_{\alpha\in I}$ in E indexed by I such that for all $y \in E$, $\lim_{\alpha,I} \|x_\alpha + y\| = \tau(y)$.*

If τ is a type over E and $(x_\alpha)_{\alpha\in I}$ is as in (iii) above, we say that $(x_\alpha)_{\alpha\in I}$ *generates* the type τ.

A subset $H \subseteq \ell_\infty$ is called *bounded* if $\sup\{\|f\| : f \in H\} < \infty$. Let H be such a set. The pointwise supremum of H is the real-valued function L defined by $L(x) = \sup\{h(x) : h \in H\}$ for every $x \in K$. We write $L = \bigvee H$ for this function. Similarly, the pointwise infimum of H is the real-valued function U defined by $U(x) = \inf\{h(x) : h \in H\}$ for every $x \in K$. This function is denoted by $\bigwedge H$. Note that both $\bigvee H$ and $\bigwedge H$ are again in $\ell_\infty(K)$.

If $H \subseteq \ell_\infty(K)$ is a bounded set of upper semicontinuous (usc) functions, then the pointwise infimum $\bigwedge H$ is usc. Similarly the pointwise supremum of a bounded set of lower semicontinuous (lsc) functions is lsc. Finally, it is clear that $f \in \ell_\infty(K)$ is continuous if and only if f is usc and lsc. Therefore, if H is a bounded set of continuous functions on K, then $\bigwedge H$ is usc and $\bigvee H$ is lsc.

3. Lemmas

This section provides lemmas and technical definitions that will be used in the proof of Theorem 1.3.

DEFINITION 3.1. For any norm-bounded net $(f_\alpha)_{\alpha\in I}$ in $C(K)$, define lower semicontinuous functions on K by setting, for every $\alpha \in I$,

$$l_\alpha = \bigvee \left\{ f \in C(K) : f \le \bigwedge_{\beta\ge\alpha} f_\beta \right\} \quad \text{and} \quad l = \bigvee_\alpha l_\alpha .$$

Similarly, define upper semicontinuous functions u and u_α on K by setting, for every $\alpha \in I$,

$$u_\alpha = \bigwedge \left\{ f \in C(K) : f \ge \bigvee_{\beta\ge\alpha} f_\beta \right\} \quad \text{and} \quad u = \bigwedge_\alpha u_\alpha .$$

All statements about l and l_α also hold for u and u_α, provided all inequalities are reversed, suprema are replaced by infima, minus by plus, etc.

Here are some basic properties of the functions l and u defined in Definition 3.1:

REMARK 3.2. Let $(f_\alpha)_{\alpha\in I}$ be a bounded net of functions and let l_α, l, u_α and u be as in Definition 3.1.

(i) If $\alpha_1, \alpha_2 \in I$ and $\alpha_1 \le \alpha_2$, then $l_{\alpha_1} \le l_{\alpha_2} \le l$ and $u_{\alpha_1} \ge u_{\alpha_2} \ge u$.

(ii) If $x \in K$ and $\varepsilon > 0$, then there exists an $\alpha_0 = \alpha(x, \varepsilon) \in I$ such that, for all indices $\alpha > \alpha_0$, $l_\alpha(x) \ge l(x) - \varepsilon$ and $u_\alpha(x) \le u(x) + \varepsilon$.

(iii) For every $\beta \in I$, every $x \in K$, every $\delta > 0$, and every neighbourhood U of x, there exists $y \in U$ and $\gamma \ge \beta$ such that $f_\gamma(y) \le l_\beta(x) + \delta$.

(iv) For every $\beta \in I$, every $x \in K$, every $\delta > 0$, and every neighbourhood U of x, there exists $y \in U$ and $\gamma \ge \beta$ such that $f_\gamma(y) \ge u_\beta(x) - \delta$.

PROOF. The statements in (i) and (ii) are immediate from the definition. We prove (iii): let $\beta \in I$, $x \in K$ and set $s = l_\beta(x)$; let U be an open neighbourhood of x and $\delta > 0$. Suppose the conclusion does not hold. Then for all $y \in U$ and all $\gamma \ge \beta$, we have $f_\gamma(y) > s + \delta$. But for every $y \in U$, there exists $f \in C(K)$ (which depends on y) such that $f \le f_\gamma$ for all $\gamma \ge \beta$ and such that $f(y) \ge s + \delta$. Then $l_\beta(y) \ge s + \delta$ for all $y \in U$, which is a contradiction. The statement in (iv) is proved with an argument dual to the one just given. $\square$

From now on we assume that $\mathrm{card}(I) \ge \kappa$, where κ is the minimum of the cardinalities of the bases of the topology of K.

LEMMA 3.3. *Let $(f_\alpha)_{\alpha\in I}$ be a bounded net of continuous functions and let l and u be as in Definition* 3.1. *Given $g \in C(K)$ and $x \in K$,*

(i) *there exists a subnet $i : I \to I$ and elements $(x_{i(\alpha)})_{\alpha\in I}$ converging to x, such that $l(x) + g(x) \ge \lim_{\alpha,I} \left(f_{i(\alpha)}(x_{i(\alpha)}) + g(x_{i(\alpha)})\right)$;*

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(ii) *there exists a subnet* $j : I \to I$ *and elements* $(x_{j(\alpha)})_{\alpha\in I}$ *converging to* x, *such that* $u(x) + g(x) \leq \lim_{\alpha,I} \left(f_{j(\alpha)}(x_{j(\alpha)}) + g(x_{j(\alpha)})\right)$.

PROOF. We show (i). For every $\alpha \in I$, let $\varepsilon_\alpha = |\alpha|^{-1}$. By assumption, every $\alpha \in I$ has only finitely many predecessors and infinitely many successors, so every ε_α is defined and $\lim_{\alpha,I} \varepsilon_\alpha = 0$. Fix a system of open neighbourhoods $(U_\alpha)_{\alpha\in I}$ of x such that for all $\alpha \leq \beta \in I$ we have $U_\alpha \supseteq U_\beta$ and $\bigcap_{\alpha\in I} U_\alpha = \{x\}$. Furthermore, assume that $|g(x) - g(y)| < \varepsilon_\alpha/2$ for all $y \in U_\alpha$.

We proceed by induction on $\alpha \in I$. Fix $\alpha \in I$ and suppose that $i(\beta)$ has been defined for all $\beta < \alpha$. Using the fact that there are only finitely many such β's we may find $\alpha_0 \in I$ such that $\alpha_0 \geq i(\beta)$ for all $\beta < \alpha$. We may assume (by (ii) of Remark 3.2) that $l(x) \geq l_\gamma(x) \geq l(x) - \varepsilon_\alpha/2$ for all $\gamma > \alpha_0$. By Remark 3.2 (iii) there exists $i(\alpha) \geq \alpha_0$ and $x_{i(\alpha)} \in U_\alpha$ such that $f_{i(\alpha)}(x_{i(\alpha)}) \leq l_{\alpha_0}(x) + \varepsilon_\alpha/2$. Remark 3.2 (i) gives

$$f_{i(\alpha)}(x_{i(\alpha)}) \leq l_{\alpha_0}(x) + \varepsilon_\alpha/2 \leq l(x) + \varepsilon_\alpha/2.$$

By construction, $i : I \to I$ is a cofinal order-preserving function. Obviously $(x_{i(\alpha)})_{\alpha\in I}$ converges to x and

$$f_{i(\alpha)}(x_{i(\alpha)}) + g(x_{i(\alpha)}) \leq l(x) + g(x) + \varepsilon_\alpha.$$

By passing to a further subnet we may assume that $\lim_{\alpha,I} f_{i(\alpha)}(x_{i(\alpha)})$ exists and

$$\lim_{\alpha,I}(f_{i(\alpha)}(x_{i(\alpha)}) + g(x_{i(\alpha)})) \leq l(x) + g(x).$$

A dual argument proves (ii). □

LEMMA 3.4. *Let* $(f_\alpha)_{\alpha\in I}$ *be a bounded net of continuous functions and let* l *and* u *be as in Definition* 3.1. *If* $g \in C(K)$ *and* $\lim_{\alpha,I} \|f_\alpha + g\|$ *exists, then*

$$\lim_{\alpha,I} \|f_\alpha + g\| \geq \max\{\|l + g\|, \|u + g\|\}.$$

PROOF. First observe that for any sc pair (l, u) and any $g \in C(K)$

$$\begin{aligned} \text{(3.1)} \qquad & \max\{\|l + g\|, \|u + g\|\} \\ & = \sup\left(\{u(x) + g(x) : x \in K\} \cup \{-l(x) - g(x) : x \in K\}\right). \end{aligned}$$

Let $x \in K$ be arbitrary. Applying Lemma 3.3 we obtain a cofinal order-preserving map $i : I \to I$ and elements $(x_{i(\alpha)})_{\alpha\in I}$ which converge to x such that

$$\lim_{\alpha,I} \left(f_{i(\alpha)}(x_{i(\alpha)}) + g(x_{i(\alpha)})\right) \leq l(x) + g(x).$$

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Therefore,

$$\lim_{\alpha,I} \|f_\alpha + g\| = \lim_{\alpha,I} \|f_{i(\alpha)} + g\| \geq \lim_{\alpha,I} \left(-f_{i(\alpha)}(x_{i(\alpha)}) - g(x_{i(\alpha)})\right)$$
$$\geq -(l(x) + g(x)).$$

A dual argument shows that $\lim_{\alpha,I} \|f_\alpha + g\| \geq u(x) + g(x)$. Applying (3.1) to these two inequalities gives the conclusion of the lemma. □

The next lemma shows that the reverse inequality also holds:

LEMMA 3.5. *Let $(f_\alpha)_{\alpha\in I}$ be a bounded net of continuous functions and let l and u be as in Definition* 3.1. *Let $g \in C(K)$, and suppose that* $\lim_{\alpha,I} \|f_\alpha + g\|$ *exists. Then*

$$\lim_{\alpha,I} \|f_\alpha + g\| \leq \max\{\|l + g\|, \|u + g\|\}.$$

PROOF. Let $r = \lim_{\alpha,I} \|f_\alpha + g\|$. For each $\alpha \in I$, choose $x_\alpha \in K$ and $s_\alpha = \pm 1$ such that $\|f_\alpha + g\| = s_\alpha(f_\alpha(x_\alpha) + g(x_\alpha))$. Using the compactness of K there exists a cofinal order-preserving map $j : I \to I$ and a constant $s = \pm 1$, such that

$$\lim_{\alpha,I} x_{j(\alpha)} = x \quad \text{and} \quad s_{j(\alpha)} = s \quad \text{for all } \alpha \in I.$$

Then $r = \lim_{\alpha,I} s(f_{j(\alpha)}(x_{j(\alpha)}) + g(x_{j(\alpha)}))$. We distinguish between two cases:

Case 1: $s = 1$. Fix $\beta \in I$. Then

$$r = \lim_{\alpha,I}(f_{j(\alpha)}(x_{j(\alpha)}) + g(x_{j(\alpha)})) = \lim_{\alpha,I;j(\alpha)\geq\beta}(f_{j(\alpha)}(x_{j(\alpha)}) + g(x_{j(\alpha)}))$$
$$\leq \limsup_{\alpha,I;j(\alpha)\geq\beta} (u_\beta(x_{j(\alpha)}) + g(x_{j(\alpha)})) \leq u_\beta(x) + g(x).$$

The last inequalities follow since $f_{j(\alpha)}(x_{j(\alpha)}) \leq u_\beta(x_{j(\alpha)})$ for $\beta \leq j(\alpha)$ and since u_β is usc. We obtain, using Remark 3.2 (ii),

$$r \leq u(x) + g(x) \leq \|u + g\|.$$

Case 2: $s = -1$. Using the same ideas as in Case 1, we show that

$$r \leq -(l(x) + g(x)) \leq \|l + g\|,$$

which gives $\lim_{\alpha,I} \|f_\alpha + g\| \leq \max\{\|l + g\|, \|u + g\|\}$. □

We will need the following theorem.

THEOREM 3.6 (Edwards [1]). *Let U be a usc function and L an lsc function on a compact Hausdorff space K, such that $U \leq L$. Then there exists a continuous function F such that $U \leq F \leq L$.*

A proof of this theorem can be found in Kaplan [4, (48.5)].

As a consequence, we obtain the following lemma:

LEMMA 3.7. *Let K be a compact Hausdorff topological space, and let $\mathfrak{W}$ be a finite open cover of K. Let $u : K \to \mathbb{R}$ be any bounded function. Then $L : K \to \mathbb{R}$ defined by*

$$L(y) = \sup\left\{u(z) : z \in \bigcap_{y\in W;\, W\in\mathfrak{W}} \overline{W}\right\}$$

for all $y \in K$ is lsc and $L \geq u$. Similarly, if $l : K \to \mathbb{R}$ is any bounded function and $U : K \to \mathbb{R}$ is defined by

$$U(y) = \inf\left\{l(z) : z \in \bigcap_{y\in W;\, W\in\mathfrak{W}} \overline{W}\right\}$$

for all $y \in K$, then U is usc and $U \leq l$.

PROOF. We only show the first statement. Observe that $L(y) \geq L(w)$ for all $w \in \bigcap_{y\in W;\, W\in\mathfrak{W}} \overline{W}$. Observe that there are only finitely many sets of the form $\bigcap_{x\in W;\, W\in\mathfrak{W}} \overline{W}$. Therefore

$$\{y \in K : L(y) \leq r\} = \bigcup_{x\in K; L(x)\leq r} \left(\bigcap_{x\in W;\, W\in\mathfrak{W}} \overline{W}\right)$$

is a finite union of closed sets, hence closed. So L is lsc. It is immediate from the definition of L that $L \geq u$. □

4. Proof of Theorem 1.3

The statement of Theorem 1.3 is repeated in the form of propositions for the convenience of the reader.

PROPOSITION 4.1. *Let τ be a type over $C(K)$. Then there exists an sc pair (l, u) such that*

$$\tau(g) = \max\{\|l+g\|, \|u+g\|\} \quad \text{for every } g \in C(K). \tag{4.1}$$

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PROOF. Given a type τ over $C(K)$ fix a net $(f_\alpha)_{\alpha\in I}$ which generates τ as in Proposition 2.2 (iii). Let l and u be obtained from this net as in Definition 3.1. Lemmas 3.4–3.5 prove (4.1).

We now show that (l, u) is an sc pair. It is immediate from Definition 3.1 that l is lsc, u is usc and $l \le u$. Suppose that x is isolated. By Remark 3.2 (iii)–(iv) applied to $U = \{x\}$ we obtain

$$\liminf_{\alpha,I} f_\alpha(x) \le l(x) \le u(x) \le \limsup_{\alpha,I} f_\alpha(x). \tag{4.2}$$

Let $r = 3\sup\{\|f_\alpha\| : \alpha \in I\}$ and define $g \in C(K)$ by setting

$$g(y) = \begin{cases} 0 & \text{if } y \ne x \\ r & \text{if } y = x. \end{cases}$$

Then $\tau(g) = \lim_{\alpha,I} \|f_\alpha + g\| = \lim_{\alpha,I} f_\alpha(x) + r$. Thus $\lim_{\alpha,I} f_\alpha(x)$ exists. Therefore, (4.2) yields $l(x) = u(x)$. $\square$

PROPOSITION 4.2. *Let (l, u) be an sc pair. Then the function $\tau : C(K) \to \mathbb{R}$ defined by* (4.1) *is a type over $C(K)$.*

PROOF. Let (l, u) be an sc pair on K and let $\tau : C(K) \to \mathbb{R}$ be defined by $\tau(g) = \max\{\|l + g\|, \|u + g\|\}$ for all $g \in C(K)$.

We use Proposition 2.2 to prove that τ is a type over $C(K)$. It suffices to show that for all $n \in \mathbb{N}$, all $g_1, \dots, g_n \in C(K)$ and all $\varepsilon > 0$ there exists $F \in C(K)$ such that $|\tau(g_i) - \|F + g_i\|| \le \varepsilon$ for all $1 \le i \le n$.

Fix $g_1, \dots, g_n \in C(K)$ and $\varepsilon > 0$. Choose a finite open cover $\mathfrak{W}$ of K, such that for all $W \in \mathfrak{W}$, all $x, y \in W$ and all $1 \le i \le n$ we have $|g_i(x) - g_i(y)| < \varepsilon/2$.

Define $L : K \to \mathbb{R}$ and $U : K \to \mathbb{R}$ by setting for all $y \in K$

$$L(y) = \sup\left\{u(z) : z \in \bigcap_{y\in W;\, W\in\mathfrak{W}} \overline{W}\right\} \quad \text{and} \quad U(y) = \inf\left\{l(z) : z \in \bigcap_{y\in W;\, W\in\mathfrak{W}} \overline{W}\right\}.$$

The function L is lsc and U is usc by Lemma 3.7. By Theorem 3.6 there exists a continuous function $f \in C(K)$ such that $U \le f \le L$. Using (3.1) we may choose a finite set $S \subseteq K$ such that for all $1 \le i \le n$

$$\max\{\|l + g_i\|, \|u + g_i\|\} = \max\{-[l(z) + g_i(z)], (u(z) + g_i(z)) : z \in S\}.$$

We write $S = \{z_p, \dots, z_q\}$, where $p, q \in \mathbb{Z}$, $p \le 0 \le q$, the points $(z_j)_{j=p}^{q}$ are pairwise distinct and for all $p \le j \le q$, z_j is isolated in K if and only if $p \le j < 0$.

For each $0 \leq j \leq q$, we choose disjoint sets V_{2j}, $V_{2j+1} \subseteq \bigcap_{z_j \in W; W \in \mathfrak{W}} W$ such that $x \in V_{2j} \cup V_{2j+1}$ implies $|f(x) - f(z_j)| < \varepsilon/2$. Further, we may assume that for all $0 \leq k \leq 2q+1$ and all $x, y \in V_k$ we have $|f(x) - f(y)| < \varepsilon/2$.

Using Urysohn's Lemma we now choose continuous functions $(f_k)_{k=p}^{2q+1}$ satisfying the following conditions: For all $p \leq j < 0$, choose f_j such that

$$f_j|_{K \setminus \{z_j\}} = 0 \quad \text{and} \quad f_j(z_j) = u(z_j) - f(z_j) = l(z_j) - f(z_j).$$

For all $0 \leq j \leq q$, choose $f_{2j} \geq 0$ and $f_{2j+1} \leq 0$ such that

$$f_{2j}|_{K \setminus V_{2j}} = 0, \qquad \|f_{2j}\| = L(z_j) - f(z_j)$$

and

$$f_{2j+1}|_{K \setminus V_{2j+1}} = 0, \quad \|f_{2j+1}\| = f(z_j) - U(z_j).$$

We set $F = f + \sum_{k=p}^{2q+1} f_k$. We would like to show that for all $1 \leq i \leq n$,

$$|\max\{\|l + g_i\|, \|u + g_i\|\} - \|F + g_i\|| \leq \varepsilon,$$

namely

$$\max\{\|l + g_i\|, \|u + g_i\|\} - \varepsilon \leq \|F + g_i\| \leq \max\{\|l + g_i\|, \|u + g_i\|\} + \varepsilon.$$

We first show the right inequality: fix $1 \leq i \leq n$. Fix $x \in K$ arbitrary and observe that $-[F(x) + g_i(x)] \leq \|F + g_i\|$ and $F(x) + g_i(x) \leq \|F + g_i\|$. We distinguish among four cases:

Case 1: $x \notin \{z_p, \ldots, z_{-1}\} \cup \bigcup_{k=0}^{2q+1} V_k$. Then $F(x) = f(x)$. We may choose $y_1, y_2 \in \bigcap_{x \in W; W \in \mathfrak{W}} \overline{W}$ such that $l(y_1) = U(x)$ and $u(y_2) = L(x)$. These choices are possible because l is lsc (u is usc, respectively) and by definition of U (L, respectively). Then

$$\begin{aligned} -[F(x) + g_i(x)] &= -f(x) - g_i(x) \leq -U(x) - g_i(x) \\ &\leq -l(y_1) - g_i(y_1) + \varepsilon/2 \leq \|l + g_i\| + \varepsilon/2 \end{aligned}$$

and

$$\begin{aligned} F(x) + g_i(x) &= f(x) + g_i(x) \leq L(x) + g_i(x) \\ &\leq u(y_2) + g_i(y_2) + \varepsilon/2 \leq \|u + g_i\| + \varepsilon/2. \end{aligned}$$

Thus, $|F(x) + g_i(x)| \leq \max\{\|l + g_i\|, \|u + g_i\|\} + \varepsilon$.

Case 2: $x = z_j$ for some $p \leq j < 0$. Then $F(x) + g_i(x) = u(x) + g_i(x) = l(x) + g_i(x)$. Therefore $|F(x) + g_i(x)| \leq \max\{\|l + g_i\|, \|u + g_i\|\} + \varepsilon$.

Case 3: $x \in V_{2j}$ for some $0 \leq j \leq q$. Observe that $F|_{V_{2j}} = f|_{V_{2j}} + f_{2j}|_{V_{2j}}$ and $F|_{V_{2j}} \geq f|_{V_{2j}}$. There exists $y_1 \in V_{2j}$ such that $f_{2j}(y_1) = L(z_j) - f(z_j)$. Further, there

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exist $y_2 \in \bigcap_{x\in W;W\in\mathfrak{W}} \overline{W}$ such that $l(y_2) = U(x)$ and $y_3 \in \bigcap_{z_j\in W;W\in\mathfrak{W}} \overline{W}$ such that $u(y_3) = L(z_j)$. Then

$$\begin{aligned}-F(x) - g_i(x) &\le -f(x) - g_i(x) \le -U(x) - g_i(x)\\ &\le -l(y_2) - g_i(y_2) + \varepsilon/2 \le \|l + g_i\| + \varepsilon/2\end{aligned}$$

and

$$\begin{aligned}F(x) + g_i(x) &= f(x) + f_{2j}(x) + g_i(x) \le f(x) + f_{2j}(y_1) + g_i(x)\\ &= f(x) + L(z_j) - f(z_j) + g_i(x)\\ &\le u(y_3) + g_i(y_3) + \varepsilon \le \|u + g_i\| + \varepsilon.\end{aligned}$$

In this last inequality, we use the assumption that $f(x) - f(z_j) \le \varepsilon/2$ because $x, z_j \in V_{2j}$, and $g_i(x) - g_i(y_3) \le \varepsilon/2$ because $x, y_3 \in \bigcap_{z_j\in W;W\in\mathfrak{W}} \overline{W}$.

Therefore, $|F(x) + g_i(x)| \le \max\{\|l + g_i\|, \|u + g_i\|\} + \varepsilon$.

Case 4: $x \in V_{2j+1}$ for some $0 \le j \le q$. This case is handled similar to the treatment of Case 3.

Combining the results from Cases 1–4 we obtain

$$|F(x) + g_i(x)| \le \max\{\|l + g_i\|, \|u + g_i\|\} + \varepsilon$$

for all $x \in K$ and all $1 \le i \le n$. Therefore, $\|F + g_i\| \le \max\{\|l + g_i\|, \|u + g_i\|\} + \varepsilon$ for all $1 \le i \le n$.

We now show that $\|F + g_i\| \ge \max\{\|l + g_i\|, \|u + g_i\|\} - \varepsilon$. Fix $1 \le i \le n$. By construction there exists $z \in S$ such that

$$\max\{-[l(z) + g_i(z)], (u(z) + g_i(z))\} = \max\{\|l + g_i\|, \|u + g_i\|\}.$$

For this choice of z we distinguish between two cases:

Case 1′: $z = z_j$ for some $p \le j < 0$. Then

$$u(z_j) + g_i(z_j) = l(z_j) + g_i(z_j) = F(z_j) + g(z_j).$$

Therefore, $\max\{\|l + g_i\|, \|u + g_i\|\} = |F(z_j) + g_i(z_j)| \le \|F + g_i\|$.

Case 2′: $z = z_j$ for some $0 \le j \le q$. Then there exist $y_0 \in V_{2j}$ and $y_1 \in V_{2j+1}$ such that $f_{2j}(y_0) = L(z_j) - f(z_j)$ and $f_{2j+1}(y_1) = -f(z_j) + U(z_j)$. We then obtain

$$\begin{aligned}\|F + g_i\| &\ge F(y_0) + g_i(y_0) \ge f(y_0) + L(z_j) - f(z_j) + g_i(z_j) - \varepsilon/2\\ &\ge L(z_j) + g_i(z_j) - \varepsilon \ge u(z_j) + g_i(z_j) - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\|F + g_i\| &\ge -[F(y_1) + g_i(y_1)] \ge -[f(y_1) - f(z_j) + U(z_j) + g_i(z_j)] - \varepsilon/2\\ &\ge -[U(z_j) + g_i(z_j)] - \varepsilon \ge -[l(z_j) + g_i(z_j)] - \varepsilon.\end{aligned}$$

We therefore obtain $\|F + g_i\| \ge \max\{\|l + g_i\|, \|u + g_i\|\}$. $\square$

The following proposition establishes the third part of Theorem 1.3.

PROPOSITION 4.3. *Let (l_1, u_1) and (l_2, u_2) be sc pairs associated with types τ_1 and τ_2 respectively as in Theorem 1.3. Then the following are equivalent*

(i) $\tau_1 = \tau_2$;
(ii) $l_1 = l_2$ *and* $u_1 = u_2$.

PROOF. The implication (ii) $\Rightarrow$ (i) is trivial. We prove the contrapositive of (i) $\Rightarrow$ (ii) and distinguish between two cases:
Case 1: $u_1 \neq u_2$. Then there exists $x \in K$ such that $u_1(x) \neq u_2(x)$. We may assume without loss of generality that $u_1(x) > u_2(x)$. Then there exists $\varepsilon > 0$ such that $u_1(x) > u_2(x) + 2\varepsilon$.

Let $U = \{y \in K : u_2(y) < u_2(x) + \varepsilon\}$. Because u_2 is usc, U is an open neighbourhood of x. By Urysohn's Lemma there exists a nonnegative continuous function g_0 with $\|g_0\| = 2r$ such that $g_0|_{K\setminus U} = 0$ and $g_0(x) = 2r$, where $r = \max\{\|u_1\|, \|u_2\|\}$. Let $s = \max\{\|l_1\|, \|l_2\|\}$; then for $i = 1, 2$ we have

$$u_i + (r+s)\mathbf{1} + g_0 \geq l_i + (r+s)\mathbf{1} + g_0 \geq 0$$

and

$$\|l_i + (r+s)\mathbf{1} + g_0\| \leq \|u_i + (r+s)\mathbf{1} + g_0\|.$$

Therefore, for $i = 1, 2$,

$$\max\{\|u_i + (r+s)\mathbf{1} + g_0\|, \|l_i + (r+s)\mathbf{1} + g_0\|\} = \|u_i + (r+s)\mathbf{1} + g_0\|.$$

Furthermore

$$\|u_1 + (r+s)\mathbf{1} + g_0\| \geq r + s + 2r + u_1(x)$$

and

$$\|u_2 + (r+s)\mathbf{1} + g_0\| \leq r + s + 2r + u_2(x) + \varepsilon.$$

Because $u_1(x) > u_2(x) + 2\varepsilon$, we obtain

$$\|u_2 + (r+2)\mathbf{1} + g_0\| < \|u_1 + (r+2)\mathbf{1} + g_0\|$$

and so

$$\begin{aligned}\max\{&\|u_2 + (r+s)\mathbf{1} + g_0\|, \|l_2 + (r+s)\mathbf{1} + g_0\|\} \\ &< \max\{\|u_1 + (r+s)\mathbf{1} + g_0\|, \|l_1 + (r+s)\mathbf{1} + g_0\|\}.\end{aligned}$$

Case 2: $l_1 \neq l_2$. This case is handled using an argument parallel to the one in the previous case. $\square$

5. Problems

This section contains suggestions for further work on this topic:

PROBLEM 5.1. *Provide a characterization of types over $C(K, \mathbb{C})$, the Banach space of all complex-valued continuous functions on K.*

The following concept provides a generalization of type:

Let E be a Banach space. Fix $n \in \mathbb{R}$. For every n-tuple $\bar{x} = (x_1, \ldots, x_n) \in E^n$ define a function $\tau_{\bar{x}} : \mathbb{R}^n \times E \to \mathbb{R}$ by setting $\tau_{\bar{x}}(a_1, \ldots, a_n, y) = \left\| \sum_{i=1}^n a_i x_i + y \right\|$.

DEFINITION 5.2. A function $\tau : \mathbb{R}^n \times E \to \mathbb{R}$ is an *n-type over E* if it is in the closure with respect to the topology of pointwise convergence of the set $\{\tau_{\bar{x}} : \bar{x} \in E^n\}$.

Let E be a Banach space. There is 1-1 correspondence between types over E (in the sense of Definition 1.1) and 1-types over E (in the sense of Definition 5.2):

Indeed, let $\tau : \mathbb{R} \times E \to \mathbb{R}$ is a 1-type. Then the function $\sigma : E \to \mathbb{R}$ defined by setting $\sigma(y) = \tau(1, y)$ for all $y \in E$ is a type over E.

Conversely, suppose $\sigma : E \to \mathbb{R}$ is a type over E. Define $\tau : \mathbb{R} \times E \to \mathbb{R}$ by setting $\tau(a, y) = |a|\sigma((1/a)x)$ if $a \neq 0$ and $\tau(0, y) = \|y\|$. Then τ is a 1-type over E.

PROBLEM 5.3. *Provide a characterization of n-types over the Banach space $C(K)$ that generalizes the characterization of* 1*-types over $C(K)$.*

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