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CRITERIA FOR FUNCTIONS TO BE BLOCH

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Criteria for a holomorphic function f defined in $|z| < 1$ to be Bloch, namely, $\sup_{|z|<1} \left(1-|z|^2\right)|f'(z)| < \infty$, are obtained in terms of the area and the length of the images of non-Euclidean disks and non-Euclidean circles, respectively.

1. Introduction

Let f be holomorphic in the disk $D = \{|z| < 1\}$ in the complex plane $\mathbb{C} = \{|z| < \infty\}$. Then f is called Bloch if the function $\left(1-|z|^2\right)|f'(z)|$ of z is bounded in D. Some necessary and sufficient conditions for f to be Bloch are known [1], [4], [5]. In the present note we shall propose new criteria for f to be Bloch.

Let

$$\sigma(z, w) = \tfrac{1}{2} \log \frac{|1-\bar{z}w|+|z-w|}{|1-\bar{z}w|-|z-w|}$$

be the non-Euclidean hyperbolic distance between z and w in D. For $0 < \rho < \infty$ and for $z \in D$, we set

$$H(z, \rho) = \{w \in D;\ \sigma(w, z) < \rho\}$$

and

$$\Gamma(z, \rho) = \{w \in D;\ \sigma(w, z) = \rho\} .$$

Let f be non-constant and holomorphic in D. Let $A_f(z, \rho)$ be the area

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of the Riemannian image $\mathsf{F}(z, \rho)$ of $H(z, \rho)$ by f , and let $A_f(z, \rho)$ be the area of the image $F(z, \rho)$ of $H(z, \rho)$ by f ; we note that $F(z, \rho)$ is the projection of $\mathsf{F}(z, \rho)$ to $\mathbb{C}$. Let $\mathsf{L}_f(z, \rho)$ be the length of the Riemannian image of $\Gamma(z, \rho)$ by f , and let $L_f(z, \rho)$ be the length of the outer boundary of $F(z, \rho)$. Here, the outer boundary of a bounded domain G in $\mathbb{C}$ means the boundary of $\mathbb{C}\backslash E$, where E is the unbounded component of the complement $\mathbb{C}\backslash G$ of G . It is easy to observe that

$$\mathsf{A}_f(z, \rho) \geq A_f(z, \rho) \quad \text{and} \quad \mathsf{L}_f(z, \rho) \geq L_f(z, \rho)$$

for each $0 < \rho < \infty$ and each $z \in D$.

THEOREM. *Let* f *be non-constant and holomorphic in* D . *Then the following are mutually equivalent:*

(I) f *is Bloch;*

(II) there exists $0 < \rho < \infty$ *such that*

$$\sup_{z \in D} \mathsf{A}_f(z, \rho) < \infty ;$$

(III) there exists $0 < \rho < \infty$ *such that*

$$\sup_{z \in D} A_f(z, \rho) < \infty ;$$

(IV) there exists $0 < \rho < \infty$ *such that*

$$\sup_{z \in D} \mathsf{L}_f(z, \rho) < \infty ;$$

(V) there exists $0 < \rho < \infty$ *such that*

$$\sup_{z \in D} L_f(z, \rho) < \infty .$$

The implication formulae *(II)* implies *(III)* and *(IV)* implies *(V)* are obvious.

2. Proof of theorem

The parts *(I)* implies *(II)* and *(I)* implies *(IV)* are not difficult to prove. Assume *(I)* with

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$$\sup_{z\in D} \left(1-|z|^2\right)|f'(z)| = k < \infty .$$

For each fixed $0 < \rho < \infty$ we set

$$\gamma = \left(e^{2\rho}-1\right)/\left(e^{2\rho}+1\right) . \tag{2.1}$$

Then

$$\begin{aligned} A_f(z, \rho) &= \iint_{H(z,\rho)} |f'(w)|^2 dxdy \quad (w = x + iy) \\ &\le k^2 \iint_{H(z,\rho)} \left(1-|w|^2\right)^{-2} dxdy \\ &= k^2 \iint_{|w|<\gamma} \left(1-|w|^2\right)^{-2} dxdy = \pi k^2\gamma^2/\left(1-\gamma^2\right) . \end{aligned}$$

Similarly

$$\begin{aligned} L_f(z, \rho) &= \int_{\Gamma(z,\rho)} |f'(w)||dw| \\ &\le k \int_{\Gamma(z,\rho)} \left(1-|w|^2\right)^{-1}|dw| \\ &= k \int_{|w|=\gamma} \left(1-|w|^2\right)^{-1}|dw| = 2\pi k\gamma/\left(1-\gamma^2\right) . \end{aligned}$$

The rest we must prove are *(III)* implies *(I)* and *(V)* implies *(I)*, and they are immediate consequences of the following:

LEMMA. *Let f be non-constant and holomorphic in D. Then for each $0 < \rho < \infty$ and for each $z \in D$,*

$$\left(1-|z|^2\right)|f'(z)| \le \left[A_f(z, \rho)/\left(\pi\gamma^2\right)\right]^{\frac{1}{2}} \tag{2.2}$$

and

$$\left(1-|z|^2\right)|f'(z)| \le L_f(z, \rho)/(2\pi\gamma) , \tag{2.3}$$

where γ is defined by (2.1).

For the proof of the lemma we may assume that $f'(z) \ne 0$. Set

$$g(w) = f\left(\frac{w+z}{1+\bar{z}w}\right) = a_0 + a_1 w + a_2 w^2 + \ldots$$

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for $|w| < 1$, where

$$a_1 = \left(1-|z|^2\right)f'(z) \neq 0 \ . \tag{2.4}$$

It follows from [3, Theorems 1 and 2], applied to g and to $r = \gamma$, that

$$\pi\gamma^2|a_1|^2 \leq A_f(z, \rho) \tag{2.5}$$

and

$$2\pi\gamma|a_1| \leq L_f(z, \rho) \ . \tag{2.6}$$

Therefore (2.2) ((2.3), respectively) follows from (2.5) ((2.6), respectively) and (2.4).

REMARK. A meromorphic analogue of a Bloch function is a normal meromorphic function. On replacing the Euclidean area by the spherical area, and the Euclidean length by the spherical length, respectively, one might expect to obtain the corresponding criteria for a meromorphic function in D to be normal in D . It is easy to observe the meromorphic analogues of *(I)* implies *(II)* and *(I)* implies *(IV)*. However, the analogue of *(III)* implies *(I)* is false. More precisely, there exists a holomorphic non-normal function f in D such that the spherical area of the image $f\left(H(z, \rho)\right)$ of $H(z, \rho)$ is bounded by π for each $z \in D$, where $0 < \rho < \infty$ is a constant. For the proof we remember Lappan's non-normal holomorphic function f in D [2] which is univalent in each $H(z, \rho)$, $z \in D$, where ρ is a constant independent of z . Whether or not the meromorphic analogue of *(V)* implies *(I)* is true is an open problem.

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