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Every symmetric Kubo–Ando connection has the order-determining property

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Abstract. In this article, the question of whether the Löwner partial order on the positive cone of an operator algebra is determined by the norm of any arbitrary Kubo–Ando mean is studied. The question was affirmatively answered for certain classes of Kubo–Ando means, yet the general case was left as an open problem. We here give a complete answer to this question, by showing that the norm of every symmetric Kubo–Ando mean is order-determining, i.e., if $A, B \in \mathcal{B}(H)^{++}$ satisfy $\|A\sigma X\| \leq \|B\sigma X\|$ for every $X \in \mathcal{A}^{++}$, where $\mathcal{A}$ is the C∗-subalgebra generated by $B - A$ and I, then $A \leq B$.

1 Introduction

Recently, in [9], the author studied the question of when the norm of a given mean, on the positive cone of an operator algebra, determines the Löwner order. As explained clearly in the introduction by the author, this problem is of relevance to the study of maps between positive cones of operator algebras that preserve a given norm of a given operator mean. Such a study has received considerable attention, as can be seen, for example, in [5–8]. The motivation of such investigations comes, first, from the study of norm additive maps or spectrally multiplicative maps, and second, from the study of the structure of certain quantum mechanical symmetry transformations relating to divergences.

Let us recall that a binary operation σ on $\mathcal{B}(H)^{++}$ is called a *Kubo–Ando connection* if it satisfies the following properties:

(i) If $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$.
(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.
(iii) If $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\sigma B_n \downarrow A\sigma B$.[1]

A *Kubo–Ando mean* is a Kubo–Ando connection with the normalization condition $I\sigma I = I$. The most fundamental connections are:

- the *sum* $(A, B) \mapsto A + B$,
- the *parallel sum* $(A, B) \mapsto A : B = \left(A^{-1} + B^{-1}\right)^{-1}$,

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[1] For a sequence (X_n) of self-adjoint operators in $\mathcal{B}(H)$, we write $X_n \downarrow X$ when (X_n) is monotonic decreasing and SOT-convergent to X. The symbol $X_n \uparrow X$ is defined dually.

- the *geometric mean*

$$(A,B) \mapsto A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}.$$

A function $f:(0,\infty)\to(0,\infty)$ is said to be *operator monotone* if $\sum_{i=1}^{n} f(a_i)P_i \le \sum_{j=1}^{m} f(b_j)Q_j$ whenever $\sum_{i=1}^{n} a_iP_i \le \sum_{j=1}^{m} b_jQ_j$, where $a_i, b_j > 0$, and the projections P_i, Q_j satisfy $\sum_{i=1}^{n} P_i = \sum_{j=1}^{m} Q_j = I$. Such a function is automatically continuous, monotonic increasing, and concave. For an operator-monotone function f, one has $f(A) \le f(B)$ whenever $A, B \in \mathcal{B}(H)^{++}$ and $A \le B$. It is easy to see that the class of operator-monotone functions is closed under addition and multiplication by positive real numbers. The transpose f° of the operator-monotone function f, defined by $f^{\circ}(x) := x f(x^{-1})$, is again operator-monotone.

Let σ be a Kubo–Ando connection on $\mathcal{B}(H)^{++}$. In the proof of [4, Theorem 3.2], it is shown that the function defined on $(0,\infty)$ by $x \mapsto I\sigma(xI)$ has the form $x \mapsto f(x)I$ for some operator-monotone function f, and it is further shown that $f(B) = I\sigma B$ for every $B \in \mathcal{B}(H)^{++}$. This gives

$$A\sigma B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for every $A, B \in \mathcal{B}(H)^{++}$. The function f is called the *representing function* of σ.

We recall that operator-monotone functions correspond to positive finite Borel measures on $[0,\infty]$[2] by Löwner's theorem (see [2]): To every operator-monotone function, f corresponds a unique positive and finite Borel measure m on $[0,\infty]$ such that

$$\begin{aligned} (1)\qquad f(x) &= \int_{[0,\infty]} \frac{x(1+t)}{x+t}\,\mathrm{d}m(t) \\ &= m(\{0\}) + x\,m(\{\infty\}) + \int_{(0,\infty)} \frac{1+t}{t}(t:x)\,\mathrm{d}m(t) \qquad (x>0). \end{aligned}$$

It is easy to see that $f(0+) = m(\{0\})$ and $f^{\circ}(0+) = m(\{\infty\})$.

Let $f:(0,\infty)\to(0,\infty)$ be an operator-monotone function, and let m be the positive and finite Borel measure on $[0,\infty]$ associated with f via Löwner's theorem by (1). The binary operation σ_f defined on $\mathcal{B}(H)^{++}$ by

$$A\sigma_f B = f(0+)\,A + f^{\circ}(0+)\,B + \int_{(0,\infty)} \frac{1+t}{t}(tA:B)\,\mathrm{d}m(t)$$

satisfies conditions (i) and (ii) of the definition of a Kubo–Ando connection. Moreover,

$$\begin{aligned} I\sigma_f A &= f(0+)\,I + f^{\circ}(0+)\,A + \int_{(0,\infty)} \frac{1+t}{t}(tI:A)\,\mathrm{d}m(t) \\ &= \int_{[0,\infty]} A(1+t)(tI+A)^{-1}\,\mathrm{d}m(t) = f(A) \end{aligned}$$

for every $A \in \mathcal{B}(H)^{++}$, and therefore

$$A\sigma_f B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

[2] We recall that every finite Borel measure on $[0,\infty]$ is regular, i.e., a Radon measure.

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for every $A, B \in \mathcal{B}(H)^{++}$. Using the fact that a continuous real-valued function is SOT (strong operator topology) continuous on bounded sets of self-adjoint operators (see [3, Proposition 5.3.2, p. 327]), it follows that if $A_n \downarrow A$ and $B_n \downarrow B$ in $\mathcal{B}(H)^{++}$, then $A_n \sigma_f B_n \downarrow A\sigma_f B$. This shows that σ_f is a Kubo–Ando connection on $\mathcal{B}(H)^{++}$.

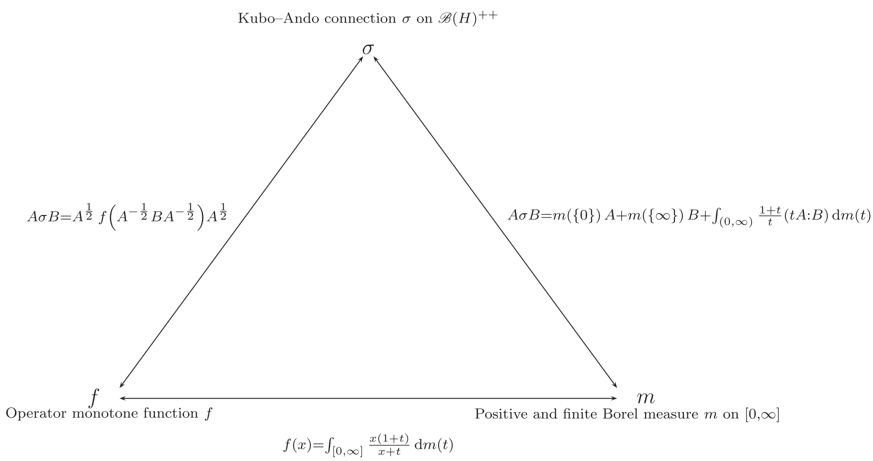
We further recall that if σ is a Kubo–Ando connection with representing function f, then the representing function of the "reversed" Kubo–Ando connection $(A, B) \mapsto B\sigma A$ is the transpose f°. The Kubo–Ando connection is said to be symmetric if it coincides with its reverse, i.e., a Kubo–Ando connection is symmetric if and only if the representing function f satisfies $f = f^\circ$ as shown in [4, Corollary 4.2]. The Kubo–Ando means are precisely the Kubo–Ando connections whose representing functions satisfy the normalizing condition $f(1) = 1$.

The most fundamental Kubo–Ando means are the power means which correspond to the operator-monotone functions

$$f_p(t) := \begin{cases} \left(\dfrac{1+t^p}{2}\right)^{\frac{1}{p}}, & \text{if } -1 \le p \le 1,\ p \neq 0, \\ \sqrt{t}, & \text{if } p = 0. \end{cases}$$

The principal cases $f_0(t) = \sqrt{t}$, $f_{-1}(t) = \frac{2t}{1+t}$, and $f_1(t) = \frac{1+t}{2}$ correspond, respectively, to the geometric mean $(A, B) \mapsto A \,\sharp\, B$, the harmonic mean $(A, B) \mapsto A!B = 2(A : B)$, and the arithmetic mean $(A, B) \mapsto A\nabla B = (A + B)/2$.

It is easy to verify that the measure m associated with the arithmetic mean is $(\delta_0 + \delta_\infty)/2$ and that associated with the harmonic mean is δ_1, where δ_x denotes the Dirac measure on the point $x \in [0, \infty]$.



A remark on the domain of definition of a Kubo–Ando connection: We have opted for having $\mathcal{B}(H)^{++}$ the defining domain of a Kubo–Ando connection (as opposed to $\mathcal{B}(H)^{+}$) in order to obtain fully consistent interchangeable relations in the diagram above. It must be said, however, that this offers no handicap because any Kubo–Ando connection σ on $\mathcal{B}(H)^{++}$ can be uniquely extended to a binary relation $\hat{\sigma}$ on $\mathcal{B}(H)^{+}$ by setting $A\hat{\sigma}B$ equal to

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$$\inf\{(A+n^{-1}I)\sigma(B+n^{-1}I) : n \in \mathbb{N}\} = \inf\{X\sigma Y : X, Y \in \mathcal{B}(H)^{++}, X \geq A, Y \geq B\},$$

and it is not hard to show that the extension $\hat{\sigma}$ satisfies (i)–(iii) of the definition of a Kubo–Ando connection. Note that the equality

$$A\hat{\sigma}B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

where f is the representing function associated with σ, holds only on $\mathcal{B}(H)^{++} \times \mathcal{B}(H)^{+}$. We further remark that the continuity properties of the function calculus (see [3, Proposition 5.3.2, p. 327]) imply the following continuity properties of $\hat{\sigma}$.

Remark 1 (i) The map

(2) $$\hat{\sigma} : \mathcal{B}(H)^{++} \times \mathcal{B}(H)^{+} \to \mathcal{B}(H)^{+} : (A, B) \mapsto A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

is continuous when the domain is equipped with the product of the relative topologies induced by the norm, and the range with the norm topology, and

(ii) for every $\varepsilon, R > 0$, the restriction of $\hat{\sigma}$ to the rectangle

$$[\varepsilon I\,,\, RI] \times [0\,,\, RI]$$

is continuous when the domain is equipped with the product of the relative topologies induced by SOT, and the range with SOT.

In the sequel, we will not distinguish between σ and $\hat{\sigma}$ any further.

2 Preliminary considerations

In this section, we collate a list of lemmas and propositions which will prove to be helpful in proving the main result.

Lemma 1 *Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function, and let m denote the positive and finite Borel measure associated with f via (1).*

(i) *If f is symmetric, $\int_{[0,\infty]} t\,\mathrm{d}m(t) = \int_{[0,\infty]} t^{-1}\,\mathrm{d}m(t)$.*

(ii) *For every Borel subset Δ of $[0, \infty]$ satisfying $m(\Delta) > 0$, the function f_Δ defined on $(0, \infty)$ by*

$$f_\Delta : x \mapsto \int_\Delta \frac{x(1+t)}{x+t}\,\mathrm{d}m(t)$$

is operator-monotone. In particular, if $m((0, \infty)) \neq 0$, the function h defined by

$$h(x) := \int_{(0,\infty)} \frac{x(1+t)}{x+t}\,\mathrm{d}m(t) = f(x) - f(0+) - f^{\circ}(0+)x \quad (x > 0)$$

is operator-monotone. If f is symmetric, then so is h.

Proof (i) By the Monotone Convergence Theorem, one has that

$$f(x) = \int_{[0,\infty]} \frac{1+t}{1+tx^{-1}}\,\mathrm{d}m(t) \uparrow \int_{[0,\infty]} 1+t\,\mathrm{d}m(t) \qquad \text{as } x \uparrow \infty,$$

and since $f^\circ(x) = xf(1/x)$, one also gets

$$f^\circ(x) = \int_{[0,\infty]} \frac{1+t}{t+x^{-1}}\,\mathrm{d}m(t) \uparrow \int_{[0,\infty]} 1+t^{-1}\,\mathrm{d}m(t) \qquad \text{as } x \uparrow \infty.$$

So, if f is symmetric, $\int_{[0,\infty]} t\,\mathrm{d}m(t) = \int_{[0,\infty]} t^{-1}\,\mathrm{d}m(t)$.

(ii) If Δ is a Borel subset of $[0,\infty]$ satisfying $m(\Delta) > 0$, the function on the Borel σ-algebra of $[0,\infty]$ defined by $m_\Delta : \Omega \mapsto m(\Delta \cap \Omega)$ is a positive finite Borel measure and therefore

$$\mathcal{B}(H)^{++} \ni A \mapsto f_\Delta(A) = \int_\Delta (1+t)(I+tA^{-1})^{-1}\,\mathrm{d}m(t) = \int_{[0,\infty]} (1+t)(I+tA^{-1})^{-1}\,\mathrm{d}m_\Delta(t)$$

is operator-monotone.

Setting $\Delta = (0,\infty)$, one obtains that

$$h(x) := \int_{(0,\infty)} \frac{x(1+t)}{x+t}\,\mathrm{d}m(t) \quad (x > 0)$$

is operator-monotone. Since $m(\{0\}) = f(0+)$ and $m(\{\infty\}) = f^\circ(0+)$,

$$h(x) = f(x) - f(0+) - f^\circ(0+)x$$

follows. It is easy to verify that if f is symmetric, so is h. ∎

Let $X \in \mathcal{B}(H)$ be self-adjoint and let Δ denote the spectrum of X. The classical measurable function calculus asserts that there exists a $*$-isomorphism Φ from the Banach algebra $L^\infty(\Delta, B(\Delta), \mu)$ into the abelian von Neumann subalgebra of $\mathcal{B}(H)$ generated by X and I, where $B(\Delta)$ is the Borel σ-algebra in Δ, and μ is the composition of any faithful normal positive weight on $\mathcal{B}(H)$ with the spectral measure associated with X. We recall that Φ is an isometry, preserves suprema/infima of monotone sequences, and maps the continuous functions on Δ onto the C$*$-subalgebra of $\mathcal{B}(H)$ generated by X and I. Denote by $\Gamma(X)$ the range of Φ. Recall that a lower semicontinuous function $f : \Omega \to [0,1]$, where Ω is a metrizable space, is the pointwise limit of an increasing sequence of continuous functions (see, for example, [10, Problem 7K, p. 49]). Let $\Gamma_1(X) \subset \Gamma(X)$ denote the image under Φ of the set of all bounded lower semicontinuous functions on Δ. In particular, $\Phi(\chi_J) \in \Gamma_1(X)$ whenever J is an open subset of Δ. For the purpose of this paper, such a projection is shortly referred to as a "Γ_1-*spectral projection of X*."

Lemma 2 *Let σ be a Kubo–Ando connection. For $A, B \in \mathcal{B}(H)^{++}$, let $\mathcal{A}$ denote the C$*$-subalgebra of $\mathcal{B}(H)$ generated by $B - A$ and I. If $\|A\sigma T\| \le \|B\sigma T\|$ holds for all $T \in \mathcal{A}^{++}$, then $\|A\sigma T\| \le \|B\sigma T\|$ holds for all positive $T \in \Gamma_1(B-A)$.*

Proof Let us first recall the general fact that whenever (T_γ) is an SOT-convergent net of positive operators, bounded from above by its SOT-limit T, then the net of norms $(\|T_\gamma\|)$ is convergent to $\|T\|$.

Invoking the continuity of σ w.r.t. the norm (as mentioned in the introduction), and since $\mathcal{A}^{++}$ is norm-dense in $\mathcal{A}^+$, it can be seen that the hypothesis implies that $\|A\sigma T\| \le \|B\sigma T\|$ holds for every $T \in \mathcal{A}^+$.

Let $X := B - A$. Let $(\Delta, B(\Delta), \mu)$ be the measure space associated with X, i.e., Δ is the spectrum of X, $B(\Delta)$ is the Borel σ-algebra in Δ, and μ is the composition of any faithful normal positive weight on $\mathcal{B}(H)$ with the spectral measure associated with X. Let Φ be the $*$-isomorphism from the Banach algebra $L^\infty(\Delta, B(\Delta), \mu)$ into the abelian von Neumann subalgebra of $\mathcal{B}(H)$ generated by X and I.

Fix a positive $T \in \Gamma_1(X)$. We want to show that $\|A\sigma T\| \leq \|B\sigma T\|$. Let f be a bounded lower semicontinuous function on Δ satisfying $\Phi(f) = T$. There exists a sequence (f_n) of continuous functions on Δ such that $f_n(x) \uparrow f(x)$ as $n \uparrow \infty$ for every $x \in \Delta$. Without loss of generality, we can assume that $0 \leq f_n(x)$ for every $n \in \mathbb{N}$ and $x \in \Delta$, i.e., we can suppose that $\Phi(f_n) \in \mathcal{A}^+$. Applying (ii) of Remark 1, one obtains $A\,\sigma\,\Phi(f_n) \uparrow A\,\sigma\,T$ and $B\,\sigma\,\Phi(f_n) \uparrow B\,\sigma\,T$ as $n \uparrow \infty$. The observation recalled in the first paragraph of the proof then yields

$$\|B\sigma T\| = \lim_n \|B\,\sigma\,\Phi(f_n)\| \geq \lim_n \|A\,\sigma\,\Phi(f_n)\| = \|A\sigma T\|. \qquad \blacksquare$$

In the subsequent lemma, the main ideas can be found in [1, Lemma 11]. We formalize them and present them here for completeness sake.

Proposition 3 *For the operators $A, B \in \mathcal{B}(H)^+$, the following assertions are equivalent:*

(i) *$A \leq B$,*
(ii) *$\|PAP\| \leq \|PBP\|$ for every Γ_1-spectral projection P of $B - A$,*
(iii) *$\{\lambda \geq 0 : \lambda P \leq PAP\} \subseteq \{\lambda \geq 0 : \lambda P \leq PBP\}$ for every Γ_1-spectral projection P of $B - A$.*

Proof The assertions (i)⇒(ii) and (i)⇒(iii) are trivial.

Let us prove that both (ii) and (iii) imply (i). Suppose that $A \nleq B$, for contradiction. Then, there exists $\varepsilon > 0$ such that the spectrum of $B - A$ has a nontrivial intersection with $(-\infty, -\varepsilon)$. Let Δ be the intersection of $(-\infty, -\varepsilon)$ with the spectrum of $B - A$, and let P_ε be the (nonzero) Γ_1-spectral projection of $B - A$ associated with the indicator function χ_Δ. Clearly, $t\,\chi_\Delta(t) \leq -\varepsilon\,\chi_\Delta(t)$ for every $t \in \mathbb{R}$, and therefore $P_\varepsilon B P_\varepsilon - P_\varepsilon A P_\varepsilon = P_\varepsilon(B - A)P_\varepsilon \leq -\varepsilon P_\varepsilon$. Rearranging the terms, we get

$$P_\varepsilon B P_\varepsilon \leq P_\varepsilon(A - \varepsilon I)P_\varepsilon. \tag{3}$$

(ii)⇒(i). It follows, by (3), that $P_\varepsilon A P_\varepsilon \geq \varepsilon P_\varepsilon$, and therefore $\|P_\varepsilon B P_\varepsilon\| \leq \|P_\varepsilon(A - \varepsilon I)P_\varepsilon\| = \|P_\varepsilon A P_\varepsilon\| - \varepsilon$.

(iii)⇒(i). First, observe that for every $A \in \mathcal{B}(H)^+$ and projection P, the supremum of $\{\lambda \geq 0 : \lambda P \leq PAP\}$ is indeed a maximum and is at most equal to $\|A\|$. Let $\lambda_0 := \max\{\lambda \geq 0 : \lambda P_\varepsilon \leq P_\varepsilon A P_\varepsilon\}$. Then (iii) and (3) imply that $\lambda_0 P_\varepsilon \leq P_\varepsilon B P_\varepsilon \leq P_\varepsilon A P_\varepsilon - \varepsilon P_\varepsilon$, which in turn shows that $(\lambda_0 + \varepsilon)P_\varepsilon \leq P_\varepsilon A P_\varepsilon$, contradicting the maximality condition of λ_0. $\blacksquare$

In the following proposition, the ideas in [9, Proposition 10] are used to generalize [9, equation (15)]. This will be of pivotal importance in proving the main result of this paper.

Proposition 4 *Let $X_s \in \mathcal{B}(H)^+$, $s > 0$ satisfy $\lim_{s\to\infty} X_s = X$ in norm, and let $P \in \mathcal{B}(H)$ be a projection. Then*

$$\lim_{s\to\infty} \|X_s + sP\| - s = \|PXP\|.$$

Proof Let $\varepsilon > 0$. It can easily be verified that

$$\left\|\left((\|PX_sP\| + \varepsilon)^{-1/2}P + s^{-1/2}(I-P)\right)X_s\left((\|PX_sP\| + \varepsilon)^{-1/2}P + s^{-1/2}(I-P)\right)\right\|$$

converges to $(\|PXP\| + \varepsilon)^{-1}\|PXP\| < 1$ as $s \to \infty$. Therefore, for sufficiently large s,

$$\left((\|PX_sP\| + \varepsilon)^{-1/2}P + s^{-1/2}(I-P)\right)X_s\left((\|PX_sP\| + \varepsilon)^{-1/2}P + s^{-1/2}(I-P)\right) \le I,$$

so by multiplying both sides of the above by the inverse of $(\|PX_sP\| + \varepsilon)^{-1/2}P + s^{-1/2}(I-P)$, the inequality $X_s \le (\|PX_sP\| + \varepsilon)P + s(I-P)$ is obtained. This implies that

$$\|X_s + sP\| - s \le \|PX_sP\| + \varepsilon, \tag{4}$$

for sufficiently large s. On the other hand, for every $s > 0$,

$$\|X_s + sP\| \ge \|PX_sP + sP\|,$$

and therefore

$$\|X_s + sP\| - s \ge \|PX_sP + sP\| - s = \|PX_sP\|. \tag{5}$$

Combining (4) and (5), for sufficiently large s, it holds that

$$\|PX_sP\| \le \|X_s + sP\| - s \le \|PX_sP\| + \varepsilon.$$

This proves that $\lim_{s\to\infty} \|X_s + sP\| - s = \|PXP\|$. ∎

Proposition 5 *[9, Lemma 2] Let $f : (0,\infty) \to (0,\infty)$ be a nontrivial (i.e., not affine) operator-monotone function satisfying $f(0+) = 0$, and let σ denote the Kubo–Ando connection associated with f via (2). For $A \in \mathcal{B}(H)^{++}$ and nonzero projection $P \in \mathcal{B}(H)$,*

$$\|A\sigma P\| = f^\circ\left(\frac{1}{\max\{\lambda \ge 0 : \lambda P \le PA^{-1}P\}}\right).$$

3 Results

Theorem 6 *Let σ be a nontrivial symmetric Kubo–Ando connection on $\mathcal{B}(H)^{++}$. Then, for every $A, B \in \mathcal{B}(H)^{++}$,*

$$A \le B \iff \|A\sigma X\| \le \|B\sigma X\|, \quad \forall X \in \mathcal{A}^{++},$$

where $\mathcal{A}$ equals the C^*-subalgebra of $\mathcal{B}(H)$ generated by $B - A$ and I.

Proof The implication $\Rightarrow$ follows trivially by the monotonicity property of Kubo–Ando connections. We shall show the converse. By Lemma 2, we can suppose that $\|A\sigma X\| \le \|B\sigma X\|$ holds for every X of the form $X = sP + tI$, where P is a Γ_1-spectral projection of $B - A$ and $s, t \in \mathbb{R}^+$. Let f be the operator-monotone function associated

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with σ, and let m be the positive and finite Borel measure associated with f by Löwner's theorem. Let $\alpha = f(0+) = m(\{0\})$. The proof will be divided in cases.

Case 1: $\alpha = 0$. Since $f = f^\circ$ is strictly monotonic increasing, this case follows immediately by Propositions 5 and 3. This result was obtained by Molnár in [9].

Case 2a: $\alpha \neq 0$ *and* $\int_{(0,\infty)} t \, dm(t) < \infty$. Let $\gamma := \int_{(0,\infty)} 1 + t \, dm(t)$. For every $s, t, \delta > 0$, $A \in \mathcal{B}(H)^{++}$ and nonzero projection $P \in \mathcal{B}(H)$:

$$\int_{(0,\infty)} \frac{1+t}{t} (tA : sP + s\delta I) \, dm(t) - A \int_{(0,\infty)} 1 + t \, dm(t)$$
$$= \int_{(0,\infty)} \left(A \left(\left(\frac{tA}{s} + P + \delta I \right)^{-1} (P + \delta I) - I \right) \right) (1+t) \, dm(t).$$

Noting that $\left\| A \left(\left(\frac{tA}{s} + P + \delta I \right)^{-1} (P + \delta I) - I \right) \right\|$ is a bounded function of s and t, and using the fact that $\int_{(0,\infty)} 1 + t \, dm(t) < \infty$, it is possible to apply the Dominated Convergence Theorem to infer that

$$\int_{(0,\infty)} \frac{1+t}{t} (tA : sP + s\delta I) \, dm(t)$$

converges in norm to γA as $s \to \infty$. This implies that

$$A\sigma(sP + s\delta I) - \beta(sP + s\delta I) \to (\alpha + \gamma) A$$

in norm, as $s \to \infty$. Noting that $\beta = m(\{\infty\}) = m(\{0\}) > 0$ and applying Proposition 4, it is deduced that

$$\lim_{s \to \infty} \left(\| A\sigma(sP + s\delta I) - \beta s \delta I \| - \beta s \right) = (\alpha + \gamma) \| PAP \|.$$

Using the fact that

$$A\sigma(sP + s\delta I) = \alpha A + \beta(sP + s\delta I) + \int_{(0,\infty)} \frac{1+t}{t} (tA : sP + s\delta I) \, dm(t) \geq \beta s \delta I,$$

it can be seen that $\| A\sigma(sP + s\delta I) - \beta s \delta I \| = \| A\sigma(sP + s\delta I) \| - \beta s \delta$. This establishes that

$$\lim_{s \to \infty} \left(\| A\sigma(sP + s\delta I) \| - \beta s (1 + \delta) \right) = (\alpha + \gamma) \| PAP \|. \tag{6}$$

So, if $A, B \in \mathcal{B}(H)^{++}$ satisfy $\| A\sigma(sP + tI) \| \leq \| B\sigma(sP + tI) \|$ for every Γ_1-spectral projection P of $B - A$ and $s, t \in \mathbb{R}^+$, it follows that $\| PAP \| \leq \| PBP \|$ holds for every Γ_1-spectral projection of $B - A$. The result follows by Proposition 3.

Case 2b: $\alpha \neq 0$ *and* $\int_{(0,\infty)} t \, dm(t) = \infty$. Denote by σ_h the (symmetric) Kubo–Ando connection associated with the function $h(x) = f(x) - \alpha - \alpha x$ (see (ii) of Lemma 1). Let m_h denote the positive and finite Borel measure associated with h. Then, $m_h(\Delta) = m(\Delta \cap (0, \infty))$ for every Borel subset Δ of $[0, \infty]$.

The inequality

$$\| A\sigma(sP) \| = \| \alpha A + \alpha s P + A\sigma_h(sP) \| \leq \| \alpha B + \alpha s P + B\sigma_h(sP) \| = \| B\sigma(sP) \| \tag{7}$$

holds for every Γ_1-spectral projection P of $B - A$ and $s > 0$.

Fix an arbitrary Γ_1-spectral projection P of $B - A$. Noting that $h(sP) = h(s)P$ for every $s > 0$, it can then be deduced that

$$A\sigma_h(sP) \leq (\|A\|I)\sigma_h(sP) = \|A\|h(\|A\|^{-1}(sP)) = \|A\|h(\|A\|^{-1}s)\,P,$$

i.e., $A\sigma_h P$ commutes with P. Thus, (7) yields

$$\begin{aligned}\|\alpha sP\| + \|A\sigma_h(sP)\| &= \|\alpha sP + A\sigma_h(sP)\|\\ &\leq \|\alpha A + \alpha sP + A\sigma_h(sP)\|\\ &\leq \|\alpha B + \alpha sP + B\sigma_h(sP)\|\\ &\leq \|\alpha B\| + \|\alpha sP\| + \|B\sigma_h(sP)\|,\end{aligned}$$

and therefore

$$\|A\sigma_h(sP)\| - \|B\sigma_h(sP)\| \leq \|\alpha B\|, \tag{8}$$

for every $s > 0$.

Let $c_A := 1/\max\{\lambda \geq 0 : \lambda P \leq PA^{-1}P\}$, and let c_B be defined similarly. Proposition 5 gives

$$\|A\sigma_h(sP)\| = s\|(s^{-1}A)\sigma_h P\| = s\,h\left(\frac{1}{s\,\max\{\lambda \geq 0 : \lambda P \leq PA^{-1}P\}}\right) = s\,h\left(c_A s^{-1}\right),$$

and since m_h is just the restriction of the measure m (associated with f) to $(0,\infty)$, we obtain

$$\|A\sigma_h(sP)\| = s\,h\left(c_A s^{-1}\right) = \int_{[0,\infty]} \frac{sc_A(1+t)}{c_A + st}\,\mathrm{d}m_h(t) = \int_{(0,\infty)} \frac{sc_A(1+t)}{c_A + st}\,\mathrm{d}m(t).$$

Similarly, $\|B\sigma_h(sP)\| = \int_{(0,\infty)} \frac{sc_B(1+t)}{c_B+st}\,\mathrm{d}m(t)$, and therefore

$$\begin{aligned}\|A\sigma_h(sP)\| - \|B\sigma_h(sP)\| &= \int_{(0,\infty)} \frac{sc_A(1+t)}{c_A+st} - \frac{sc_B(1+t)}{c_B+st}\,\mathrm{d}m(t)\\ &= (c_A - c_B)\int_{(0,\infty)} \frac{s^2 t(1+t)}{(c_A+st)(c_B+st)}\,\mathrm{d}m(t).\end{aligned}$$

The Monotone Convergence Theorem implies that as $s \uparrow \infty$, the integral increases to $\int_{(0,\infty)} \frac{1}{t} + 1\,\mathrm{d}m(t)$. The relation between the two measures m and m_h, part (i) of Lemma 1, and the hypothesis then yield that

$$\begin{aligned}\|A\sigma_h(sP)\| - \|B\sigma_h(sP)\| &\to (c_A - c_B)\int_{(0,\infty)} \frac{1}{t} + 1\,\mathrm{d}m(t)\\ &= (c_A - c_B)\int_{[0,\infty]} \frac{1}{t} + 1\,\mathrm{d}m_h(t)\\ &= (c_A - c_B)\int_{[0,\infty]} t + 1\,\mathrm{d}m_h(t)\\ &= (c_A - c_B)\int_{(0,\infty)} t + 1\,\mathrm{d}m(t),\end{aligned}$$

as $s \uparrow \infty$. Since $\int_{(0,\infty)} t + 1\,\mathrm{d}m(t) = \infty$, it follows that $c_A - c_B \leq 0$ since otherwise one would get a contradiction with (8).

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This shows that

$$\max\{\lambda \geq 0 : \lambda P \leq PA^{-1}P\} \geq \max\{\lambda \geq 0 : \lambda P \leq PB^{-1}P\}$$

for every Γ_1-spectral projection of $B - A$, and therefore $A^{-1} \geq B^{-1}$ by Proposition 3. ∎

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