

Note on Triangle Transformations.

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The investigation given in the following Note was suggested by a passage in the paper by Mr Lemoine, presented to the Society by Dr. Mackay at a recent meeting. The main subject of that paper is what he terms the "*Transformation continue dans le triangle et dans le tétraèdre*"; for the explanation of that phrase and other terms connected with it, the reader is referred to the paper just mentioned. In the notation for the quantities connected with the triangle, however, I shall follow Dr. Mackay's system as explained in his paper in Vol. I. of our *Proceedings*.

I shall use, moreover, the letters α, β, γ provisionally as symbols of operation, to denote what Mr Lemoine calls "*la transformation continue en A, en B, en C*" respectively. Thus, as shown in Lemoine's paper,

$$aa = a, ab = -b, aA = -A, aB = \pi - B, \text{ etc.}$$

A compound operation such as $\beta a . A$ I shall use to mean $\beta(aA)$, and $\frac{1}{a}$ to denote the reverse of the operation a . Thus $aB = \pi - B$, and $B = \frac{1}{a}(\pi - B)$ are taken as equivalent equations.

When an equation like $\alpha = \beta$ occurs, it shall mean that, with respect to the functions considered, the operation α is equivalent to the operation β ; and $\alpha = 1$ shall mean that the operation α leaves the functions under consideration unchanged.

Now in Mr Lemoine's paper he gives a list of four kinds of cases, with respect to the variety of results got by applying α, β and γ to triangle-identities, which may be indicated by the following typical cases :

- (1) $\alpha = \beta = \gamma = 1$
- (2) $\alpha, \beta, \gamma, 1$ all different
- (3) $\alpha = 1 \neq \beta = \gamma$
- (4) $\alpha = \beta = \gamma \neq 1$

He states that all these are found to occur, but that he has not yet found any case of the following type :

$$(5) \quad \alpha = 1, \quad \beta, \gamma, 1 \text{ all different.}$$

The object of this Note is to account for the non-occurrence of such cases. We may complete the list of typical cases by

$$(6) \quad \alpha \neq 1, \quad \beta = \gamma = 1$$

$$(7) \quad \alpha \neq 1, \quad \beta = \gamma \neq 1.$$

We have

$$\begin{aligned} \gamma(a) &= -a \\ \beta\gamma(a) &= \beta(-a) = a \\ \alpha\beta\gamma(a) &= \alpha(a) = a \end{aligned}$$

Similarly $\alpha\beta\gamma(r) = -r, \quad \alpha\beta\gamma(r_1) = -r_1, \text{ etc.}$

In fact, if F denote any function whatever, we have

$$\begin{aligned} &\alpha\beta\gamma\{F(a, b, c, s, s_1, s_2, s_3, r, r_1, r_2, r_3, h_1, h_2, h_3, \Delta, R)\} \\ &= \{F(a, b, c, s, s_1, s_2, s_3, -r, -r_1, -r_2, -r_3, \\ &\quad -h_1, -h_2, -h_3, -\Delta, -R)\} \end{aligned} \quad \Bigg\}$$

Thus the compound operator $\alpha\beta\gamma$ changes the signs of certain letters, but leaves them otherwise unaltered.

Now if any identity connecting the functions of the general triangle be written in the form $F = 0$, it is clear that the identity will not be altered by applying the operation $\alpha\beta\gamma$, so long as F consists of terms which, with respect to the letters r, r_1, \dots, R (whose signs are changed thereby) are either all of *even* dimensions, or all of *odd* dimensions. Such terms will hereafter be referred to, for brevity's sake, as *even*, or *odd* simply. The letters $ab \dots s_3$ whose signs are unchanged by the operation $\alpha\beta\gamma$ are not to be reckoned in this connection. Omitting for the present, then, all reference to identities in which F is a *mixed* function, we may say that $\alpha\beta\gamma = 1$, *i.e.* the operator $\alpha\beta\gamma$ reproduces the same identity as we start with.

Now it is easily proved that $\alpha^2 F = F$, whatever the function F may be, or, as we may write it $\alpha^2 = 1$, whence $\alpha = \frac{1}{\alpha}$. And the same is true as applied to an identical equation.

But since $a\beta\gamma = 1$, restricting ourselves to identities which are not *mixed*, we have $a \cdot a\beta\gamma = a$

$$\therefore a^2\beta\gamma = a \qquad \therefore \beta\gamma = a$$

Hence if $a = 1$, $\beta\gamma = 1 \qquad \therefore \beta = \frac{1}{\gamma} = \gamma$

This shows that case (5) cannot occur.

Conversely if $\beta = \gamma$, we have $\beta\gamma = \gamma^2 = 1 \qquad \therefore a = 1$

This shows that cases (6) and (7) cannot occur.

We may sum up these results by saying that if $a = 1$, then $\beta = \gamma$; and *vice versa*.

Now it may be asked, Are there not *mixed* identities which give results of the types (5), (6), (7)? The answer is that there are; but they are always composite identities, which may be reduced to two or more simpler ones. In fact, if any identity be expressed by $F = 0$ when $F \equiv F_1 + F_2$, F_1 being an *odd*, and F_2 an *even* function, as explained above, then applying $a\beta\gamma$ to the identity $F_1 + F_2 = 0$ we get a new identity $-F_1 + F_2 = 0$ whence $F_1 = 0$ and $F_2 = 0$ identically; *i.e.* $F = 0$ is composed by adding together two identities $F_1 = 0$ and $F_2 = 0$. It is easy to manufacture such cases, *e.g.*,

$$sr - s_1r_1 + rr_1 - s_2s_3 = 0.$$

This is an identity which belongs to the type (5).

I have in the foregoing omitted all consideration of the angles A, B, C , for the reason that they introduce further complication. In fact

$$a\beta\gamma F(A, B, C) = F(-A, -B + 2\pi, -C)$$

so that the characteristic property of $(a\beta\gamma)$ is less simple. It is clear, however, that so long as only such functions of A, B, C as are unaltered except in sign by substituting $-A + 2\pi$ for $-A$, etc., the results will be the same as before, A, B and C being reckoned along with r, r_1 etc., in counting dimensions.

Thus any formula may contain any trigonometrical function of A or of any multiple of A , also $\cot \frac{A}{2}, \tan \frac{A}{2}$, without making any difference in our conclusions. But if such functions as $\cos \frac{A}{2}, \sin \frac{A}{3}$ occur, the above conclusions cannot be drawn; and I doubt whether in such cases we could even depend on the validity of the "transformation continue" itself, without special precautions.