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# A NOTE ON DERIVATIONS OF GROUP RINGS

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ABSTRACT. Let $RG$ denote the group ring of a group $G$ over a semiprime ring $R$. We prove that, if the center of $G$ is of finite index and some natural restrictions hold, then every $R$-derivation of $RG$ is inner. We also give an example of a group $G$ which is both locally finite and nilpotent and such that, for every field $F$, there exists an $F$-derivation of $FG$ which is not inner.

1. **Introduction.** Let $RG$ denote the group ring of a group $G$ over a ring with unity $R$. In this paper, we shall study $R$-derivations of $RG$ *i.e.*, derivations $d: RG \longrightarrow RG$ such that $d(R) = 0$, in case where $R$ is a semiprime ring and $G$ is a torsion group.

A ring $R$ is said to be of *characteristic* 0 if $R$ has no non-zero torsion elements. Otherwise, there exists a set $\mathcal{P}$ of prime integers such that, for every prime $p \in \mathcal{P}$ there exists a non-zero ideal $I$ of $R$ verifying $pI = 0$. Any of the primes $p \in \mathcal{P}$ is called *a characteristic* of $R$.

We recall that a derivation $d$ of a ring $S$ is said to be *inner* if there exists an element $s \in S$ such that $d(x) = sx - xs$ for every element $x \in S$. Our main result in this paper is the following.

THEOREM 1.1. *Let $R$ be a semiprime ring and $G$ a torsion group such that $[G : Z(G)] < \infty$, where $Z(G)$ denotes the center of $G$. Suppose that either* $\operatorname{char} R = 0$ *or for every characteristic $p$ of $R$, $p \nmid o(g)$, for all $g \in G$. Then every $R$-derivation of $RG$ is inner.*

We remark that the same conclusion does not hold in general. In fact, we shall give an example of a locally finite nilpotent group $G$ such that, for any field $F$, there exists an $F$-derivation of $FG$ which is not inner.

We also note that if $d$ is any derivation of the group ring $RG$ such that $d(R) \subset R$, then $d$ can be written as $d = d_1 + d_2$, where $d_1$ is a derivation of $R$ (with $d_1(G) = 0$) and $d_2$ is an $R$-derivation of $RG$. Thus, results about $R$-derivations of $RG$ can be extended to derivations $d$ such that $d(R) \subset R$. We recall that derivations that are not inner also appear in group rings of torsion free groups [2].

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2. **The results.** We first observe that if $Z = Z(R)$ is the center of $R$, then $d$ induces a $Z$-derivation of $ZG$. In fact, given $r \in R$ and $g \in G$, we have that $rg = gr$ so $rd(g) = d(g)r$ meaning that $d(g) \in ZG$. We shall denote by $d_Z$ the restriction of $d$ to $ZG$.

Now, let $T$ be another ring such that $R \subset T$ and $Z(R) \subset Z(T)$. Then, identifying $TG$ with $T \otimes_Z ZG$, $d$ can be extended in a natural way to a $T$-derivation $d_T$ of $TG$ by defining $d_T = 1 \otimes d\colon T \otimes_Z ZG \longrightarrow T \otimes_Z ZG$.

PROPOSITION 2.1. *Let $R \subset T$ be rings with the same unit, such that $Z(R) \subset Z(T)$ and let $d$ be an $R$-derivation of a group ring $RG$. Then $d$ is inner if and only if $d_T$ is inner.*

PROOF. Assume first that $d$ is an inner derivation induced by an element $\alpha \in RG$. For every element $r \in R$ we have that $0 = d(r) = \alpha r - r\alpha$ so it follows that $\alpha \in ZG$. Thus, given $t \in T$ and $g \in G$, we have that $d_T(tg) = t(\alpha g - g\alpha) = \alpha(tg) - (tg)\alpha$ and $d_T$ is also inner.

Conversely, assume now that $d_T$ is inner induced by an element $\alpha \in TG$. As before, we obtain that $\alpha \in Z(T)G$.

Write $\alpha = \sum_{g\in G} \alpha_g g$, with $\alpha_g \in Z(T)$, for all $g \in G$. Given an arbitrary element $h \in G$, we compute:

$$d(h)h^{-1} = d_T(h)h^{-1} = (\alpha h - h\alpha)h^{-1} = \sum_{g\in G} \alpha_g(gh - hg)h^{-1},$$

*i.e.*

$$d(h)h^{-1} = \sum_{g\in G} \alpha_g(g - hgh^{-1}) = \sum_{g\in G}(\alpha_g - \alpha_{h^{-1}gh})g.$$

Since $d(h)h^{-1} \in RG$, it follows that $\alpha_g - \alpha_{h^{-1}gh} = r_{gh} \in R$, for all $g, h \in G$. Set $\operatorname{supp}(\alpha) = \{g \in G \mid \alpha_g \neq 0\}$; then, the relation above says that if $g \in \operatorname{supp}(\alpha)$ and $\alpha_g \notin R$, we have that $\alpha_{h^{-1}gh} \neq 0$, for all $h \in G$. Hence, $g$ has finitely many conjugates and thus $g \in \Phi$, the FC-subgroup of $G$.

Now, write $\alpha = \beta + \gamma$, where $\beta = \sum_{g\notin\Phi} \alpha_g g$ and $\gamma = \sum_{g\in\Phi} \alpha_g g$. As shown above, if $g \notin \Phi$ then $\alpha_g \in R$ so $\beta \in RG$. Moreover, if $\Gamma$ is a set of representatives of the conjugacy classes of $G$ contained in $\Phi$, we can write:

$$\begin{aligned}
\gamma &= \sum_{g\in\Phi} \alpha_g g = \sum_{g\in\Gamma}(\alpha_g g + \alpha_{x^{-1}gx}x^{-1}gx + \cdots) \\
&= \sum_{g\in\Gamma}\Big(\alpha_g g + (\alpha_g - r_{gx})x^{-1}gx + \cdots\Big) \\
&= \sum_{g\in\Gamma} \alpha_g(g + x^{-1}gx + \cdots) - \sum_{g\in\Gamma}(r_{gx}x^{-1}gx + \cdots) \\
&= \sum_{g\in\Gamma} \alpha_g K_g + \delta,
\end{aligned}$$

where $K_g$ is the class sum of the conjugacy class of $g \in \Phi$ and $\delta \in RG$.

Hence, $\alpha = \beta + \delta + \sum_{g\in\Gamma} \alpha_g K_g$, where $\beta, \delta \in RG$. Since class sums lie in the center of $RG$ and $\alpha_g \in Z(T)$, it follows that $\sum_{g\in\Gamma} \alpha_g K_g \in Z(TG)$. Thus, the inner derivation

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induced by $\alpha$ coincides with the one induced by $\beta + \delta \in RG$ and consequently, $d$ is inner in $RG$. ∎

We are now ready to prove our main result.

PROOF OF THE THEOREM. First, assume that $G$ is a finite group and that $p \not| \, |G|$, for every characteristic $p$ of $R$. Because of the proposition above, it is enough to prove the result when $R$ is a commutative semiprime ring. We claim that we also may assume that $R$ is Noetherian. In fact, let $R'$ be the subring generated by the finitely many elements of $R$ which occur as coefficients of elements in $d(G)$. Then $d$ restricts to a derivation of $R'G$ and $R'$ is semiprime and Noetherian and it follows again from our proposition that it suffices to prove that $d$ is inner in $R'G$.

Let $P_1, P_2, \ldots, P_n$ be the finitely many minimal prime ideals of $R$ and let $F_i$ be an algebraically closed field containing $R/P_i$, $1 \leq i \leq n$. Since $R$ is semiprime, we have that $\bigcap_{i=1}^n P_i = 0$; hence, $R$ can be imbedded in $T = \bigoplus_{i=1}^n F_i$. Thus, by the proposition, it is enough to prove that $d_T$ is inner in $TG$.

Note that if the characteristic of $F_i$ is a prime integer $p_i$, then $p_i \in P_i$ so $p_i(\bigcap_{j \neq i} P_j) = 0$ and thus $p_i$ is a characteristic of $R$; therefore, $p_i \not| \, |G|$, $1 \leq i \leq n$. Hence, Maschke's Theorem shows that $TG$ is a direct sum of full matrix rings over fields, say $TG = I_1 \oplus \cdots \oplus I_k$, where each $I_j$ is generated, as an ideal, by a central idempotent. Since it is easily seen that $d_T(e) = 0$ for every central idempotent $e \in TG$, it follows that $d_T(I_j) \subset I_j$ and thus $d_T$ gives, by restriction, a derivation of each component, which is inner, induced by an element $a_j \in I_j$ [1, p. 100]. Consequently $d_T$ is the inner derivation of $TG$ induced by $a = a_1 + \cdots + a_k$.

Now we consider the general case. We first notice that $d\big(Z(G)\big) = 0$. In fact, given $z \in Z(G)$ with $o(z) = m$, we have that $0 = d(z^m) = mz^{m-1}d(z)$ and the assumption implies that $d(z) = 0$.

Now, let $X = \{g_1, \ldots, g_n\}$ be a transversal of $Z(G)$ in $G$. For every index $i$, $1 \leq i \leq n$, we write:

$$d(g_i) = \sum_{i,j,k} \alpha_{ijk} z_{ijk} g_k, \quad z_{ijk} \in Z(G), \ \alpha_{ijk} \in R.$$

Also, for $i,j = 1, \ldots, n$ let $g_i g_j = c_{ij} g_k$, $c_{ij} \in Z(G)$. Denote by $H$ the subgroup of $G$ generated by all the elements $z_{ijk}$, $c_{ij}$, $g_l$. Since $Z(G)$ is abelian and $G$ is torsion, it follows that $H$ is finite. Also, the restriction $d|_{RH}$ is an $R$-derivation of $RH$. By the first part, there exists an element $\alpha \in RH$ such that $d|_{RH}$ is the inner derivation induced by $\alpha$.

Now, given an element $g \in G$, write $g = zg_i$, with $z \in Z(G)$, $1 \leq i \leq n$. Then:

$$d(g) = zd(g_i) = z(\alpha g_i - g_i \alpha) = \alpha(zg_i) - (zg_i)\alpha = \alpha g - g\alpha.$$

Consequently, $d$ is inner in $RG$, induced by $\alpha$.

3. **An example.** Let $H$ be a finite, non-abelian $p$-group and let $G = \prod_i H_i$ be the direct sum of infinitely many copies of $H$. Then $G$ is a locally finite nilpotent group and it can be written in the form $G = \cup_n G_n$, where $G_n = \prod_{i=1}^n H_i$. Consider any field $F$ and the group algebra $FG$. Let $x_i$ be any non-central element of $H_i$ and define $d: FG \longrightarrow FG$ by $d(\alpha) = \sum_{i=1}^{\infty} [x_i, \alpha]$, where $[x_i, \alpha] = x_i\alpha - \alpha x_i$. Notice that, if $\alpha \in FG_n$, then all summands after the $n$-th are zero and hence $d(\alpha) \in FG_n$. Indeed, $d$ restricted to $FG_n$ is just the inner derivation induced by $x_1 + \cdots + x_n$. Thus $d$ is a derivation of $FG$. Furthermore, it is not an inner derivation since $d(H_i) \neq 0$ for all indices $i$.

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