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A GENERALISATION OF THE QUINTUPLE PRODUCT IDENTITY

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Abstract

The quintuple product identity has appeared many times in the literature. Indeed, no fewer than 12 proofs have been given. We establish a more general identity from which the quintuple product identity follows in two ways.

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1. The quintuple product identity

$$
\begin{aligned}
&\prod_{n\geqslant 1}(1+aq^n)(1+a^{-1}q^{n-1})(1-a^2q^{2n-1})(1-a^{-2}q^{2n-1})(1-q^n)\\
(1)\qquad &\quad=\prod_{n\geqslant 1}(1-a^{-3}q^{3n-1})(1-a^3q^{3n-2})(1-q^{3n})\\
&\qquad+a^{-1}\prod_{n\geqslant 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-q^{3n})
\end{aligned}
$$

has appeared many times in the literature, in several guises, and has a long and involved pedigree.

The quintuple product identity was first brought to prominence by G. N. Watson (1929) in the course of proving certain results concerning continued fractions which had been stated without proof by Ramanujan, and again by Watson (1938), in the course of proving that $p(n)$, the number of partitions of n, satisfies certain congruences modulo powers of 5 and 7 conjectured by Ramanujan. The quintuple product identity is, however, implicit in the earlier work of Weierstrass on elliptic functions; it is one of the basic identities satisfied by

Weierstrass's sigma function (Schwartz (1893), page 47). It was also stated, in terms of theta functions, by R. Fricke (1916), pages 432–433. W. N. Bailey (1951), aware of Watson's 1929 paper, gave a simple proof of a slightly different version of the quintuple product identity, and employed it to simplify certain identities of the Rogers Ramanujan type. D. B. Sears (1952) showed that the quintuple product identity, as well as another identity given by Bailey, follows from an identity he had stated earlier, in Sears (1951), and which is, as he pointed out, essentially the result of Weierstrass referred to above. This derivation of the quintuple product identity was later given in her book by L. J. Slater (1966). Another proof of the quintuple product identity was given by A. O. L. Atkin and P. Swinnerton-Dyer (1954) in their proof of the Dyson conjectures for $p(n)$.

B. Gordon (1961), apparently unaware of all the foregoing, rediscovered the quintuple product identity and gave some applications of it, while L. J. Mordell (1961) believing that the identity was new with Gordon, also gave a proof. L. Carlitz, in his review of Gordon's paper, pointed out that Gordon had been anticipated by Bailey. M. V. Subbarao and M. Vidyasagar (1970) gave two identities equivalent to the quintuple product identity. Their work inspired L. Carlitz (1972) to give two proofs of the quintuple product identity, and these were soon followed by a simple proof in Carlitz and Subbarao (1972). George E. Andrews (1974) showed that the quintuple product identity follows from a very powerful result due to Bailey, namely his summation of a well poised ${}_6\psi_6$. The quintuple product identity was proved yet again by C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson (1985) in their analysis of Chapter 16 of Ramanujan's second notebook.

Finally, the referee informs me that K. G. Ramanathan has recently found a version of the quintuple product identity in the so-called "lost notebook" of Ramanujan, confirming a conviction expressed by Watson in his 1938 paper and by Adiag *et al.* in 1985 that Ramanujan was aware of the identity.

The object of this note is to establish the more general identity

$$
\begin{aligned}
&\prod_{n\geqslant 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n})\\
&\qquad\times(1+bq^{4n-2})(1+b^{-1}q^{4n-2})(1-q^{4n})\\
&\quad=\prod_{n\geqslant 1}(1+ab^{-1}q^{6n-3})(1+a^{-1}bq^{6n-3})(1-q^{6n})\\
&\qquad\qquad\times(1+a^2bq^{12n-6})(1+a^{-2}b^{-1}q^{12n-6})(1-q^{12n})\\
&\quad+aq\prod_{n\geqslant 1}(1+ab^{-1}q^{6n-1})(1+a^{-1}bq^{6n-5})(1-q^{6n})\\
&\qquad\qquad\times(1+a^2bq^{12n-2})(1+a^{-2}b^{-1}q^{12n-10})(1-q^{12n})\\
&\quad+a^{-1}q\prod_{n\geqslant 1}(1+ab^{-1}q^{6n-5})(1+a^{-1}bq^{6n-1})(1-q^{6n})\\
&\qquad\qquad\times(1+a^2bq^{12n-10})(1+a^{-2}b^{-1}q^{12n-2})(1-q^{12n}),
\end{aligned}
\tag{2}
$$

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from which the quintuple product identity follows in two ways. Thus we obtain (1) if we set aq for a, $-a^{-2}$ for b, q for q^2 and divide by $\prod_{n\geqslant 1}(1-q^{2n}) = \prod_{n\geqslant 1}(1-q^{6n-4})(1-q^{6n-2})(1-q^{6n})$ or if we set aq for a, $-a$ for b, q for q^2, divide by $\prod_{n\geqslant 1}(1-q^n) = \prod_{n\geqslant 1}(1-q^{3n-2})(1-q^{3n-1})(1-q^{3n})$ then set q for q^2 again.

2. Proof of identity (2)

We have, by Jacobi's triple product identity,

$$\prod_{n\geqslant 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n}) \times(1+bq^{4n-2})(1+b^{-1}q^{4n-2})(1-q^{4n}) = \sum_{r,\,s=-\infty}^{\infty} a^r b^s q^{r^2+2s^2}.$$

We perform this summation by first summing 'diagonally':

$$\prod_{n\geqslant 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n}) \times(1+bq^{4n-2})(1+b^{-1}q^{4n-2})(1-q^{4n}) = \sum_{n=-\infty}^{\infty}\sum_{r+s=n} a^r b^s q^{r^2+2s^2}.$$

We now consider the three cases $n = 3m,\ 3m+1,\ 3m-1$. In the first case set $r = 2m+t$, $s = m-t$, in the second set $r = 2m+t+1$, $s = m-t$, and in the third set $r = 2m+t-1$, $s = m-t$. We obtain

$$\begin{aligned}
&\prod_{n\geqslant 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n}) \\
&\qquad\times(1+bq^{4n-2})(1+b^{-1}q^{4n-2})(1-q^{4n}) \\
&= \sum_{m,\,t=-\infty}^{\infty} a^{2m+t}b^{m-t}q^{(2m+t)^2+2(m-t)^2} \\
&\quad+ \sum_{m,\,t=-\infty}^{\infty} a^{2m+t+1}b^{m-t}q^{(2m+t+1)^2+2(m-t)^2} \\
&\quad+ \sum_{m,\,t=-\infty}^{\infty} a^{2m+t-1}b^{m-t}q^{(2m+t-1)^2+2(m-t)^2} \\
&= \sum_{-\infty}^{\infty}(a^2b)^m q^{6m^2}\sum_{-\infty}^{\infty}(ab^{-1})^t q^{3t^2} + aq\sum_{-\infty}^{\infty}(a^2bq^4)^m q^{6m^2}\sum_{-\infty}^{\infty}(ab^{-1}q^2)^t q^{3t^2} \\
&\quad+ a^{-1}q\sum_{-\infty}^{\infty}(a^2bq^{-4})^m q^{6m^2}\sum_{-\infty}^{\infty}(ab^{-1}q^{-2})^t q^{3t^2}
\end{aligned}$$

which, by Jacobi again, yields (2).

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