ENUMERATION OF BICOLOURABLE GRAPHS

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In a previous paper **(2)**, one of us has derived a formula for the counting series for bicoloured graphs.[2] These are graphs each of whose points *has been* coloured with exactly one of two colours in such a way that every two adjacent points have different colours.

In this paper we first enumerate bicoloured graphs without isolated points and connected bicoloured graphs. This leads us to corresponding problems for bicolourable graphs. Such a graph has the property that its points *can be* coloured with two colours so as to obtain a bicoloured graph. The enumeration of connected bicolourable graphs and bicolourable graphs without isolated points solves a problem proposed to us by Pólya.

In **(3)**, an outline for a programme to attempt to settle the four-colour conjecture by enumeration was presented. This involves the derivation of generating functions for all planar graphs and for planar four-colourable graphs, followed by the confrontation of these two counting series. Thus the present paper may be a small step in this project. Recently Tutte **(9)** has counted the number of triangulations of a triangle, thereby making a beginning towards the enumeration of planar graphs.

1. Introduction. A *graph* G consists of a finite set of *points* $v_1, v_2, \ldots, v_p$ and a collection of q *lines* each of which joins two distinct points, with at most one line joining the same pair of points. Two points of G are *adjacent* if they are joined by a line. Two *graphs* are *isomorphic* if there is a 1–1 correspondence between their sets of points which preserves adjacency. A *bicoloured graph,* or more briefly, a *bigraph,* is a graph each of whose points has been assigned one of two colours so that every two adjacent points have different colours. Two bigraphs G and H are *isomorphic* if there exists an isomorphism θ from G onto H (as ordinary graphs) with the property that $\theta(v_1)$ and $\theta(v_2)$ have the same colour in H if and only if v_1 and v_2 have the same colour in G.

Consider two bigraphs G and H having the same two colours. Then G and H are *colour-isomorphic* if there exists an isomorphism θ from G onto H (as ordinary graphs) such that for every point v of G, $\theta(v)$ has the same colour as v.

The *colour image* of G is obtained by changing the colour of every point of G. Thus G is isomorphic to its colour image. If G and its colour image are

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[2]A knowledge of the paper **(2)** is a prerequisite for the present article.

colour-isomorphic, then G is called *symmetric.* If G has an equal number of points of each colour but is not symmetric, we say that G is *non-symmetric.* Figure 1 shows a symmetric bigraph; Figure 2, a non-symmetric one. In these two figures the letters r and g placed near the points stand for the colours red and green respectively. Every bigraph in this paper will be understood to have colours red and green.

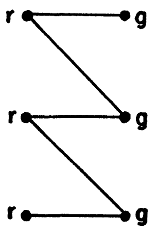

FIGURE 1

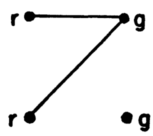

FIGURE 2

2. Counting series for bigraphs. We now develop the notation required for the counting formulas to be derived. Let b_{ijk} be the number of non-isomorphic bigraphs with i lines, j points of one colour, and k points of the other colour, $0 \leqslant j \leqslant k$, $k > 0$, and let

$$B(x, y, z) = \sum b_{ijk}\, x^i\, y^j\, z^k$$

be the counting series for these graphs. A formula for $B(x, y, z)$ may be obtained by a routine modification of the results in **(2)**. This formula makes use of Pólya's counting theorem **(6)**, and of the cartesian product $A \times B$ and exponentiation B^A of two permutation groups A and B. The meanings of these operations and the proof of equation (1) may be found in **(2)**. The definition of the cycle index $Z(A)$ of the permutation group A is given in **(1)** and **(6)**. As usual, S_n denotes the symmetric group of degree n.

The counting series for non-isomorphic bigraphs is given by

$$(1) \qquad B(x, y, z) = \sum_{1 \leqslant m < n} Z(S_m \times S_n, 1 + x) y^m z^n + \sum_{n \geqslant 1} Z(S_n^{S_2}, 1 + x) y^n z^n + \sum_{n \geqslant 1} z^n.$$

Let b^*_{ijk} be the number of colour-non-isomorphic bigraphs with i lines, j green points, and k red points. Let

$$B^*(x, y, z) = \sum b^*_{ijk}\, x^i y^j z^k.$$

LEMMA 1. *Let G and H be isomorphic bigraphs. Then either G and H are colour-isomorphic or G and the colour image of H are colour-isomorphic.*

It follows immediately from this obvious but useful result that the number of colour-non-isomorphic bigraphs with m red points and n green points,

$m < n$, is equal to the number of colour-non-isomorphic bigraphs with m points of one colour and n points of the other colour, and hence is counted by

$$Z(S_m \times S_n, 1 + x)y^m z^n.$$

In the proof in **(2)** of Formula (1) we incidentally obtained the counting series for colour-non-isomorphic bigraphs with the same number n of points of each colour. This series is $Z(S_n \times S_n, 1 + x)y^n z^n$. Hence we may state the next result.

The counting series for colour-non-isomorphic bigraphs is given by

$$B^*(x, y, z) = \sum_{n \geqslant 1} y^n + \sum_{m,n \geqslant 1} Z(S_m \times S_n, 1 + x)y^m z^n + \sum_{n \geqslant 1} z^n. \tag{2}$$

Let $E(x, y, z)$ be the subseries of $B(x, y, z)$ which counts those bigraphs with an equal number of points of each colour, and let $E^*(x, y, z)$ be the corresponding subseries of $B^*(x, y, z)$. It follows from equations (1) and (2) that

$$E(x, y, z) = \sum_{n \geqslant 1} Z(S_n^{S_2}, 1 + x)y^n z^n, \tag{3}$$

$$E^*(x, y, z) = \sum_{n \geqslant 1} Z(S_n \times S_n, 1 + x)y^n z^n. \tag{4}$$

Now let $M(x, y, z)$ and $\bar{M}(x, y, z)$ be the counting series for symmetric and non-symmetric bigraphs respectively, with the same number of points of each colour, so that

$$M(x, y, z) + \bar{M}(x, y, z) = E(x, y, z).$$

LEMMA 2. *The counting series for symmetric and non-symmetric bigraphs are given by the formulas:*

$$\bar{M}(x, y, z) = E^*(x, y, z) - E(x, y, z), \tag{5}$$

$$M(x, y, z) = E^*(x, y, z) - 2\bar{M}(x, y, z) = 2E(x, y, z) - E^*(x, y, z). \tag{6}$$

3. Colour-non-isomorphic bigraphs with no isolated points.

By adding a red point to any bigraph G, we obtain a bigraph with at least one isolated red point. Similarly, we can add a green point to G to obtain a bigraph with at least one isolated green point. Therefore, the series

$$y(1 + B^*(x, y, z)) + z(1 + B^*(x, y, z))$$

counts all colour-non-isomorphic bigraphs with at least one isolated point. However, some bigraphs are counted twice by this series, namely those that have both a red and a green isolated point. As these bigraphs are counted by the series

$$yz(1 + B^*(x, y, z)),$$

we obtain the following formula.

THEOREM 1. *The counting series $I^*(x, y, z)$ for colour-non-isomorphic bigraphs without isolated points is given by*

$$1 + I^*(x, y, z) = (1 - y - z + yz)(1 + B^*(x, y, z)). \tag{7}$$

4. Connected graphs and indecomposable figures. In order to enumerate in the next section those colour-non-isomorphic bigraphs which are connected, we need a well-known formula which counts connected graphs in terms of all graphs. This result was first reported by Riddell and Uhlenbeck **(7)** and also appears in Riordan **(8**, p. 147**)** and in Harary **(1)**.

Let $g(x, y)$ be the counting series for all graphs and let $c(x, y)$ be the counting series for connected graphs:

$$1 + g(x, y) = Z(S_\infty, c(x, y)), \tag{8}$$

where, following **(4)**,

$$Z(S_\infty, c(x, y)) = \sum_{n=0}^{\infty} Z(S_n, c(x, y))$$

and by definition $Z(S_0, c(x, y)) = 1$.

In Section 7, we shall need the following generalization of equation (8). Let Φ be a collection of sets ϕ_i, which in analogy with the presentation in Pólya **(6)** we call "figures." It is assumed that the content of a figure ϕ is an ordered n-tuple $(i_1, i_2, \ldots, i_n)$ of non-negative integers. We stipulate that if $\phi_1, \phi_2 \in \Phi$ are disjoint and $\phi_1 \cup \phi_2 \in \Phi$, then the content of $\phi_1 \cup \phi_2$ is the vector sum of the contents of ϕ_1 and of ϕ_2. Let $f_{i_1 \ldots i_n}$ be the number of figures in Φ whose content is the n-tuple $(i_1, \ldots, i_n)$. Then the figure counting series of the figure collection Φ is defined to be

$$f(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n = 0}^{\infty} f_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}. \tag{9}$$

Let Φ be a figure collection satisfying the above conditions which does not contain the empty set as a figure. The figure ϕ is *indecomposable* if ϕ is not the union of two disjoint figures.

THEOREM 2. *Let Φ be a figure collection which does not contain the empty set and is closed under the union of disjoint figures. The counting series $c(x_1, \ldots, x_n)$ of all the indecomposable figures in the figure collection can be obtained recursively from the identity*

$$1 + f(x_1, \ldots, x_n) = Z(S_\infty, c(x_1, \ldots, x_n)). \tag{10}$$

There is a well-known combinatorial identity which facilitates computation of actual numbers from equation (8) or (10):

$$Z(S_\infty, f(x_1, \ldots, x_n)) = \exp \sum_{r=1}^{\infty} \frac{1}{r} f(x_1^r, \ldots, x_n^r). \tag{11}$$

5. Colour-non-isomorphic connected bigraphs. A specialization of equation (10) readily yields the counting series $C^*(x, y, z)$ for colour-non-isomorphic connected bigraphs.

THEOREM 3. *The counting series $C^*(x, y, z)$ is obtained implicitly in terms of $B^*(x, y, z)$ by the equation*

$$1 + B^*(x, y, z) = Z(S_\infty, C^*(x, y, z)). \tag{12}$$

6. Bigraphs with no isolated points. The object of this section is to calculate the counting series $I(x, y, z)$ for bigraphs with no isolated points (*isolates*). This will be accomplished by introducing several additional counting series and manipulating these. One such counting series is that which enumerates bigraphs without isolates and with the same number of points of each colour, i.e., those bigraphs that are counted by both of the counting series $I(x, y, z)$ and $E(x, y, z)$. We denote this counting series by $(E \cap I)(x, y, z)$ or more briefly by $EI(x, y, z)$.

Similarly, let $MI(x, y, z)$ be the counting series for bigraphs which are symmetric and have no isolates and let $\bar{M}I(x, y, z)$ count bigraphs that are non-symmetric and have no isolates. Another counting series which is useful is $E^*I^*(x, y, z)$, which counts colour-non-isomorphic bigraphs with the same number of points of each colour and without isolates.

Our plan is to derive $I = I(x, y, z)$ by finding both summands of

$$I = (I - EI) + EI.$$

By definition, $I - EI = I(x, y, z) - EI(x, y, z)$ counts those non-isomorphic bigraphs with a different number of points of each colour. But this is precisely the same as the colour-non-isomorphic bigraphs with a different number of points of each colour. However, these bigraphs are counted by the subseries of $I^*(x, y, z)$ consisting of all terms in which the exponent of y is less than that of z. Thus the terms of $I - EI$ can be read off from an expansion of $I^*(x, y, z)$ obtained from equation (7).

In order to find $EI(x, y, z)$, we require the equations of the next lemma. These are obtained in the same manner as the equations of Lemma 2 by considering the series analogous to the ones occurring in equations (5) and (6) which emunerate bigraphs with no isolates.

LEMMA 3. *The counting series $\bar{M}I$ and MI satisfy the following identities:*

$$\bar{M}I(x, y, z) = E^*I^*(x, y, z) - EI(x, y, z), \tag{13}$$

$$MI(x, y, z) = E^*I^*(x, y, z) - 2\bar{M}I(x, y, z). \tag{14}$$

From these two equations we see at once that

$$EI(x, y, z) = \tfrac{1}{2}[E^*I^*(x, y, z) + MI(x, y, z)]. \tag{15}$$

By equation (15), the counting series EI is known as soon as the two series E^*I^* and MI are determined. But E^*I^* is that subseries of I^* obtained from equation (7) by taking all terms in which y and z have equal exponents.

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The counting series $M = M(x, y, z)$ is already known by equation (6). We now derive an expression for the counting series MI in terms of M.

LEMMA 4. *The counting series MI satisfies the following identity:*

$$1 + MI(x, y, z) = (1 - yz)(1 + M(x, y, z)). \tag{16}$$

To prove equation (16), note that if a symmetric bigraph has one isolate, then it has a second one of the other colour. Hence the series $yz(1+M(x, y, z))$ counts all symmetric bigraphs with at least one isolate, proving the lemma.

On substituting the series MI obtained from equation (16) into equation (15), we obtain the series EI, which can then be added to the series $(I - EI)$ to yield the desired counting series $I(x, y, z)$.

THEOREM 4. *The counting series for bicoloured graphs with no isolates is given by:*

$$I(x, y, z) = [I(x, y, z) - EI(x, y, z)] + \tfrac{1}{2}E^*I^*(x, y, z) + \tfrac{1}{2}[(1 - yz)M(x, y, z) - yz],$$

where $I - EI$ and E^*I^* can be read off from equation (7) *and M is determined by equation* (6).

7. Connected bigraphs. The object of this section is to calculate the counting series $C(x, y, z)$ for connected bigraphs. This replaces the incorrect formulas given in Section 5 of **(2)**. We shall require four auxiliary counting series which are denoted in analogy with the preceding section by

$$EC = EC(x, y, z),\quad MC,\quad \bar{M}C,\quad \text{and}\quad E^*C^*.$$

The first part of the derivation of series $C(x, y, z)$ proceeds analogously to the derivation of the preceding section and will be presented in outline form. We begin by writing

$$C = (C - EC) + EC.$$

The series $C - EC$ is read off from the known series $C^*(x, y, z)$ found in equation (12). The next three equations are immediately derived by analogy with equations (13), (14), and (15):

$$\bar{M}C(x, y, z) = E^*C^*(x, y, z) - EC(x, y, z), \tag{17}$$

$$MC(x, y, z) = E^*C^*(x, y, z) - 2\bar{M}C(x, y, z), \tag{18}$$

$$EC(x, y, z) = \tfrac{1}{2}[E^*C^*(x, y, z) + MC(x, y, z)]. \tag{19}$$

As before, the series E^*C^* is obtained at once from C^*. Since this settles the term EC, we still need only to determine the counting series MC. At this point the procedure deviates from that of the preceding section.

Lemma 7 will provide a recursive formula for MC. But we require two preliminary results for its proof, the first of which makes use of Theorem 2.

By Theorem 2 we can immediately obtain the counting series for symmetric

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bigraphs which are indecomposable with respect to the collection of all symmetric bigraphs. These indecomposable bigraphs are exactly those symmetric bigraphs which are not the union of two disjoint symmetric bigraphs. We shall refer to such a bigraph briefly as *indecomposable.*

Clearly, every connected symmetric bigraph is indecomposable. But Figure 3 shows a symmetric bigraph which is not connected and is indecomposable, since its components are not symmetric.

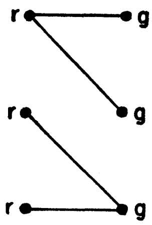

FIGURE 3

LEMMA 5. *Let $f(x, y, z)$ be the counting series for all indecomposable symmetric bigraphs. Then $f(x, y, z)$ can be obtained recursively from*

$$1 + M(x, y, z) = Z(S_\infty, f(x, y, z)). \tag{20}$$

We next require a characterization of those indecomposable symmetric bigraphs which are disconnected.

LEMMA 6. *A bigraph G is symmetric, indecomposable, and disconnected if and only if G is the union $H \cup H'$ of a connected bigraph H which is not symmetric and its disjoint colour image H'.*

To prove the sufficiency, we first note that G is obviously disconnected since it has exactly two components H and H'. Since the colour image of $H \cup H'$ is $H' \cup H$, it follows that G is symmetric. Finally, G is indecomposable by definition since neither of its two components is symmetric.

Conversely, let H be a component of the disconnected bigraph G. Then H is not symmetric because G is indecomposable. Thus the colour image H' of H is also a component of G and is disjoint with H. As $H \cup H'$ is indecomposable by definition, G contains no other components beside H and H'; hence $G = H \cup H'$.

LEMMA 7. *The counting series $MC(x, y, z)$ can be determined recursively in terms of $f(x, y, z)$ by the following equations:*

$$f(x, y, z) = \tfrac{1}{2}C^*(x^2, yz, yz) - \tfrac{1}{2}MC(x^2, y^2, z^2) + MC(x, y, z). \tag{21}$$

To prove this lemma we first count all disconnected indecomposable bigraphs using Lemma 6. Let $d(x, y, z)$ be that counting series. By definition $f(x, y, z)$ counts all indecomposable bigraphs. Hence

$$MC(x, y, z) = f(x, y, z) - d(x, y, z). \tag{22}$$

By Lemma 6 each figure ψ counted by $d(x, y, z)$ is the union of a figure ϕ (which is a connected bigraph that is not symmetric) and its colour image ϕ'. Let the content of ϕ be (i, j, k). Then the content of ϕ' is (i, k, j) and that of ψ is $(2i, j + k, j + k)$. The series $C^* - M^*C^*$ counts twice those connected bigraphs which are not symmetric. Hence we see that

$$(23)\qquad d(x, y, z) = \tfrac{1}{2}[C^*(x^2, yz, yz) - M^*C^*(x^2, yz, yz)].$$

But the series MC and M^*C^* are the same because the number of non-isomorphic connected symmetric bigraphs is equal to the number of colour-non-isomorphic connected symmetric bigraphs since the colour image of any symmetric bigraph is itself. Hence $M^*C^*(x^2, yz, yz) = MC(x^2, yz, yz)$. As symmetric bigraphs have the same number of points of each colour, $MC(x^2, yz, yz) = MC(x^2, y^2, z^2)$. Thus equation (23) becomes:

$$(24)\qquad d(x, y, z) = \tfrac{1}{2}C^*(x^2, yz, yz) - \tfrac{1}{2}MC(x^2, y^2, z^2).$$

Combining (22) and (24) yields equation (21), proving Lemma 7.

On substituting the series MC obtained recursively from equation (21) into equation (19), we obtain the series EC, which can then be added to the series $(C - EC)$ to yield the desired counting series $C(x, y, z)$.

THEOREM 5. *The counting series for connected bicoloured graphs is given by*

$$C(x, y, z) = [C(x, y, z) - EC(x, y, z)] + \tfrac{1}{2}[E^*C^*(x, y, z) + MC(x, y, z)],$$

where $C - EC$ and E^*C^* can be read off from equation (12) *and MC can be obtained recursively from equation* (21).

8. Bicolourable graphs. We wish to exploit a relationship between connected bicoloured graphs and connected bicolourable graphs. It has been proved by König (**5**, p. 170) that a graph is bicolourable if and only if every cycle is of even length. It is easy to see that this condition is equivalent to the statement that a graph is bicolourable if and only if all paths joining the same pair of points have odd length or they all have even length. Thus if G is a connected bicolourable graph, then by colouring any one point of G green, the colour of every point is uniquely determined. As an immediate consequence, we obtain the following statement.

THEOREM 6. *The number of non-isomorphic connected bicolourable graphs is equal to the number of non-isomorphic connected bicoloured graphs.*

It follows at once that the counting series $C(x, y, y)$ counts connected bicolourable graphs, where the coefficient of $x^q y^p$ is the number of non-isomorphic connected bicolourable graphs with p points and q lines. Let $c(x, y) = C(x, y, y)$ be the counting series for connected bicolourable graphs. Let $b(x, y)$ be the corresponding counting series for all bicolourable graphs, connected or not. Then an application of equation (8) shows that the following equation holds.

THEOREM 7. *The counting series for bicolourable graphs is expressed implicitly in terms of that for connected bicolourable graphs by the equation:*

$$1 + b(x, y) = Z(S_\infty, c(x, y)). \tag{25}$$

The only other result we shall obtain is the counting series $i(x, y)$ for bicolourable graphs with no isolated points. This is obtained as an immediate corollary of the preceding theorem:

$$1 + i(x, y) = (1 - y)(1 + b(x, y)). \tag{26}$$

In the first appendix to this article, explicit expressions will be stated for various counting series of bicoloured graphs and bicolourable graphs with up to six points. The second appendix presents the diagrams of all connected bicolourable graphs with up to six points.

Appendix 1. Explicit expressions for counting series of bicoloured graphs and bicolourable graphs with up to six points.

Colour-non-isomorphic bigraphs.

$$\begin{aligned} B^*(x, y, z) = {} & [y + z] + [y^2 + yz(1 + x) + z^2] + [y^3 + y^2z(1 + x + x^2) \\ & + yz^2(1 + x + x^2) + z^3] + [y^4 + y^3z(1 + x + x^2 + x^3) \\ & + y^2z^2(1 + x + 3x^2 + x^3 + x^4) + yz^3(1 + x + x^2 + x^3) + z^4] \\ & + [y^5 + y^4z(1 + x + x^2 + x^3 + x^4) \\ & \qquad + y^3z^2(1 + x + 3x^2 + 3x^3 + 3x^4 + x^5 + x^6) \\ & + y^2z^3(1 + x + 3x^2 + 3x^3 + 3x^4 + x^5 + x^6) \\ & \qquad + yz^4(1 + x + x^2 + x^3 + x^4) + z^5] \\ & + [y^6 + y^5z(1 + x + x^2 + x^3 + x^4 + x^5) \\ & + y^4z^2(1 + x + 3x^2 + 3x^3 + 6x^4 + 3x^5 + 3x^6 + x^7 + x^8) \\ & + y^3z^3(1 + x + 3x^2 + 6x^3 + 7x^4 + 7x^5 + 6x^6 + 3x^7 + x^8 + x^9) \\ & + y^2z^4(1 + x + 3x^2 + 3x^3 + 6x^4 + 3x^5 + 3x^6 + x^7 + x^8) \\ & + yz^5(1 + x + x^2 + x^3 + x^4 + x^5) + z^6] + \dots \end{aligned}$$

Non-isomorphic bigraphs.

$$\begin{aligned} B(x, y, z) = {} & [z] + [yz(1 + x) + z^2] + [yz^2(1 + x + x^2) + z^3] \\ & + [y^2z^2(1 + x + 2x^2 + x^3 + x^4) + yz^3(1 + x + x^2 + x^3) + z^4] \\ & + [y^2z^3(1 + x + 3x^2 + 3x^3 + 3x^4 + x^5 + x^6) \\ & \qquad + yz^4(1 + x + x^2 + x^3 + x^4) + z^5] \\ & + [y^3z^3(1 + x + 2x^2 + 4x^3 + 5x^4 + 5x^5 + 4x^6 + 2x^7 + x^8 + x^9) \\ & + y^2z^4(1 + x + 3x^2 + 3x^3 + 6x^4 + 3x^5 + 3x^6 + x^7 + x^8) \\ & + yz^5(1 + x + x^2 + x^3 + x^4 + x^5) + z^6] + \dots \end{aligned}$$

Symmetric bigraphs.

$$\begin{aligned} M(x, y, z) = {} & yz(1 + x) + y^2z^2(1 + x + x^2 + x^3 + x^4) \\ & + y^3z^3(1 + x + x^2 + 2x^3 + 3x^4 + 3x^5 + 2x^6 + x^7 + x^8 + x^9) + \dots \end{aligned}$$

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Non-symmetric bigraphs.

$$\bar{M}(x, y, z) = y^2z^2x^2 + y^3z^3(x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + x^7) + \ldots$$

Colour-non-isomorphic bigraphs with no isolates.

$$\begin{aligned} I^*(x, y, z) = {} & yz(x) + [y^2z(x^2) + yz^2(x^2)] \\ & + [y^3z(x^3) + y^2z^2(x^2 + x^3 + x^4) + yz^3(x^3)] \\ & + [y^4z(x^4) + y^3z^2(x^3 + 2x^4 + x^5 + x^6) \\ & + y^2z^3(x^3 + 2x^4 + x^5 + x^6) + yz^4(x^4)] \\ & + [y^5z(x^5) + y^4z^2(2x^4 + 2x^5 + 2x^6 + x^7 + x^8) \\ & \qquad + y^3z^3(x^3 + 2x^4 + 5x^5 + 4x^6 + 3x^7 + x^8 + x^9) \\ & + y^2z^4(2x^4 + 2x^5 + 2x^6 + x^7 + x^8) + yz^5(x^5)] + \ldots \end{aligned}$$

Symmetric bigraphs with no isolates.

$$\begin{aligned} MI(x, y, z) = {} & yz(x) + y^2z^2(x^2 + x^3 + x^4) \\ & \qquad + y^3z^3(x^3 + 2x^4 + 3x^5 + 2x^6 + x^7 + x^8 + x^9) + \ldots \end{aligned}$$

Bigraphs with no isolates.

$$\begin{aligned} I(x, y, z) = {} & yzx + yz^2(x^2) + [y^2z^2(x^2 + x^3 + x^4) + yz^3(x^3)] \\ & + [y^2z^3(x^3 + 2x^4 + x^5 + x^6) + yz^4(x^4)] \\ & + [y^3z^3(x^3 + 2x^4 + 4x^5 + 3x^6 + 2x^7 + x^8 + x^9) \\ & + y^2z^4(2x^4 + 2x^5 + 2x^6 + x^7 + x^8) + yz^5(x^5)] + \ldots \end{aligned}$$

Colour-non-isomorphic connected bigraphs.

$$\begin{aligned} C^*(x, y, z) = {} & [y + z] + [yz(x)] + [y^2z(x^2) + yz^2(x^2)] \\ & + [y^3z(x^3) + y^2z^2(x^3 + x^4) + yz^3(x^3)] \\ & + [y^4z(x^4) + y^3z^2(2x^4 + x^5 + x^6) + y^2z^3(2x^4 + x^5 + x^6) + yz^4(x^4)] \\ & + [y^5z(x^5) + y^4z^2(2x^5 + 2x^6 + x^7 + x^8) \\ & + y^3z^3(4x^5 + 4x^6 + 3x^7 + x^8 + x^9) + y^2z^4(2x^5 + 2x^6 + x^7 + x^8) \\ & + yz^5(x^5)] + \ldots \end{aligned}$$

Indecomposable symmetric bigraphs.

$$\begin{aligned} f(x, y, z) = {} & yz(1 + x) + y^2z^2(x^3 + x^4) \\ & \qquad + y^3z^3(x^4 + 2x^5 + 2x^6 + x^7 + x^8 + x^9) + \ldots \end{aligned}$$

Connected symmetric bigraphs.

$$MC(x, y, z) = yzx + y^2z^2(x^3 + x^4) + y^3z^3(2x^5 + 2x^6 + x^7 + x^8 + x^9) + \ldots$$

Connected bigraphs.

$$\begin{aligned} C(x, y, z) = {} & z + yzx + yz^2(x^2) + [y^2z^2(x^3 + x^4) + yz^3(x^3)] \\ & + [y^2z^3(2x^4 + x^5 + x^6) + yz^4(x^4)] \\ & + [y^3z^3(3x^5 + 3x^6 + 2x^7 + x^8 + x^9) \\ & + y^2z^4(2x^5 + 2x^6 + x^7 + x^8) + yz^5(x^5)] + \ldots \end{aligned}$$

Connected bicolourable graphs.

$$c(x, y) = y + y^2x + y^3x^2 + y^4(2x^3 + x^4) + y^5(3x^4 + x^5 + x^6) + y^6(6x^5 + 5x^6 + 3x^7 + 2x^8 + x^9) + \ldots$$

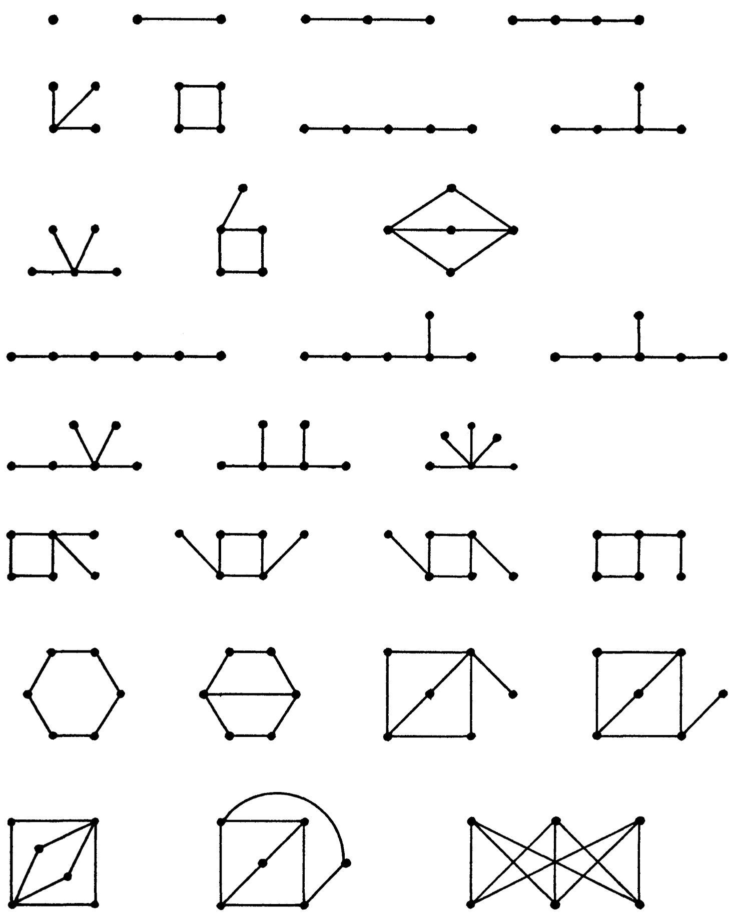
Bicolourable graphs.

$$b(x, y) = y + y^2(1 + x) + y^3(1 + x + x^2) + y^4(1 + x + 2x^2 + 2x^3 + x^4) + y^5(1 + x + 2x^2 + 3x^3 + 4x^4 + x^5 + x^6) + y^6(1 + x + 2x^2 + 4x^3 + 7x^4 + 8x^5 + 6x^6 + 3x^7 + 2x^8 + x^9) + \ldots$$

Bicolourable graphs with no isolates.

$$i(x, y) = y^2x + y^3x^2 + y^4(x^2 + 2x^3 + x^4) + y^5(x^3 + 3x^4 + x^5 + x^6) + y^6(x^3 + 3x^4 + 7x^5 + 5x^6 + 3x^7 + 2x^8 + x^9) + \ldots$$

Appendix 2. Diagrams of all connected bicolourable graphs with up to six points.



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References

1. F. Harary, *The number of linear, directed, rooted, and connected graphs*, Trans. Amer. Math. Soc., *78* (1955), 445–463.
2. ——— *On the number of bicolored graphs*, Pacific J. Math., *8* (1958), 743–755.
3. ——— *Unsolved problems in the enumeration of graphs*, Publ. Math. Inst. Hung. Acad. Sci., *5* (1960), 63–95.
4. F. Harary and G. Prins, *The number of homeomorphically irreducible trees, and other species*, Acta Math., *101* (1959), 141–162.
5. D. König, *Theorie der endlichen und unendlichen Graphen* (Leipzig, 1936; reprinted New York, 1950).
6. G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen*, Acta Math., *68* (1937), 145–254.
7. R. J. Riddell and G. E. Uhlenbeck, *On the theory of the virial development of the equation of state of monoatomic gases*, J. Chem. Phys. *21* (1953), 2056–2064.
8. J. Riordan, *An introduction to combinatorial analysis* (New York, 1958), p. 147.
9. W. T. Tutte, *A census of planar triangulations*, Can. J. Math. *14* (1962), 21–38.

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