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Successive Minima and Radii

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

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Abstract. In this note we present inequalities relating the successive minima of an o-symmetric convex body and the successive inner and outer radii of the body. These inequalities join known inequalities involving only either the successive minima or the successive radii.

1 Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, *i.e.,* compact convex sets with non-empty interior, in the n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathcal{K}_0^n$ be the family of all o-symmetric convex bodies, *i.e.,* $K \in \mathcal{K}^n$ with $K = -K$. Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ be the standard inner product and Euclidean norm in $\mathbb{R}^n$, respectively. We denote the n-dimensional unit ball by B_n. The volume of a set $M \subset \mathbb{R}^n$, *i.e.,* its n-dimensional Lebesgue measure, is denoted by $\mathrm{V}(M)$ and we set $\kappa_n = \mathrm{V}(B_n)$. If $K \subset \mathbb{R}^n$ is an i-dimensional convex body, *i.e.,* its affine hull is an i-dimensional plane, we write $\mathrm{V}^i(K)$ to denote its i-dimensional volume.

The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$. For $L \in \mathcal{L}_i^n$, $L^\perp$ denotes its orthogonal complement and for $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_i^n$ the orthogonal projection of K onto L is denoted by $K|L$. For $M \subset \mathbb{R}^n$, $\operatorname{lin} M$ and $\operatorname{conv} M$ denote respectively the linear and the convex hulls of M.

The diameter, the minimal width, the circumradius, and the inradius of a convex body K are denoted by $\mathrm{D}(K)$, $\omega(K)$, $\mathrm{R}(K)$, and $\mathrm{r}(K)$, respectively. For more information on these functionals and their properties we refer to [3, pp. 56–59]. If f is a functional on $\mathcal{K}^n$ depending on the dimension of the space in which a convex body K is embedded, and if K is contained in an affine space A, then we write $f(K; A)$ to denote that f has to be evaluated with respect to the space A. With this notation we define the successive outer and inner radii.

Definition 1.1 For $K \in \mathcal{K}^n$ and $i = 1, \ldots, n$ let

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) \quad \text{and} \quad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}(K \cap (x + L); x + L).$$

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So $\mathrm{R}_i(K)$ is the smallest radius of a solid cylinder which has an i-dimensional spherical cross section and contains K, and $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K. We obviously have

$$\mathrm{R}_n(K) = \mathrm{R}(K), \quad \mathrm{R}_1(K) = \frac{\omega(K)}{2}, \quad \mathrm{r}_n(K) = \mathrm{r}(K), \quad \text{and} \quad \mathrm{r}_1(K) = \frac{\mathrm{D}(K)}{2}.$$

Notice that the outer radii are increasing in i, whereas the inner radii are decreasing in i. We also have for $i \in \{1, \dots, n\}$ and any convex body K,

$$1 \leq \frac{\mathrm{R}_i(K)}{\mathrm{r}_{n-i+1}(K)} < i + 1. \tag{1.1}$$

For the lower bound, which is best possible, we refer to [2, Lemma 2.1]. Determining the optimal upper bound is still an open problem, even in the o-symmetric case. The bound presented above is given in [15] (see also [14]). The following relation between the in- and outer radii and the volume of an arbitrary convex body $K \in \mathcal{K}^n$ can be found in [2, Corollary 2.1]:

$$\frac{2^n}{n!}\mathrm{r}_1(K)\cdots\mathrm{r}_n(K) \leq \mathrm{V}(K) \leq 2^n\mathrm{R}_1(K)\cdots\mathrm{R}_n(K). \tag{1.2}$$

In the case when K is o-symmetric, we also have

$$\frac{2^n}{n!}\mathrm{R}_1(K)\cdots\mathrm{R}_n(K) \leq \mathrm{V}(K) \leq 2^n\mathrm{r}_1(K)\cdots\mathrm{r}_n(K), \tag{1.3}$$

(see [2, Theorem 2.1])

For more information on successive radii, their size for special bodies as well as computational aspects of these radii, we refer the reader to [1, 2, 4–7, 12].

Here we are mainly interested in the relations of these radii to the successive minima of an o-symmetric convex body with respect to the integer lattice, which we introduce next.

We denote by $\mathbb{Z}^n$ the integer lattice, *i.e.,* the lattice of all points with integral coordinates in $\mathbb{R}^n$. Then any lattice Λ of $\mathbb{R}^n$ can be obtained as $\Lambda = B\mathbb{Z}^n$ with $B \in \mathrm{GL}_n(\mathbb{R})$, and the determinant of the lattice is defined as $\det \Lambda = |\det B|$. As a general reference for lattices we refer to [9].

For $K \in \mathcal{K}_0^n$ and a lattice Λ, the i-th successive minimum $\lambda_i(K, \Lambda)$ of K with respect to Λ, $i = 1, \dots, n$, is defined as

$$\lambda_i(K, \Lambda) = \min\{\lambda \in \mathbb{R} : \lambda > 0, \dim(\lambda K \cap \Lambda) \geq i\}.$$

Clearly $\lambda_1(K, \Lambda) \leq \cdots \leq \lambda_n(K, \Lambda)$. The second fundamental theorem of Minkowski (see [9, §9.1, 9.4], [11, 13]) relates the successive minima with the volume of a convex body $K \in \mathcal{K}_0^n$:

$$\frac{2^n}{n!}\det\Lambda \leq \lambda_1(K, \Lambda)\cdots\lambda_n(K, \Lambda)\mathrm{V}(K) \leq 2^n\det\Lambda. \tag{1.4}$$

In the case of the integer lattice $\mathbb{Z}^n$ we will just write $\lambda_i(K)$ instead of $\lambda_i(K, \mathbb{Z}^n)$. In this paper we relate the successive minima with the inner and outer radii. A quite obvious attempt to do that would be via relations of the type $\lambda_i(K)\mathrm{r}_j(K)$ or $\lambda_i(K)\mathrm{R}_j(K)$. The next proposition shows, however, that in general we cannot bound these products.

Proposition 1.2 *Let $K \in \mathcal{K}_0^n$. Then*

$$\frac{1}{\mathrm{R}(K)} \leq \lambda_i(K) \leq \frac{1}{\mathrm{r}(K)}, \quad 1 \leq i \leq n.$$

In all other cases, the products $\lambda_i(K)\mathrm{r}_j(K)$ and $\lambda_i(K)\mathrm{R}_j(K)$ cannot be bounded, either from above or from below, by a constant depending only on the dimension.

Therefore we consider products of several radii and successive minima.

Theorem 1.3 *Let $K \in \mathcal{K}_0^n$. For $i = 1, \dots, n-1$ we have*

$$\lambda_{i+1}(K) \cdots \lambda_n(K)\mathrm{V}(K) \leq 2^n \mathrm{r}_1(K) \cdots \mathrm{r}_i(K), \tag{1.5}$$

$$\lambda_1(K) \cdots \lambda_i(K)\mathrm{V}(K) \geq \frac{2^n}{n!}\mathrm{R}_1(K) \cdots \mathrm{R}_{n-i}(K). \tag{1.6}$$

None of these inequalities can be improved in the sense that 2^n or $2^n/n!$ cannot be replaced, respectively, by $2^n - \varepsilon$ or $2^n/n! + \varepsilon$ for any $\varepsilon > 0$.

By (1.1) we have $\mathrm{r}_{n-j+1}(K) \leq \mathrm{R}_j(K)$, and so we get the following corollary.

Corollary 1.4 *Let $K \in \mathcal{K}_0^n$. For $i = 1, \dots, n-1$ we have*

$$\lambda_{i+1}(K) \cdots \lambda_n(K)\mathrm{V}(K) \leq 2^n \mathrm{R}_{n-i+1}(K) \cdots \mathrm{R}_n(K), \tag{1.7}$$

$$\lambda_1(K) \cdots \lambda_i(K)\mathrm{V}(K) \geq \frac{2^n}{n!}\mathrm{r}_{i+1}(K) \cdots \mathrm{r}_n(K). \tag{1.8}$$

None of these inequalities can be improved in the sense of Theorem 1.3.

For inequality (1.5) and inequality (1.7) (inequality (1.6) and inequality (1.8)), the "limit" case $i = 0$ ($i = n$) (*i.e.,* when no radii appear in the inequalities) is Minkowski's inequality (1.4). The "limit" case $i = n$ ($i = 0$), *i.e.,* when no successive minima appear in the formulae, gives the upper (lower) bounds for the volume in (1.2) and (1.3). Thus, these inequalities build a bridge between Minkowski's inequality and the known inequalities involving inner and outer radii.

In the next section we present the proofs of the main results, as well as some consequences for general (not necessarily o-symmetric) convex bodies.

2 Proofs of the Main Results

For a convex body $K \in \mathcal{K}^n$ containing the origin in its interior, the polar body of K is the convex body $K^* = \left\{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \text{ for all } x \in K \right\}$ (see [17, §1.6]). The inner and outer radii of an o-symmetric convex body $K \in \mathcal{K}_0^n$ and its polar are related by the following identity, for which we refer to [6, (1.2)]:

$$\mathrm{R}_i(K^*)\,\mathrm{r}_i(K) = 1 \quad \text{for } i = 1, \dots, n. \tag{2.1}$$

Proof of Proposition 1.2 Since $\mathrm{r}(K)B_n \subseteq K$, we obviously have

$$\lambda_i(K) \leq \lambda_i\big(\mathrm{r}(K)B_n\big) = \frac{1}{\mathrm{r}(K)}\lambda_i(B_n) = \frac{1}{\mathrm{r}(K)}$$

for $1 \leq i \leq n$. Analogously, from $K \subseteq \mathrm{R}(K)B_n$ we find $\lambda_i(K) \geq 1/\mathrm{R}(K)$ and so we trivially get the inequalities in the proposition.

Next we show that the inequalities above are the only possible upper and lower bounds for the products $\lambda_i(K)\mathrm{r}_j(K)$ and $\lambda_i(K)\mathrm{R}_j(K)$. In order to see that there is no upper bound on $\lambda_i(K)\mathrm{r}_j(K)$, $j = 1, \dots, n-1$, we consider the j-dimensional unit ball B_j embedded in a j-dimensional irrational plane $L \in \mathcal{L}^n_j$, *i.e.*, $L \cap \mathbb{Z}^n = \{0\}$. Taking the convex hull of B_j and suitable points with irrational coordinates, close enough to L, we can find an n-dimensional convex body K_0 with $\mathrm{r}_j(K_0) = 1$ but arbitrarily large $\lambda_i(K_0)$.

The non-existence of lower bounds on $\lambda_i(K)\mathrm{r}_j(K)$, $j = 2, \dots, n$, is shown by the following cross-polytope $P_n(m)$. For $m \in \mathbb{N}$ and $i = 1, \dots, n$, let

$$v_i := (m^{i-1}, \dots, m, 1, 0, \dots, 0)^\intercal \in \mathbb{R}^n,$$

and $P_n(m) := \mathrm{conv}\{\pm v_i : i = 1, \dots, n\}$. Thus $P_n(m)$ is an o-symmetric lattice cross-polytope containing the origin as the only interior lattice point. Hence

$$\lambda_i(P_n(m)) = 1$$

for all $i = 1, \dots, n$ and next we show the inner radii $\mathrm{r}_j(P_n(m))$, $j = 2, \dots, n$, can be arbitrarily small. Since r_j are decreasing in j, it suffices to verify this fact for r_2. Moreover, from $\mathrm{r}_2(P_n(m)) \leq \mathrm{R}_{n-1}(P_n(m))$ (*cf.* (1.1)) we just have to check that for a suitable projection π the lengths of the projected vertices $\pi(v_i)$ can be made arbitrarily small. Let π be the orthogonal projection onto the hyperplane orthogonal to v_n. The k-th coordinate of the projection $\pi(v_i) = v_i - \langle v_i, v_n\rangle / |v_n|^2 v_n$ of v_i is given by

$$(\pi(v_i))_k = \begin{cases} m^{i-k}\dfrac{1 + m^2 + \dots + m^{2(n-i-1)}}{1 + m^2 + \dots + m^{2(n-1)}} & \text{for } k = 1, \dots, i, \\ -m^{2n-i-k}\dfrac{1 + m^2 + \dots + m^{2(i-1)}}{1 + m^2 + \dots + m^{2(n-1)}} & \text{for } k = i+1, \dots, n. \end{cases}$$

Hence,

$$\pi(v_i) = v_i - \frac{\langle v_i, v_n\rangle}{|v_n|^2}v_n \to (0, \dots, 0)^\intercal \quad \text{when } m \to \infty,$$

and so $\mathrm{R}_{n-1}(P_n(m))$ tends to zero as m approaches infinity.

In order to deal with the outer radii we use polarity. By (2.1) we may write

$$\lambda_i(K)\mathrm{R}_j(K) = \frac{\lambda_i(K)\lambda_{n-i+1}(K^*)}{\lambda_{n-i+1}(K^*)\mathrm{r}_j(K^*)}.$$

By classical results in the geometry of numbers, we know that the numerator is bounded from above and below (*cf.* [8, Theorem 23.2]). Hence, by taking K as the polar body of $P_n(m)$ and the foregoing discussion on the inner radii, we see that $\lambda_i(K)\mathrm{R}_j(K)$ is not bounded from above for $j \geq 2$; by taking $K = K_0^*$, we get that $\lambda_i(K)\mathrm{R}_j(K)$ is not bounded from below for $j \leq n-1$. ■

Next we come to the proof of Theorem 1.3, providing upper and lower bounds for products of successive minima in terms of the inner and outer radii.

Proof of Theorem 1.3 We start with inequality (1.5). Let $z_1, \dots, z_i \in K$ be i linearly independent points with $\lambda_j(K)z_j \in \lambda_j(K)K \cap \mathbb{Z}^n$. We consider a suitable $(n-i)$-dimensional coordinate plane

$$L_{n-i} = \{x \in \mathbb{R}^n : x_{j_1} = \cdots = x_{j_i} = 0, \, j_k \in \{1, \dots, n\}\}$$

such that

$$\mathrm{lin}\{z_1, \dots, z_i\} \cap L_{n-i} = \{0\}. \tag{2.2}$$

Denoting by $\Lambda_{n-i} = \mathbb{Z}^n \cap L_{n-i}$ the sublattice of all points in L_{n-i} with integer coordinates, Minkowski's second fundamental theorem assures that

$$\lambda_1(K \cap L_{n-i}, \Lambda_{n-i}) \cdots \lambda_{n-i}(K \cap L_{n-i}, \Lambda_{n-i})\mathrm{V}^{n-i}(K \cap L_{n-i}) \leq 2^{n-i}.$$

From (2.2) we know that $\lambda_j(K \cap L_{n-i}, \Lambda_{n-i})K$ contains $i + j$ linearly independent points of $\mathbb{Z}^n$, for $j = 1, \dots, n-i$. Therefore,

$$\lambda_{i+j}(K) \leq \lambda_j(K \cap L_{n-i}, \Lambda_{n-i}), \quad j = 1, \dots, n-i,$$

and hence

$$\lambda_{i+1}(K) \cdots \lambda_n(K)\mathrm{V}^{n-i}(K \cap L_{n-i}) \leq 2^{n-i}. \tag{2.3}$$

With $L_i = L_{n-i}^{\perp}$ we get by the o-symmetry of K (see [10])

$$\mathrm{V}^{n-i}(K \cap L_{n-i}) \geq \frac{\mathrm{V}(K)}{\mathrm{V}^i(K|L_i)}.$$

Since $K|L_i$ is an i-dimensional o-symmetric convex body, we have that (see [2, Theorem 2.1]) $\mathrm{V}^i(K|L_i) \leq 2^i \mathrm{r}_1(K|L_i) \cdots \mathrm{r}_i(K|L_i)$. Together with $\mathrm{r}_j(K|L_i) \leq \mathrm{r}_j(K)$ (see [2, Lemma 2.1]), we get $\mathrm{V}^i(K|L_i) \leq 2^i \mathrm{r}_1(K) \cdots \mathrm{r}_i(K)$. Therefore

$$\mathrm{V}^{n-i}(K \cap L_{n-i}) \geq \frac{\mathrm{V}(K)}{2^i \mathrm{r}_1(K) \cdots \mathrm{r}_i(K)},$$

and using (2.3) we obtain

$$\lambda_{i+1}(K) \cdots \lambda_n(K)\mathrm{V}(K) \leq 2^n \mathrm{r}_1(K) \cdots \mathrm{r}_i(K).$$

In order to show that inequality (1.5) cannot be improved, it suffices to consider the tightness of inequality (1.7) in Corollary 1.4. Let $Q_n(\mu)$ be the orthogonal parallelepiped with edge-lengths $\mu, \mu^2, \dots, \mu^n$, for $\mu \geq 1$. The successive minima of such a box are $\lambda_j(Q_n(\mu)) = 2/\mu^{n-j+1}$, $j = 1, \dots, n$, the outer radii R_j are given by $\mathrm{R}_j(Q_n(\mu)) = (1/2)\left(\sum_{k=1}^{j} \mu^{2k}\right)^{1/2}$ (see [5, Theorem 4.4]) and for the volume we find $\mathrm{V}\left(Q_n(\mu)\right) = \mu \cdots \mu^n$. Thus

$$\frac{\prod_{j=i+1}^{n} \lambda_j(Q_n(\mu))}{\prod_{j=n-i+1}^{n} \mathrm{R}_j(Q_n(\mu))} \mathrm{V}(Q_n(\mu)) = 2^i \frac{2^{n-i} \mu^{n-i+1} \cdots \mu^n}{\prod_{j=n-i+1}^{n} \left(\sum_{k=1}^{j} \mu^{2k}\right)^{1/2}},$$

which tends to 2^n as μ approaches infinity.

Now we prove inequality (1.6). Again let $z_1, \dots, z_i \in K$ be i linearly independent points with $\lambda_j(K)z_j \in \lambda_j(K)K \cap \mathbb{Z}^n$. We denote by $u_j := \lambda_j(K)z_j$, and we consider the i-dimensional sublattice Λ_i of $\mathbb{Z}^n$ determined by $\{u_1, \dots, u_i\}$. Clearly, $\det \Lambda_i \geq 1$. Minkowski's lower bound in (1.4) gives

$$\frac{2^i}{i!} \leq \frac{2^i}{i!} \det \Lambda_i \leq \lambda_1(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) \cdots \lambda_i(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) \mathrm{V}^i(K \cap \operatorname{lin} \Lambda_i).$$

Since $\lambda_j(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) = \lambda_j(K)$, $1 \leq j \leq i$, we can write

$$\frac{2^i}{i!} \leq \lambda_1(K) \cdots \lambda_i(K) \mathrm{V}^i(K \cap \operatorname{lin} \Lambda_i). \tag{2.4}$$

With $L_{n-i} = (\operatorname{lin} \Lambda_i)^{\perp}$ we know that

$$\mathrm{V}^i(K \cap \operatorname{lin} \Lambda_i) \mathrm{V}^{n-i}(K | L_{n-i}) \leq \binom{n}{i} \mathrm{V}(K),$$

(see [16]). Since $K|L_{n-i}$ is an $(n-i)$-dimensional o-symmetric convex body, we have (see [2, Theorem 2.1])

$$\mathrm{V}^{n-i}(K|L_{n-i}) \geq \frac{2^{n-i}}{(n-i)!} \mathrm{R}_1(K|L_{n-i}) \cdots \mathrm{R}_{n-i}(K|L_{n-i}),$$

and since $\mathrm{R}_j(K|L_{n-i}) \geq \mathrm{R}_j(K)$ (see [2, Lemma 2.1]), we arrive at

$$\mathrm{V}^{n-i}(K|L_{n-i}) \geq \frac{2^{n-i}}{(n-i)!} \mathrm{R}_1(K) \cdots \mathrm{R}_{n-i}(K).$$

Therefore,

$$\mathrm{V}^i(K \cap \operatorname{lin} \Lambda_i) \leq \binom{n}{i} \frac{\mathrm{V}(K)}{\mathrm{V}^{n-i}(K|L_{n-i})} \leq \frac{n!}{i! 2^{n-i}} \frac{\mathrm{V}(K)}{\mathrm{R}_1(K) \cdots \mathrm{R}_{n-i}(K)},$$

and with (2.4) we get

$$\frac{2^n}{n!} \mathrm{R}_1(K) \cdots \mathrm{R}_{n-i}(K) \leq \lambda_1(K) \cdots \lambda_i(K) \mathrm{V}(K).$$

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To show that inequality (1.6) cannot be improved, it suffices to consider the tightness of inequality (1.8) in Corollary 1.4. We consider for $\mu > 1$ the orthogonal cross-polytope $C_n^*(\mu) := \operatorname{conv}\{\pm\mu^i e_i : i = 1, \dots, n\}$, where e_i denotes the i-th canonical unit vector. The successive minima of such a cross-polytope are

$$\lambda_j\big(C_n^*(\mu)\big) = 1/\mu^{n-j+1},\ j = 1, \dots, n,$$

the inner radii r_j are given by $\mathrm{r}_j(C_n^*(\mu)) = (\sum_{k=n-j+1}^{n} \mu^{-2k})^{-1/2}$ (see [5, Theorem 4.4]) and for its volume we find $\mathrm{V}(C_n^*(\mu)) = (2^n/n!)\mu \cdots \mu^n$. Thus

$$\frac{\prod_{j=1}^{i} \lambda_j\big(C_n^*(\mu)\big)}{\prod_{j=i+1}^{n} \mathrm{r}_j\big(C_n^*(\mu)\big)} \mathrm{V}\big(C_n^*(\mu)\big) = \frac{2^n}{n!} \frac{\mu \cdots \mu^{n-i}}{\prod_{j=i+1}^{n}\big(\sum_{k=n-j+1}^{n} \mu^{-2k}\big)^{-1/2}},$$

which tends to $2^n/n!$ when $\mu \to \infty$. ■

Next we want to present some inequalities as in Theorem 1.3 for arbitrary convex bodies. To this end we consider the *difference body* $DK = K + (-K)$ of a convex body $K \in \mathcal{K}^n$, which is certainly o-symmetric, and for further properties we refer for instance to [8, §9.5]. It is also well known that $\mathrm{V}(DK) \geq 2^n \mathrm{V}(K)$ and moreover, for the outer radii R_j and the inner radii r_j it has been proved that $\mathrm{R}_j(DK) \leq 2\mathrm{R}_j(K)$ (see [12, Lemma 2.1]) and $\mathrm{r}_j(DK) \geq 2\mathrm{r}_j(K)$ (see [12, Remark 2.1]), $j = 1, \dots, n$. These properties together with Corollary 1.4 imply the following result for general convex bodies.

Corollary 2.1 *Let $K \in \mathcal{K}^n$. For $i = 1, \dots, n-1$ we have*

$$\lambda_{i+1}(DK) \cdots \lambda_n(DK)\mathrm{V}(K) \leq 2^i \mathrm{R}_{n-i+1}(K) \cdots \mathrm{R}_n(K). \tag{2.5}$$

$$\lambda_1(DK) \cdots \lambda_i(DK)\mathrm{V}(DK) \geq \frac{2^{2n-i}}{n!} \mathrm{r}_{i+1}(K) \cdots \mathrm{r}_n(K). \tag{2.6}$$

None of these inequalities can be improved, in the sense of Theorem 1.3.

Remark 1. In order to express inequality (2.6) in terms of the volume of K, the well-known Rogers–Shephard inequality $\mathrm{V}(DK) \leq \binom{2n}{n}\mathrm{V}(K)$ (see [17, §7.3]) can be applied. The resulting bound, however, is not best possible.

Finally, we remark that identity (2.1) allows us to express the inequalities in Theorem 1.3 in terms of the inner and outer radii of the polar body.

Remark 2. Let $K \in \mathcal{K}_0^n$. For $i = 1, \dots, n-1$ we have

$$\lambda_{i+1}(K) \cdots \lambda_n(K)\, \mathrm{R}_1(K^*) \cdots \mathrm{R}_i(K^*)\mathrm{V}(K) \leq 2^n,$$

$$\lambda_1(K) \cdots \lambda_i(K)\, \mathrm{r}_1(K^*) \cdots \mathrm{r}_{n-i}(K^*)\mathrm{V}(K) \geq \frac{2^n}{n!}.$$

None of these inequalities can be improved, in the sense of Theorem 1.3.

In the same way we can rewrite Corollary 1.4 and Corollary 2.1 in terms of the radii of the polar body.

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