A Geometrical Construction for the Rainbow Formula.

By Dr David Robertson.

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Internal Reflections in a Sphere. A ray of light PQ (Fig. 1) incident at any point Q on the surface of a transparent sphere is partly reflected and partly refracted along QR_1. At R_1 it is partly reflected along R_1R_2, and partly emerges along R_1S_1. The same thing occurs at R_2, R_3, etc., on which the successive reflected

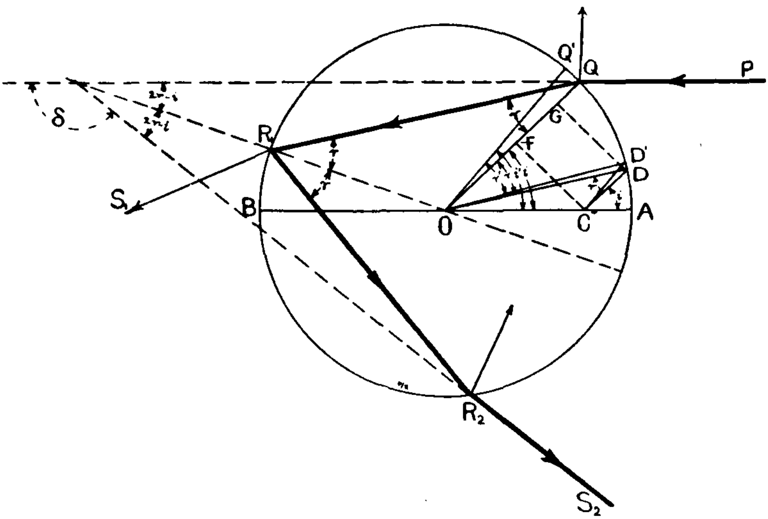

Fig. 1. *Internal Reflections in a Sphere.*

portions fall. AB is the diameter parallel to the incident light, and OQ the radius to the point of incidence. Let μ denote the index of refraction of the material of the sphere relative to the surrounding medium. Divide OA in C so that $OC : OA = 1 : \mu$.

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Draw CD ∥ OQ and meeting the circle in D. Draw CF and DG ⊥ OQ.

Then, $\sin DOQ = \frac{GD}{OD} = \frac{FC}{\mu . OC} = \frac{1}{\mu}\sin COF = \frac{1}{\mu}\sin i = \sin r.$

$\therefore \quad \angle DOQ = r = \angle OQR.$

$\therefore$ DO is parallel to the ray refracted at Q.

The deviation of the ray is $(i - r)$ at incidence, and also at emergence; it is $(\pi - 2r)$ at each reflection, and so the total deviation with n reflections is :—

$$\begin{aligned} \delta &= 2(i - r) + n(\pi - 2r) \\ &= n\pi - 2\{(n+1)r - i\}. \end{aligned}$$

Minimum Deviation. Let Q move to an adjacent point Q′ further from A; then

$$\begin{aligned} \delta - \delta' &= -2\{(n+1)(r - r') - (i - i')\} \\ &= -2\{(n+1)(QOQ' - DOD') - QOQ'\} \\ &= -2\{nQOQ' - (n+1)DOD'\} \\ &= -2\{nDCD' - (n+1)DOD'\} \\ &= -2QOQ'\left\{n - (n+1)\frac{DOD'}{DCD'}\right\} \\ &= -2QOQ'\left\{n - (n+1)\frac{CD}{OD\cos r}\right\} \\ &= -2QOQ'\left\{n - (n+1)\frac{CD}{OA\cos r}\right\}. \end{aligned}$$

If Q be very close to A, this becomes

$$\begin{aligned} &-2QOQ'\left\{n - (n+1)\frac{CA}{OA}\right\} \\ &= -\frac{2}{\mu}QOQ'\left\{(n+1) - \mu\right\}. \end{aligned}$$

Since μ is less than 2, this last expression is always negative, which shows that the deviation at first diminishes as Q moves away from A, provided there be one or more internal reflections. It will continue to diminish with a further displacement of Q from

A until $(\delta - \delta')$ changes sign by passing through zero, after which it will increase. Hence, for minimum deviation

$$\delta - \delta' = 0$$

or

$$CD = \frac{n}{n+1} OA\cos r.$$

$$\frac{n}{n+1} = \frac{CD}{OD\cos r} = \frac{\sin(i-r)}{\sin i \,.\, \cos r}$$

$$= \frac{\sin i \,.\, \cos r - \cos i \,.\, \sin r}{\sin i \,.\, \cos r} = 1 - \frac{1}{\mu}\,\frac{\cos i}{\cos r}.$$

$$\therefore \quad \frac{\mu}{n+1} = \frac{\cos i}{\cos r}.$$

$$\therefore \quad (n+1)^2\cos^2 i = \mu^2\cos^2 r = \mu^2 - \mu^2\sin^2 r = \mu^2 - \sin^2 i$$

$$= \mu^2 - 1 + \cos^2 i.$$

$$\therefore \quad \cos i = \sqrt{\frac{\mu^2 - 1}{n(n+2)}}.$$

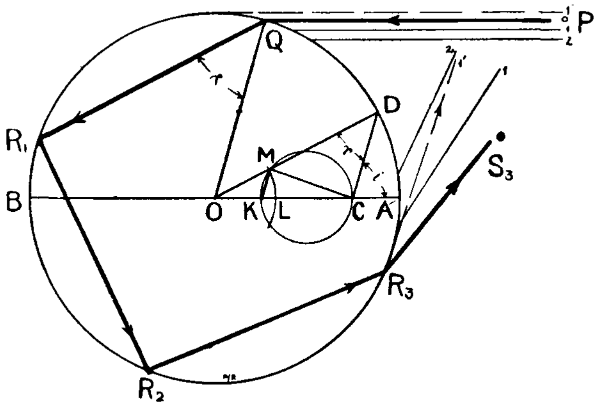

Fig. 2. *Minimum Deviation in a Sphere after two Reflections.*

Construction for Ray of Least Deviation. The large circle in Fig. 2 represents the sphere, and AB is the diameter parallel to the incident light.

Divide OA in C so that OC : OA = 1 : μ.

,, OA in L ,, OL : LA = 1 : n.

,, OC in K ,, OK : KC = 1 : n = OL : LA.

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Draw a circle on CK as diameter. Draw the arc LM with centre at O, and cutting the last circle at M. KM is parallel to the radius to the point of incidence, and OM is parallel to the direction after refraction of the ray which is at least deviated by the sphere after n internal reflections.

Proof. Continue OM to meet the outer circle at D, and join CD, CM, KM. Draw OQ ∥ CD. We have already proved that if OQ be parallel to CD, DO is the direction of the ray refracted at Q.

Then $\quad \mathrm{OM} : \mathrm{MD} = \mathrm{OL} : \mathrm{LA} = \mathrm{OK} : \mathrm{KC},$

$\therefore \quad \mathrm{KM} \parallel \mathrm{CD} \parallel \mathrm{OQ}, \quad \text{and} \quad \therefore \quad \mathrm{CD} \perp \mathrm{CM}.$

Hence $\quad \mathrm{CD} = \mathrm{DM}\cos\mathrm{CDM} = \mathrm{LA}\cos r$

$$= \frac{n}{n+1}\mathrm{AO}\cos r,$$

which is the condition for minimum deviation.