I am sure this book would be very useful for anyone turning back to study mathematics with any kind of break since their first degree. But I think it would also be useful for undergraduates who are finding themselves rather overwhelmed by their courses, and who perhaps would benefit from the overviews, definitions and examples given here, rather than the greater detail demanded of their studies.

Many of my A-level students do continue to study Mathematics at an undergraduate level. This would be an interesting parting gift for them, on the lines of "one day, all this will be clear", and I am sure this would be a useful reference book for them.

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Algebra, notes from the underground by Paolo Aluffi, pp. 488, £29.99 (paper), ISBN 978-1-108-95823-3, Cambridge University Press (2021)

Despite the Dostoevskian title, this is nothing more sinister than a rigorous undergraduate textbook whose subversiveness consists in presenting rings before groups. The author, who is based at Florida State University, argues that a readership without significant experience in abstract algebra will find it more natural to deal with, say, the ring of integers by not having to ignore one of the two natural operations, and claims that, as groups require fewer axioms than rings, groups represent greater generalisation and hence greater difficulty. He puts much emphasis on modules, quoting Atiyah and Macdonald who wrote in 1970 that 'following the modern trend, we put more emphasis on modules ...', and commenting that such a trend has not reached introductory algebra teaching in 2020.

Aluffi's style is rigorous but also chatty. (An extreme example: 'Wait ... this seems to depend on the choice of representative a for the class. Don't we have to check that it is well-defined, that is, independent of the representative? You bet we do. Check it now. OK, OK, let's check it together.'). He starts with two chapters of number theory. To him the Well-Ordering Principle is crucial; he claims that proofs based on induction are better dealt with through the Principle (provided you grant its truth), not liking the use of 'and so on forever' in ordinary mathematical induction. He gives a careful justification of the Euclidean Algorithm, but refers to 'irreducible' numbers rather than primes. Shortly he says that this is why $\mathbb{Z}/0\mathbb{Z}$ is an integral domain but not a field ('0 is a prime number but is not irreducible'); I would think this sort of argument might look to beginners like begging the question. Quotient sets are defined early and clearly, but the author's injunction to 'resist with all your might the temptation to think of '$\mathbb{Z}/n\mathbb{Z}$ as a subset of $\mathbb{Z}$' comes rather too early, I think, to be very meaningful to the beginning reader. There is also a section on RSA encryption.

By the end of Chapter 3 we have seen that $\mathbb{Z}/n\mathbb{Z}$ is a field if, and only if, n is irreducible, and that a finite integral domain is a field. Chapter 4 introduces Cartesian products and ring homomorphisms; I was glad to see Aluffi explaining that isomorphic rings were 'just different manifestations of the "same" concept.' In chapter 5 we meet a ('canonical') decomposition of functions, obviously dear to the author's heart, into a composition of an injective function, a bijective function and a surjective function. This motivates the introduction of kernels and ideals ('$\mathbb{Z}$ is not an ideal in $\mathbb{Q}$'); typical of Aluffi's clarity is his demonstration that the quotient ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$. He also proves the Chinese Remainder Theorem–that is, the theorem itself and not the algorithm derived from it to which its name is usually given. Chapter 6 is on integral domains, including fields of fractions, and the last chapter in Part I is on polynomial rings and factorisation.

Part II is entitled Modules. The opening gambit moves from vector spaces ('an ideal of a ring R [is] some kind of "vector space over R"') to $\mathbb{Z}$-modules: 'A more common name for a $\mathbb{Z}$-module is an abelian group.' Modules are shown to have the same canonical decomposition as rings. There is a chapter on modules over integral domains, and then come abelian groups proper. By this time of course one doesn't need much new theory; most of the key results have already been proved in the ring/module context. Part III, 'Groups', moves quickly through the main families of groups, quotient groups and isomorphism theorems, before going on to subgroups and Lagrange's Theorem, after which Sylow's theorems are reached almost at once. By now it is clear where the book is heading, and the last three chapters (Part IV, 'Fields') duly culminate in the fundamental theorem of Galois theory.

For its chosen route, the book does a largely impressive job, though the author cannot resist putting in a good many nuggets of more advanced mathematics. There are roughly twenty or thirty exercises at the end of each chapter (not each section of each chapter), and about a third of these have solutions given in an appendix.

So why teach rings first? Certainly the sequence presented here provides an intellectually coherent body of fully-developed mathematics, presented as a highly polished monolith, but I must admit to considerable reservations as to whether it represents best pedagogical practice. My own experience of teaching and learning suggests that many students like to begin with the tangibility of geometrical examples of groups, uncomplicated by more than one operation–I have never found any learner to be fazed by this. They also find it easier to appreciate the concept of isomorphism when there are several familiar groups with identical structures, a more common scenario than in rings. Nor do I see why Aluffi's criterion for greater abstraction is merely 'fewer axioms'. To me the whole approach seems teacher-centred rather than learner-centred, and I was reminded of how in my undergraduate days the intellectual purity of a course seemed to count for more than the interests of students (with the tacit implication that 'if you don't like it, you don't belong here'). I was also reminded of being told by a Cambridge professor, himself a superb teacher, that he thought first-year undergraduate abstract algebra should not be taught by specialist algebraists. No doubt students of ability high enough to cope with an initial flood of definitions (rings, PIDs, UFDs, fields, modules, etc) and a purely algebraic style offering little visualisation will succeed, but then such students would succeed with other approaches too. However, I am very ready to accept that my opinion is based on limited evidence, and no doubt there will be readers of this review who will disagree with me. If you have good reason to believe that it is in the best interests of your students to learn abstract algebra like this, then good luck–this book does its self-appointed job well. Meanwhile it also provides a clearer introduction to rings than one generally finds in 'groups-first' texts, and it would be valuable merely for that.

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Essential mathematics for undergraduates by Simon G. Chiossi, pp 490, £54.99 (hard), ISBN 978-3-030-87173-4, Springer Verlag (2021).

Students of mathematics know a fair amount of the subject, but then in every subject worthy of serious study there is always much more to be learned. Many topics, however desirable for inclusion in the curriculum, will be omitted from any particular course. The book being reviewed is a collection of what the author considers to be essential material for