TOPOLOGICAL CONVEXITY SPACES

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1. Preliminaries

We shall start by recalling the definition and some basic properties of a convexity space; a topological convexity space (tcs) will then be a convexity space together with an admissible topology, and will be a generalisation of a topological vector space (tvs). After showing that the usual tvs results connecting the linear and topological properties extend to this new setting we then prove a form of the Krein-Milman theorem in a tcs.

Let X be a non-empty set, let $a, b, \ldots$ be elements of X, and let $A, B, \ldots$ be subsets of X. We do not distinguish between a point of X and the singleton subset which it defines. Thus in X the notation $\in$ is redundant and is replaced by $\subset$. Also we write $A \approx B$ for *A meets B* or $A \cap B \neq \emptyset$. A *join* on X is a mapping $\cdot: X \times X \to 2^X$, i.e. it associates with each ordered pair (a, b) of elements of X a subset $a \cdot b$ (or simply ab) of X. Given a join and $a, b \subset X$ we define $a/b = \{c \subset X\colon\ a \subset bc\}$. The operations $\cdot$ and $/$ can be extended to subsets of X in the obvious way, for example $AB = \bigcup_{\substack{a \subset A \\ b \subset B}} ab$. In particular for $a, b, c \subset X$, $a(bc)$ is a subset of X. Note also that $A \approx BC$ if and only if $A/B \approx C$.

Definition. A pair $(X, \cdot)$ is a *convexity space* if $\cdot$ is a join on X satisfying

(i) $ab \neq \emptyset$, $a/b \neq \emptyset$;

(ii) $aa = a = a/a$;

(iii) $a(bc) = (ab)c$;

(iv) $a/b \approx c/d \Rightarrow ad \approx bc$;

for all $a, b, c, d \subset X$.

The most familiar example of a convexity space is a real vector space with join defined by $ab = \{\lambda a + (1-\lambda)b\colon\ 0 < \lambda < 1\}$, and other examples are given in (**2**). If $(X, \cdot)$ and $(Y, \cdot)$ are convexity spaces and a join is defined on $X \times Y$ by $(a, b).(c, d) = a.c \times b.d$ $(a, c \subset X;\ b, d \subset Y)$, then it is easily verified that this product space is a convexity space. Although the above axioms are algebraic in nature they have a strong geometric motivation based on the vector space example. The properties of joins are discussed in more detail in (**2**), (**5**) and some consequences of these axioms are given in (**1**), (**3**). We have included here only those properties necessary for our subsequent study of topologies on $(X, \cdot)$. One consequence of the axiom $aa = a = a/a$ is that $a \subset ab$ if and only

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if $a = b$. Thus ab is to be thought of as the relative open line segment joining a to b.

A set A is *convex* if whenever $a, b \subset A$ it follows that $ab \subset A$, or more succinctly if $AA \subset A$. Similarly, a set A is *star-shaped from* $a \subset A$ if $aA \subset A$. The *kernel* A^* of A is the set of points in A from which it is star-shaped, i.e.

$$A^* = \{a \subset A \colon aA \subset A\}.$$

The *core* $\tilde{A}$ and *linear access* $\hat{A}$ of A are defined by

$$\tilde{A} = \{a : \forall b \exists c \subset ab \text{ with } ac \subset A\},\ \hat{A} = \{a : \exists b \text{ with } ab \subset A\}.$$

Some easy consequences of the axioms are that $AB = BA$, $(A/B)C \subset AC/B$ and $(A/B)/C = A/BC$. In particular if A and B are convex then so are AB, A/B and $AB \cup B$. Also any intersection of convex sets is convex. The intersection of all convex sets containing a set A is called its *convex hull* and is denoted by $[A]$. Full details of these results are given in (2), (5). The concepts of core and linear access are familiar in a vector space and can be found, for example, in (8).

2. Topologies on $(X, \cdot)$

Definition. A *topological convexity space* $(X, \cdot, \tau)$ is a convexity space $(X, \cdot)$ together with a topology τ on X satisfying

(i) $a \subset \overline{ab}$ for all $a, b \subset X$;

(ii) if $ab \approx U \in \tau$ then there exist $V, W \in \tau$ with $a \subset V$, $b \subset W$ and such that $a'b' \approx U$ whenever $a' \subset V$, $b' \subset W$;

(iii) if $a/b \approx U \in \tau$ then there exist $V, W \in \tau$ with $a \subset V$, $b \subset W$ and such that $a'/b' \approx U$ whenever $a' \subset V$, $b' \subset W$.

We mentioned earlier that a real vector space was an example of a convexity space. We now show that if τ is a topology making a vector space X into a tvs, then $(X, \cdot, \tau)$ is a tcs. We verify (i)-(ii) of the above definition:

(i) Let $a, b \subset X$ and $a \subset U \in \tau$. Then by the continuity of the function $R \to X$ given by $\lambda \mapsto \lambda a + (1-\lambda)b$ it follows that $\lambda a + (1-\lambda)b \subset U$ for some λ, $0 < \lambda < 1$. Thus $ab \approx U$ and $a \subset \overline{ab}$.

(ii) If $ab \approx U \in \tau$ then $\lambda a + (1-\lambda)b \subset U$ for some λ, $0 < \lambda < 1$. Thus by the continuity of the function $X \times X \to X$ given by $(x, y) \to \lambda x + (1-\lambda)y$ there exist $V, W \in \tau$ with $a \subset V$, $b \subset W$ and such that $a' \subset V$, $b' \subset W$ imply $\lambda a' + (1-\lambda)b' \subset U$ and $a'b' \approx U$. ((iii) similarly.)

Given two tcs′ their product space together with their product topology is a tcs. So, in particular, our theory incorporates the non-trivial extension to products of tvs′.

Theorem 1. *If A is open in a tcs, then so are AB and A/B.*

Proof. Let $(X, \cdot, \tau)$ be a tcs with $A, B \in \tau$ and let $c \subset AB$. Then $c \subset Ab$ for some $b \subset B$ and $c/b \approx A \in \tau$. Thus there exist $V, W \in \tau$ with $c \subset V$, $b \subset W$

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and such that $c'/b' \approx A$ whenever $c' \subset V$, $b' \subset W$. In particular, $c'/b \approx A$ and $c' \subset Ab$ for each $c' \subset V$. Thus $c \subset V \subset Ab \subset AB$ and $c \subset (AB)^0$. It follows that $AB \subset (AB)^0$ and so $AB \in \tau$.

Similarly, if $d \subset A/B$ then $d \subset A/b$ and $db \approx A \in \tau$ for some $b \subset B$. Hence there exist V', $W' \in \tau$ with $d \subset V'$, $b \subset W'$ and such that $d'b' \approx A$ whenever $d' \subset V'$, $b' \subset W'$. In particular, $d'b \approx A$ and $d' \subset A/b$ for each $d' \subset V'$. Thus $d \subset V' \subset A/b \subset A/B$ and $d \subset (A/B)^0$. It follows that $A/B \subset (A/B)^0$ and so $A/B \in \tau$ as required.

Now given $(X, \cdot)$ we shall construct τ on X which makes $(X, \cdot, \tau)$ a tcs.

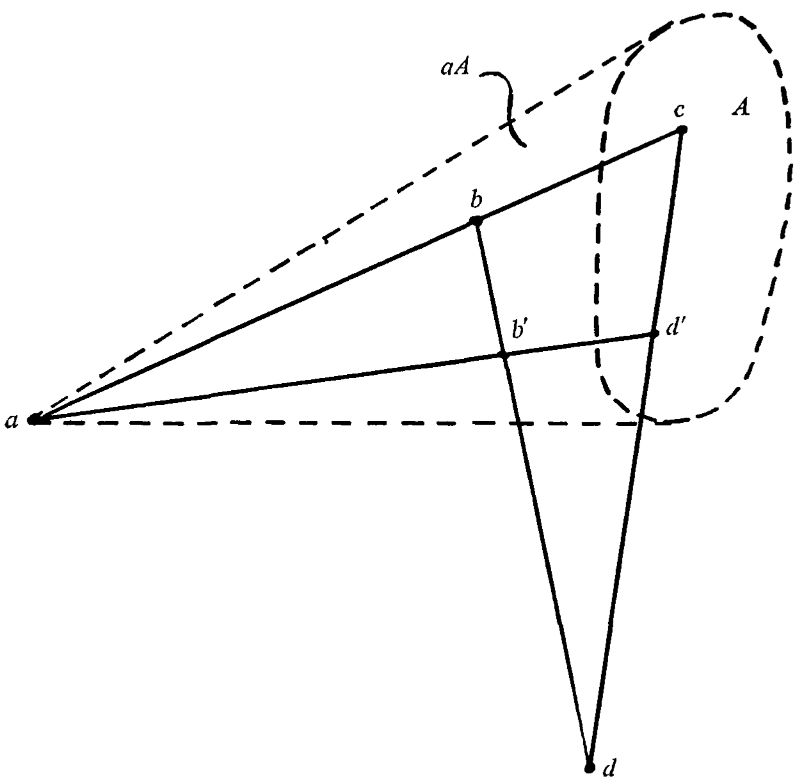
Lemma. *Given a convexity space $(X, \cdot)$ let α consist of those convex subsets A of X with $A = \tilde{A}$. Then*

(i) $A \in \alpha$ *and* $a \subset X \Rightarrow aA \in \alpha$ *and* $a/A \in \alpha$;

(ii) $A, B \in \alpha \Rightarrow A \cap B \in \alpha$.

Proof. (i) Let $A \in \alpha$, $a \subset X$ and $b \subset aA$ (see figure). We show that $b \in \widetilde{aA}$. For given $d \subset X$ there exist $c \subset b/a \cap A = b/a \cap \tilde{A}$ and $d' \subset cd$ with $cd' \subset A$. But then $b/a \approx d'/d$ (in c) and so there exists $b' \subset bd \cap ad'$. Thus

$$b'b \subset (ac)(ad') = a(cd') \subset aA,$$



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and for each $d \subset X$ we have exhibited $b' \subset bd$ with $bb' \subset aA$. It follows that $b \subset \widetilde{aA}$ and $aA \subset \widetilde{aA}$. Since a and A are convex and $\widetilde{aA} \subset aA$ it follows that aA is convex with $\widetilde{aA} = aA$ and so $aA \in \alpha$ as required.

In a similar way we show that given $A \in \alpha$, $a \subset X$ and $b \subset a/A$ then for each $d \subset X$ there exists $b' \subset bd$ with $bb' \subset a/A$. For there exist $c \subset a/b \cap A = a/b \cap \tilde{A}$, $d' \subset c/d$ and $d'' \subset d'c \cap A \subset (c/d)c \cap A \subset c/d \cap A$. But then $a \subset bc \subset bdd''$ and so there exists $b' \subset a/d'' \cap bd$. Hence by the convexity of A (and a/A)

$$bb' \subset (a/d'')(a/c) \subset (a/A)(a/A) = a/A$$

as required. Thus, as in the first part of the proof, $a/A \in \alpha$.

(ii) If A, $B \in \alpha$ then clearly $A \cap B$ is convex and $\widetilde{A \cap B} \subset A \cap B$, so we simply show that $A \cap B \subset \widetilde{A \cap B}$. Let $a \subset A \cap B$ and $b \subset X$. Then we construct $c \subset ab$ with $ac \subset A \cap B$. For $a \subset A \cap B = \tilde{A} \cap \tilde{B}$ and so there exist c_1, $c_2 \subset ab$ with $ac_1 \subset A$ and $ac_2 \subset B$. Thus $c_1/a \approx c_2/a$ (in b) and so $ac_1 \approx ac_2$ (in c, say). Hence $ac \subset a(ac_1 \cap ac_2) \subset a(ac_1) \cap a(ac_2) = ac_1 \cap ac_2 \subset A \cap B$ as required.

Theorem 2. *Let $(X, \cdot)$ be a convexity space, let α consist of those convex subsets A of X with $A = \tilde{A}$, and let β consist of all unions of members of α. Then $(X, \cdot, \beta)$ is a tcs.*

Proof. It is clear that $\varnothing = \tilde{\varnothing}$, $X = \tilde{X}$ and $\varnothing$, $X \in \alpha \subset \beta$. Also if A, $B \in \alpha$ then by the lemma $A \cap B \in \alpha$. It follows that β is a topology on X. To verify that $(X, \cdot, \beta)$ is a tcs we prove properties (i)-(iii) of the definition:

(i) Given a, $b \subset X$ and $a \subset U \in \alpha$ it follows that $a \subset U = \tilde{U}$ and $ab \approx U$. Hence $a \subset \overline{ab}$.

(ii) If $ab \approx U \in \beta$ then for some $U' \in \alpha$ with $U' \subset U$ there exists $c \subset ab \cap U'$. Let $d \subset a/c$, $e \subset b/c$, $V = dU'$ and $W = eU'$. Then, by the Lemma, V, $W \in \alpha \subset \beta$, and $a \subset dc \subset dU' = V$, $b \subset W$. Also $c \subset ab \subset (dc)(ec) = (de)c$ and so $c \subset de$ and $de \approx U'$. Hence if $a' \subset V = dU'$ and $b' \subset W = eU'$, then by the convexity of U'

$$U' \approx de \subset (a'/U')(b'/U') \subset a'b'/U'U' = a'b'/U'.$$

Thus $a'b' \approx U'U' = U' \subset U$ and $a'b' \approx U$ as required.

(iii) Similarly, given a, $b \subset X$ and $U \in \beta$ with $a/b \approx U$ there exists $U' \subset U$ with $U' \in \alpha$ and $c \subset a/b \cap U'$. Let $d \subset ab$, $V = dU'$ and $W = d/U'$. Then, as in the Lemma, V, $W \in \alpha \subset \beta$. Also $a/c \approx d/a$ (in b) whence $a = aa \approx dc$ and $a \subset dc \subset V$. Similarly $d \subset ab \subset dcb$, $d \subset cb$ and $b \subset d/c \subset W$. Finally, if $a' \subset V = dU'$ and $b' \subset W = d/U'$ then $d \subset (a'/U') \cap b'U'$ whence $a'/b' \approx U'U' = U' \subset U$ as required.

In the case when X is a vector space, β is the familiar convex core topology (**4**). Note that all the linear topologies discussed in (**4**), (**7**) can be extended to a convexity space.

A tcs $(X, \cdot, \tau)$ is called *locally star-shaped* if given $a \subset U \in \tau$ there exists $V \in \tau$ with $a \subset V^*$ (i.e. V star-shaped from a) and $a \subset V \subset U$. Similarly $(X, \cdot, \tau)$ is *locally convex* if given $a \subset U \in \tau$ there exists $V \in \tau$ with V convex and $a \subset V \subset U$. Further $(X, \cdot, \tau)$ is *strongly convex* if given A compact, convex and $A \subset U \in \tau$ there exists $V \in \tau$ with V convex and $A \subset V \subset U$. It is clear that strongly convex $\Rightarrow$locally convex$\Rightarrow$locally star-shaped. Also, a locally convex tvs is strongly convex. For if A is compact, convex and contained in an open set U in a locally convex tcs, then U^c is closed and disjoint from A. Hence there exists a convex open set B containing the origin with $(A+B) \cap (U^c+B) = \varnothing$ (**6**, page 65). It follows that $A+B$ is convex and open with $A \subset A+B \subset U$. Also, the product of locally convex tvs' is strongly convex.

The tcs $(X, \cdot, \beta)$ constructed above is clearly locally convex since β has a basis of convex sets. Furthermore, we shall see in the next section that β is the finest locally convex topology making $(X, \cdot)$ into a tcs.

3. Linear properties

Unless otherwise stated, the following results are in a tcs $(X, \cdot, \tau)$. The equivalent results in a tvs can be found, for example, in (**8**).

Theorem 3. *If A is Convex then $\bar{A}A^0 = A^0$.*

Proof. Let A be convex, $a \subset \bar{A}$, $b \subset A^0$ and assume that $ab \approx (A^0)^c$ (i.e. the complement of A^0). Then there exists $c \subset ab \cap (A^0)^c$ and $c/a \approx A^0 \in \tau$. Hence there exist $V, W \in \tau$ with $c \subset V$, $a \subset W$ and such that $c'/a' \approx A^0$ whenever $c' \subset V$, $a' \subset W$. Since $c \subset (A^0)^c$ it follows that $V \approx A^c$, and since $a \subset \bar{A}$ it follows that $W \approx A$. Thus, by the convexity of A, $A^c/A \approx A^0 \subset A$ and $A^c \approx AA = A$ which is a contradiction. Hence $a \subset \bar{A}$, $b \subset A^0$ imply $ab \subset A^0$ and so $\bar{A}A^0 \subset A^0$. Also $A^0 \subset A^0 A^0 \subset \bar{A}A^0$ and the result follows.

Corollary. *If A is convex then so is A^0.*

Proof. If A is convex then $A^0 A^0 \subset \bar{A}A^0 = A^0$.

Theorem 4. *If A is convex then so is $\bar{A}$.*

Proof. Let $a, b \subset \bar{A}$, $c \subset ab$ and $c \subset U \in \tau$. We shall show that $U \approx A$ whence $c \subset \bar{A}$. For $ab \approx U$ and so there exist $V, W \in \tau$ with $a \subset V$, $b \subset W$ and such that $a'b' \approx U$ whenever $a' \subset V$, $b' \subset W$. Now since $a, b \subset \bar{A}$ and A is convex it follows that $V \approx A$, $W \approx A$ and $U \approx AA = A$. Thus $c \subset \bar{A}$, $ab \subset \bar{A}$, $\bar{A}\bar{A} \subset \bar{A}$ and $\bar{A}$ is convex.

Theorem 5. *In a locally star-shaped tcs $A^0 \subset \tilde{A}$ and $\hat{A} \subset \bar{A}$.*

Proof. If $a \subset A^0$ in a locally star-shaped space then there exists $U \in \tau$ with $a \subset U \subset A^0$ and $a \subset U^*$. Now for any $b \subset X$ we have $a \subset \overline{ab}$ and so there exists $c \subset ab \cap U$. But then $ac \subset U^* U \subset U \subset A$, $a \subset \tilde{A}$ and $A^0 \subset \tilde{A}$ as required.

Secondly, if $a \subset \hat{A}$ then $ab \subset A$ for some b. Hence $a \subset \overline{ab} \subset \bar{A}$ and $\hat{A} \subset \bar{A}$ as required.

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Theorem 6. *If A is convex and $A^0 \neq \emptyset$ in a locally star-shaped tcs then* $\bar{A} = \hat{A} = \overline{A^0}$ *and* $A^0 = \tilde{A}$.

Proof. Let A be convex and $a \subset A^0$ in a locally star-shaped tcs. Then we show $\tilde{A} \subset A^0$ and $\bar{A} \subset \overline{A^0} \cap \hat{A}$, which together with Theorem 5 gives the required results. If $b \subset \tilde{A}$ and $b' \subset b/a$ then $bb' \approx A$ (in c, say). Thus $c \subset bb' \subset b(b/a) \subset b/a$ and by Theorem 3, $b \subset ac \subset A^0 A \subset A^0 \bar{A} = A^0$. Hence $\tilde{A} \subset A^0$.

Secondly, if $b \subset \bar{A}$ then again by Theorem 3, $ab \subset A^0 \bar{A} = A^0$ and

$$b \subset \overline{ab} \cap \widehat{ab} \subset \overline{A^0} \cap \hat{A}.$$

Hence $\bar{A} \subset \overline{A^0} \cap \hat{A}$.

Theorem 7. *If $(X, \cdot)$ is a convexity space then β is the finest topology making it into a locally convex tcs, i.e. if $(X, \cdot, \tau)$ is a locally convex tcs then $\tau \subset \beta$.*

Proof. By Theorem 2, $(X, \cdot, \beta)$ is a locally convex tcs. Now if $(X, \cdot, \tau)$ is another such and $U \in \tau$, then for each $a \subset U$ there exists $V_a \in \tau$ with V_a convex and $a \subset V_a \subset U$. But then by Theorem 5, $V_a = V_a^0 \subset \tilde{V}_a \subset V_a$ and so $V_a = \tilde{V}_a$. Thus $V_a \in \alpha$, $U = \bigcup_{a \subset U} V_a \in \beta$ and $\tau \subset \beta$ as required.

4. Faces and the Krein-Milman theorem

Here we only deal with the separation properties and facial structure of convex sets insomuch that they concern the Krein-Milman theorem in a tcs. Full details of separation in a convexity space can be found in (3) and publications on faces and on finite-dimensional results are under preparation.

If F, A are convex in $(X, \cdot)$ then $F \subset A$ is a *face* of A if whenever $a, b \subset A$ and $F \approx ab$ it follows that $a, b \subset F$. If a is a face of A then a is called an *extreme point* of A and the collection of all such is the *profiile* $\mathring{A}$ of A. It is easy to check that if G is a face of F and F is a face of A then G is a face of A. For if $G \subset F \subset A$, $x, y \subset A$ and $xy \approx G$, then $xy \approx F$ and, since F is a face of A, $xy \subset F$. Thus $x, y \subset F$, $xy \approx G$ and, since G is a face of F, $x, y \subset G$. Also if F_i $(i \in I)$ is a family of faces of A, then $\bigcap_{i \in I} F_i$ is a face of A. For if $x, y \subset A$ and $xy \approx F = \bigcap_{i \in I} F_i$ then $xy \approx F_i$ for each i, $x, y \subset F_i$ for each i and $x, y \subset F$.

Lemma. *Let $(X, \cdot, \tau)$ be a strongly convex tcs with τ Hausdorff, let A be compact, convex and let B be closed with $\emptyset \neq B \subset A$, $B \neq A$. Then there exists a non-empty closed face G of A with $G \cap B = \emptyset$. Further if B is a singleton then the result holds in a locally convex space.*

Proof. We prove the part of the lemma concerning strongly convex spaces, the proof of the remaining part being very similar. Let $(X, \cdot, \tau)$, A and B be as stated and let $a \subset A \backslash B$. Then since a^c is open there is a $U \in \tau$ with U convex and $B \subset U \subset a^c$. Consider the non-empty collection $\mathscr{U}$ of all convex open sets

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containing B but not A, and partially order $\mathscr{U}$ by inclusion. Then any totally-ordered subfamily $\mathscr{U}_0$ of $\mathscr{U}$ has an upper bound, namely their union, since by compactness $A \not\subset \bigcup_{U \in \mathscr{U}_0} U$. Hence by Zorn's Lemma $\mathscr{U}$ has a maximal member U_0, say. Let $G = A \cap U_0^c (\neq \emptyset)$. We shall show that G is the required face of A.

Firstly, G is convex. For if $c, d \subset G$ with $cd \not\subset G$ then there exists $e \subset cd \cap U_0$. Then choosing $c' \subset c/e$ and putting $U' = U_0 \cup c'U_0$ gives U' convex (Section 1), open (Theorem 1), containing B, not containing A ($d \subset A \backslash U'$) and properly containing U_0 ($c \subset U' \backslash U_0$). This contradicts the maximality of U_0 in $\mathscr{U}$ and so G is convex as claimed.

Next, G is a face of A. For if $a, b \subset A$ and $c \subset ab \cap G$ then we claim that $a, b \subset G$. If say, $a \not\subset G$ then $a \subset U_0$ and by the convexity of U_0 and the fact that $c \subset U_0^c$ it follows that $b \subset U_0^c$. But then in a similar fashion to the above $U_0 \cup bU_0 \in \mathscr{U}$ contradicts the maximality of U_0. Thus G is a face of A.

Finally G is closed and $G \cap B = \emptyset$ since $U_0 \in \mathscr{U}$. Thus G is the required face of A.

Theorem 8. *If A is compact and convex with non-empty closed face F in a locally convex Hausdorff tcs then $F \approx \mathring{A}$.*

Proof. Note firstly that if F_0 is a minimal such face of A then F_0 is a singleton. For if $a \subset F_0$, $a \neq F_0$, then by the second part of the Lemma applied to $a \subset F_0$ there exists a non-empty closed face G of F_0 with $a \not\subset G$ and $G \neq F_0$. But then G is a non-empty closed face of A properly contained in F_0, and F_0 is not minimal.

Now let F be as stated and let $\mathscr{F}$ be the collection of all families of non-empty closed faces of A satisfying

(*a*) $F \in \mathscr{F}$;

(*b*) $F_1, \ldots, F_k \in \mathscr{F} \Rightarrow F_1 \cap \ldots \cap F_k \neq \emptyset$.

Then $\mathscr{F}$ is non-empty since the family consisting of F alone is in $\mathscr{F}$. Partially-order $\mathscr{F}$ by inclusion of the families and observe that each totally-ordered sub-collection of $\mathscr{F}$ has an upper bound (namely its family-wise union). Thus by Zorn's Lemma $\mathscr{F}$ has a maximal member $\mathscr{F}_0$, say. Let $F_0 = \bigcap_{B \in \mathscr{F}_0} B$. Then, by the compactness of A, $F_0 \neq \emptyset$ and hence it is a minimal non-empty closed face of A. Thus by our earlier comments $F_0 = a \subset \mathring{A}$ and by construction $F_0 \subset F$. Thus $F \approx \mathring{A}$ as required.

Theorem 9. (Krein-Milman). *If A is compact and convex in a strongly convex Hausdorff tcs then $A = \overline{[\mathring{A}]}$ (i.e. the closure of the convex hull of the profile of A).*

Proof. Let B $= \overline{[\mathring{A}]} \subset \bar{A} = A$. Then, if $B \neq A$, by the Lemma there exists a non-empty closed face G of A with $G \cap B = \emptyset$. But then by Theorem 8 $G \approx \mathring{A} \subset B$ which is a contradiction. Thus $A = B = \overline{[\mathring{A}]}$ as required.

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