Two Circular Notes.

By R. Tucker, M.A.

I.

1. In Fig. 8, take

$$\angle BFD = \theta = \angle CDE = \angle AEF,$$

and denote DE, EF, FD by x, y, z respectively; then

$$\begin{aligned} &x\sin B\sin(C+\theta) && + z\sin C\sin\theta = a\sin B\sin C = P, \\ &x\sin A\sin\theta + y\sin C\sin(A+\theta) && = P, \\ & \qquad y\sin B\sin\theta + z\sin A\sin(B+\theta) && = P. \end{aligned}$$

From these equations

$$x[\sin^3\theta + \sin(A+\theta)\sin(B+\theta)\sin(C+\theta)]$$
$$= x\sin A\sin B\sin C\sin(\theta+\omega)/\sin\omega = P\sin A,$$

i.e., $$x = a\sin\omega/\sin(\theta+\omega).$$

Hence $\triangle$DEF is similar to ABC, and the modulus of similarity is $\sin\omega/\sin(\theta+\omega)$.

2. The preceding result is got, in an elegant geometrical manner, in Milne's *Companion* (Simmons, Cap. V., the Tucker Circles, pp. 127–137); *c.f.* also H. M. Taylor, On the relations of the intersections of a circle with a triangle, L.Math. Soc. *Proceedings*, vol. XV., pp. 122–139.

3. We readily deduce the equation to the circle DEF to be

$$(\lambda^2 \equiv a^2b^2 + b^2c^2 + c^2a^2)$$

$$\lambda^2\sin^2(\theta+\omega)\,.\,\Sigma(a\beta\gamma) = \Sigma(a\alpha)\,.\,\Sigma\{b^2c^2\sin(A+\theta)\,.\,\alpha\}\,.\,\sin\theta. \quad \text{(i.)}$$

This circle cuts the sides BC, CA, AB, in points D′, E′, F′ such that $\angle AF'E' = \theta = \angle BD'F' = \angle CE'D'$.

4. The equation to this group of circles is given in the form

$$(\alpha - Ka)(\beta - Kb)(\gamma - Kc) - \alpha\beta\gamma = 0.$$

(Simmons l.c. p. 136, and Casey, *Conics* 2nd edition, p. 421. In the 1st edition the form was utterly wrong.)

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5. For the "T.R." circle, $\theta = \omega$, and (i.) becomes, since

$$\sin(A+\omega) = (b^3 + c^2)\sin A/\lambda,$$
$$K^2\Sigma(\alpha\beta\gamma) = \Sigma(a\alpha) \, . \, \Sigma\{bc(b^2+c^2)\alpha\} \, ;$$

which is a neater form than either of the forms given by me in Quar. Jour. of Math. (The "Triplicate-Ratio" Circle, vol. XIX., p. 346).

6. For the "Cosine" Circle, $\theta = \frac{\pi}{2}$, and (i.) becomes

$$K^2\Sigma(\alpha\beta\gamma) = 4\Sigma(a\alpha) \, . \, \Sigma\{b^2c^2\cos A \, . \, \alpha\} \, .$$

II.

1. In Fig. 9 I connect any point O with the vertices A, B, C of the triangle and then bisect the angles BOC, COA, AOB by lines meeting BC, CA, AB in D, E, F.

I propose to consider some properties of the group of triangles DEF.

2. For AO, BO, CO write l, m, n; and denote the triangles BOC, COA, AOB by δ_a, δ_b, δ_c, then the trilinear co-ordinates of O are

$$2\delta_a/a, \quad 2\delta_b/b, \quad 2\delta_c/c \, .$$

3. From Euc. vi. 3

AD, BE, CF cointersect in a point, P, and the points D, E, F are given by

$$\left.\begin{array}{llll} D, & o & cn & bm \\ E, & cn & o & al \\ F, & bm & al & o \end{array}\right\} .$$

Hence the equations to

$$\left.\begin{array}{l} AD, \\ BE, \\ CP, \end{array}\right\} \text{are} \quad \left.\begin{array}{l} bm\beta = cn\gamma \, , \\ cn\gamma = al\alpha \, , \\ al\alpha = bm\beta \, , \end{array}\right\} \quad \text{and the point P is} \quad al\alpha = bm\beta = cn\gamma \, . \qquad \text{(i.)}$$

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4. Now
$$\frac{\mathrm{BD}}{m}=\frac{\mathrm{CD}}{n}=\frac{a}{m+n},$$
$$\frac{\mathrm{CE}}{n}=\frac{\mathrm{AE}}{l}=\frac{b}{n+l},$$
$$\frac{\mathrm{AF}}{l}=\frac{\mathrm{BF}}{m}=\frac{c}{l+m};$$

and therefore
$$2\triangle\mathrm{BDF}=\mathrm{BD}\,.\,\mathrm{BF}\,.\,\sin\mathrm{B}$$
$$=m^2\,.\,2\triangle/(m+n\,.\,l+m);$$
$$\therefore\ \triangle\mathrm{DEF}=\triangle\,.\,2lmn/(l+m\,.\,m+n\,.\,n+l). \qquad \text{(ii.)}$$

5. The equation to the circle DEF is
$$2abc\,.\,l+m\,.\,m+n\,.\,n+l\,.\,\Sigma(a\beta\gamma)= \qquad \text{(iii.)}$$
$$\Sigma(a\alpha)\,.\,\Sigma[al\,.\,\alpha\{b^2\,.\,l+m\,.\,m+n+c^2\,.\,m+n\,.\,n+l-a^2\,.\,n+l\,.\,l+m\}].$$

6. The equations to DE, EF, FD are
$$\left.\begin{aligned} al\alpha+bm\beta-cn\gamma=0,\\ -al\alpha+bm\beta+cn\gamma=0,\\ al\alpha-bm\beta+cn\gamma=0.\end{aligned}\right\} \qquad \text{(iv.)}$$

7. If D′, E′, F′ are the Harmonic Conjugates to D, E, F respectively, we have
$$\frac{\mathrm{BD}'}{m}=\frac{\mathrm{CD}'}{n}=\frac{a}{m-n},\ \text{etc.,}$$
and the points
$$\left.\begin{matrix}\mathrm{D}'\\ \mathrm{E}'\\ \mathrm{F}'\end{matrix}\right\}\ \text{are given by}\ \begin{vmatrix} o & -cn & bm\\ cn & o & -al\\ -bm & al & o\end{vmatrix}$$

Hence the line D′E′F′, the axis of Perspective of DEF and ABC, has for its equation
$$al\alpha+bm\beta+cn\gamma=0.$$

8. If the ratio $l:m:n$ is given, O is, of course, determined by the intersections of the circles on DD′, EE′, FF′ as diameters.

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9. The equations to the circles on DD′, FF′, referred to BC, BA as axes are

$$(m^2-n^2)(x^2+y^2+2xy\cos B)-2m^2ax-2m^2ay\cos B+m^2a^2=0\,,$$
$$(m^2-l^2)(x^2+y^2+2xy\cos B)-2m^2cx\cos B-2m^2cy+m^2c^2=0\,. \quad \text{(v.)}$$

10. The circle (iii.) cuts BC again in d, so that

$$\left.\begin{aligned}&\frac{Bd}{c^2(m+n)(n+l)+a^2(n+l)(l+m)-b^2(l+m)(m+n)}\\ &=\frac{Cd}{a^2(n+l)(l+m)+b^2(l+m)(m+n)-c^2(m+n)(n+l)}\end{aligned}\right\}.$$

11. If $l=m=n$, O is the circumcentre, P the centroid, and (iii.) the nine-point circle.

If O is Ω, $\quad l:m:n=cb^2:ac^2:ba^2\,,$

and P is $\quad b\alpha=c\beta=a\gamma\,;$

and if O is Ω′ $\quad l:m:n=c^2b:a^2c:b^2a\,,$

and P is $\quad c\alpha=a\beta=b\gamma\,.$

12. If P is Ω, then $l:m:n=ca:ab:bc\,,$

and if P is Ω′, then $l:m:n=ab:bc:ca\,.$

13. If P is the circumcentre, then

$$l:m:n=\operatorname{cosec}2A:\operatorname{cosec}2B:\operatorname{cosec}2C\,;$$

and if P is the orthocentre, then

$$l:m:n=\cot A:\cot B:\cot C\,,$$

and (iii.) is, of course, the nine-point circle.

14. If O is the orthocentre, then

$$l:m:n=\cos A:\cos B:\cos C\,,$$

and P is $\quad \alpha\sin 2A=\beta\sin 2B=\gamma\sin 2C\,.$

15. If $l:m:n=s-a:s-b:s-c$, then P is the Gergonne point, and (iii.) is the In-circle.

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16. If P is the Symmedian-point, then (iii.) is

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2)\Sigma(a\beta\gamma)=\Sigma(a\alpha)\,.\,\Sigma[bca(b^4+c^4-a^4+\lambda^2)],$$

where $\lambda^2 \equiv a^2b^2+b^2c^2+c^2a^2$ as before.

17. If $n=0$, $m=a$, $l=b$, (iii.) becomes

$$(a+b)\Sigma(a\beta\gamma)=\Sigma(a\alpha)\,.\,[b\cos\mathrm{A}\alpha+a\cos\mathrm{B}\beta]$$

which cuts AB in the points where the bisector of $\angle$ C and the perpendicular from C meet it.

18. If the locus of P is the line

$$p\alpha+q\beta+r\gamma=0\,,$$

then the envelope of DE is found from the equations

$$\left.\begin{aligned} &\frac{p}{al}+\frac{q}{bm}+\frac{r}{cn}=0\,, \\ \text{and}\qquad &al\alpha+bm\beta-cn\gamma=0\,. \end{aligned}\right\}$$

It is readily seen to be the in-conic

$$p^2\alpha^2+q^2\beta^2+r^2\gamma^2-2pq\alpha\beta+2qr\beta\gamma+2rp\gamma\alpha=0\,.$$

The chords of contact are

$$\left.\begin{aligned} p\alpha-q\beta+r\gamma&=0\,, \\ -p\alpha+q\beta+r\gamma&=0\,, \\ p\alpha+q\beta+r\gamma&=0\,. \end{aligned}\right\}$$

In like manner, with the same condition, the envelope of D′E′F′ is

$$p^2\alpha^2+q^2\beta^2+r^2\gamma^2-2qr\beta\gamma-2rp\gamma\alpha-2pq\alpha\beta=0\,.$$

19. If P, P′ are inverse points, and O, O′ are given by (l, m, n), (l', m', n'), then we have

$$a^2ll'=b^2mm'=c^2nn'. \qquad \text{(vi.)}$$

The equation to PP′, in the general case, is

$$all'\alpha[m'n-mn']+\ldots+\ldots=0$$

hence, if (vi.) is satisfied, and the line passes through the symmedian point, we must have

$$\Sigma[m'n-mn']=0\,. \qquad \text{(vii.)}$$

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20. If O, O′, O″ are points such that P, P′, P″ are collinear, then we have

$$\Sigma l'l''mn[m''n' - m'n''] = 0\,.$$

21. If P, P′ are inverse points, the equation (vii.) is satisfied by the cubic

$$\Sigma\left(\frac{m-n}{a^2l}\right) = 0 \quad :$$

a value is

$$\frac{1}{l} : \frac{1}{m} : \frac{1}{n} = \lambda^2 - 2b^2c^2 : \lambda^2 - 2c^2a^2 : \lambda^2 - 2a^2b^2\,.$$

An Arithmetical Problem.

By Dr Wm. Peddie.

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