Proceedings of the Edinburgh Mathematical Society (2014) **57**, 299–304
DOI:10.1017/S0013091513000862

THETA GROUPS AND PRODUCTS OF ABELIAN AND RATIONAL VARIETIES

YURI G. ZARHIN[1,2]

[1] *Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA*
[2] *Department of Mathematics, Weizmann Institute of Science, POB 26, Rehovot 7610001, Israel* (zarhin@math.psu.edu)

Dedicated to Slava Shokurov on the occasion of his 60th birthday

Abstract We prove that an analogue of Jordan's theorem on finite subgroups of general linear groups does not hold for the groups of birational automorphisms of products of an elliptic curve and the projective line. This gives a negative answer to a question posed by Vladimir L. Popov.

Keywords: birational automorphisms; line bundles; abelian varieties

2010 *Mathematics subject classification:* Primary 14E07
Secondary 14K05

1. Introduction

Throughout this paper, k is an algebraically closed field of characteristic zero, $\mathbb{A}^1$ and $\mathbb{P}^1$ are the affine line and projective line respectively (both over k). If U is an irreducible algebraic variety over k, then we write $k[U]$, $k(U)$ and $\mathrm{Bir}(U)$ for its ring (k-algebra) of regular functions, the field of rational functions and the group of birational k-automorphisms respectively.

The following definition was inspired by the classical theorem of Jordan about finite subgroups of general linear groups.

Definition 1.1 (Popov [3, Definition 2.1]). A group B is called a *Jordan group* if there exists a positive integer J_B such that every finite subgroup B_1 of B contains a normal commutative subgroup, whose index in B_1 is at most J_B.

Popov [**3**, § 2] posed the question of whether $\mathrm{Bir}(Y)$ is a Jordan group when Y is an irreducible surface. He obtained a positive answer to his question for almost all surfaces. (The case of rational surfaces was done earlier by J.-P. Serre.) One of the few remaining cases is a product $E \times \mathbb{P}^1$ of an elliptic curve E and the projective line.

Our main result is the following statement, which gives a negative answer to Popov's question.

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Theorem 1.2. *If E is an elliptic curve over k, then* $\mathrm{Bir}(E\times\mathbb{P}^1)$ *is not a Jordan group.*

Since $U\times\mathbb{A}^1$ is birationally isomorphic to $U\times\mathbb{P}^1$, the groups $\mathrm{Bir}(U\times\mathbb{A}^1)$ and $\mathrm{Bir}(U\times\mathbb{P}^1)$ are isomorphic and Theorem 1.2 becomes equivalent to the assertion that $\mathrm{Bir}(E\times\mathbb{A}^1)$ is not a Jordan group, which, in turn, is a special case of the following statement.

Theorem 1.3. *Let X be an abelian variety of positive dimension over k. Then* $\mathrm{Bir}(X\times\mathbb{A}^1)$ *is not a Jordan group.*

Corollary 1.4. *Let X be an abelian variety of positive dimension over k and Z be a rational variety of positive dimension over k. Then* $\mathrm{Bir}(X\times Z)$ *is not a Jordan group.*

Proof of Corollary 1.4 (modulo Theorem 1.3). Since Z is birationally isomorphic to the d-dimensional affine space $\mathbb{A}^d$ with $d=\dim(Z)\geqslant 1$, the groups $\mathrm{Bir}(X\times Z)$ and $\mathrm{Bir}(X\times\mathbb{A}^d)$ are isomorphic. So, it suffices to check that $\mathrm{Bir}(X\times\mathbb{A}^d)$ is *not* a Jordan group. If $d=1$, the result follows from Theorem 1.3. If $d>1$, then $X\times\mathbb{A}^d=(X\times\mathbb{A}^1)\times\mathbb{A}^{d-1}$ and one may view $\mathrm{Bir}(X\times\mathbb{A}^1)$ as the certain subgroup of $\mathrm{Bir}(X\times\mathbb{A}^d)$, and again Theorem 1.3 gives us the desired result. □

The paper is organized as follows. Section 2 deals with the certain subgroup $\mathrm{Bir}_1(X\times\mathbb{A}^1)$ of $\mathrm{Bir}(E\times\mathbb{A}^1)$ that is generated by translations on X and multiplications of the global coordinate t on $\mathbb{A}^1$ by non-zero rational functions on X. We assert that $\mathrm{Bir}_1(X\times\mathbb{A}^1)$ is not a Jordan group; obviously, this assertion implies that $\mathrm{Bir}(X\times\mathbb{A}^1)$ is also not a Jordan group. In § 3 we discuss a *symplectic geometry* related to certain analogues of Heisenberg groups that were introduced by Mumford [**1**, § 1]. In § 4, using results of Mumford [**1**, § 1], we realize these analogues as subgroups of $\mathrm{Bir}_1(X\times\mathbb{A}^1)$, which allows us to prove that $\mathrm{Bir}_1(X\times\mathbb{A}^1)$ is not a Jordan group.

2. Birational automorphisms of products of an abelian variety and the affine line

Let X be an abelian variety of positive dimension over k. If $y\in X(k)$, then we write T_y for the translation map

$$T_y\colon X\to X,\qquad x\mapsto x+y.$$

As usual, we write $\mathrm{div}(f)$ for the divisor of a rational function $f\in k(X)^*$. Clearly, T_y^*f is the rational function $x\mapsto f(x+y)$, whose divisor coincides with $T_y^*(\mathrm{div}(f))$. Let t be the global coordinate on $\mathbb{A}^1$.

We write $\mathrm{Bir}_1(X\times\mathbb{A}^1)\subset\mathrm{Bir}(X\times\mathbb{A}^1)$ for the set of birational automorphisms of the form

$$A(y,f)\colon X\times\mathbb{A}^1 \dashrightarrow X\times\mathbb{A}^1,\qquad (x,t)\mapsto(x+y,f(x)\cdot t)=(T_y(x),f(x)\cdot t),$$

https://doi.org/10.1017/S0013091513000862 Published online by Cambridge University Press

where y runs through $X(k)$, and f through $k(X)^*$. Actually, $\mathrm{Bir}_1(X \times \mathbb{A}^1)$ is a subgroup of $\mathrm{Bir}(X \times \mathbb{A}^1)$. Indeed, one may easily check that

$$A(y_2, f_2)A(y_1, f_1) = A(y_1 + y_2, T^*_{y_1}(f_2) \cdot f_1) \in \mathrm{Bir}_1(X \times \mathbb{A}^1)$$

and that the inverse of $A(y, f)$ in $\mathrm{Bir}(X \times \mathbb{A}^1)$ coincides with $A(-y, T^*_{-y}(1/f)) \in \mathrm{Bir}_1(X \times \mathbb{A}^1)$.

Now Theorem 1.3 becomes an immediate corollary of the following statement.

Theorem 2.1. *Let X be an abelian variety of positive dimension over k. Then $\mathrm{Bir}_1(X \times \mathbb{A}^1)$ is not a Jordan group.*

We prove Theorem 2.1 in § 4.

3. Group theory

Let $\boldsymbol{K}$ be a finite commutative group. Let $\hat{\boldsymbol{K}} := \mathrm{Hom}(\boldsymbol{K}, k^*)$ be the group of *characters* of $\boldsymbol{K}$. We write the group law on $\boldsymbol{K}$ additively and on $\hat{\boldsymbol{K}}$ multiplicatively. In particular, we write $\mathbf{1}$ for the trivial character of $\boldsymbol{K}$. Clearly, the groups $\boldsymbol{K}$ and $\hat{\boldsymbol{K}}$ are isomorphic (non-canonically); in particular, they have the same order, which we denote by $N = N_{\boldsymbol{K}}$.

Let $\mu_N \subset k^*$ be the (sub)group of Nth roots of unity. Clearly, for every non-zero $x \in \boldsymbol{K}$ there exists $\ell \in \hat{\boldsymbol{K}}$ with $\ell(x) \neq 1$. On the other hand,

$$Nx = 0, \quad \ell(x) \in \mu_N, \quad \forall x \in \boldsymbol{K},\ \ell \in \hat{\boldsymbol{K}}.$$

Let us consider the commutative finite group $\boldsymbol{H}_{\boldsymbol{K}} = \boldsymbol{K} \times \hat{\boldsymbol{K}}$ and the non-degenerate alternating bi-additive form

$$e_{\boldsymbol{K}} \colon \boldsymbol{H}_{\boldsymbol{K}} \times \boldsymbol{H}_{\boldsymbol{K}} \to k^*, \qquad ((x, \ell), (x', \ell')) \mapsto \ell'(x)/\ell(x').$$

Clearly, all the values of $e_{\boldsymbol{K}}$ lie in μ_N.

Let E be an *isotropic* subgroup of $\boldsymbol{H}_{\boldsymbol{K}}$ with respect to $e_{\boldsymbol{K}}$. Let $E^\perp$ be the orthogonal complement of E in $\boldsymbol{H}_{\boldsymbol{K}}$ with respect to $e_{\boldsymbol{K}}$. Then $E \subset E^\perp$ and the non-degeneracy of $e_{\boldsymbol{K}}$ gives rise to a group isomorphism

$$\boldsymbol{H}_{\boldsymbol{K}}/E^\perp \cong \mathrm{Hom}(E, k^*) = \hat{E}.$$

In particular, E and $\boldsymbol{H}_{\boldsymbol{K}}/E^\perp$ have the same order. The inclusions $E \subset E^\perp \subset \boldsymbol{H}_{\boldsymbol{K}}$ imply that

$$\#(E)^2 = \#(E) \cdot \#(\boldsymbol{H}_{\boldsymbol{K}}/E^\perp)$$

divides $\#(\boldsymbol{H}_{\boldsymbol{K}}) = N^2$ and therefore $\#(E)$ divides N. Since

$$N^2 = \#(\boldsymbol{H}_{\boldsymbol{K}}) = \#(E) \cdot \#(\boldsymbol{H}_{\boldsymbol{K}}/E),$$

the index of E in $\boldsymbol{H}_{\boldsymbol{K}}$ is divisible by N. This means that *the index of every isotropic subgroup in $\boldsymbol{H}_{\boldsymbol{K}}$ is divisible by N and therefore is greater than or equal to N.*

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Following [**1**, § 1], let us consider the set

$$\mathfrak{G}_{\boldsymbol{K}} = k^* \times \boldsymbol{H}_{\boldsymbol{K}} = k^* \times \boldsymbol{K} \times \hat{\boldsymbol{K}},$$

and introduce on it the group structure by defining the product

$$(a, x, \ell)(a', x', \ell') := (aa'\ell'(x), x + x', \ell\ell').$$

One may naturally identify k^* with the central subgroup $\{(a, 0, \mathbf{1}) \mid a \in k^*\}$. In fact, $\mathfrak{G}_{\boldsymbol{K}}$ sits in the short exact sequence

$$0 \to k^* \to \mathfrak{G}_{\boldsymbol{K}} \xrightarrow{\pi} \boldsymbol{H}_{\boldsymbol{K}} \to 0,$$

where $\pi\colon \mathfrak{G}_{\boldsymbol{K}} \twoheadrightarrow \boldsymbol{H}_{\boldsymbol{K}}$ sends (a, x, ℓ) to (x, ℓ). One may easily check that if $g, g' \in \mathfrak{G}_{\boldsymbol{K}}$, then

$$gg'g^{-1}{g'}^{-1} = e_{\boldsymbol{K}}(\pi(g), \pi(g')) \in k^* \subset \mathfrak{G}_{\boldsymbol{K}}.$$

It follows that a subgroup $\tilde{E} \subset \mathfrak{G}_{\boldsymbol{K}}$ is *commutative* if and only if its image $\pi(\tilde{E})$ is an *isotropic* subgroup in $\boldsymbol{H}_{\boldsymbol{K}}$; if this is the case, then the index of $\pi(\tilde{E})$ in $\boldsymbol{H}_{\boldsymbol{K}}$ is greater than or equal to $N = N_{\boldsymbol{K}}$.

Clearly, the subset

$$\mathfrak{G}^1_{\boldsymbol{K}} = \mu_N \times \boldsymbol{H}_{\boldsymbol{K}} = \mu_N \times \boldsymbol{K} \times \hat{\boldsymbol{K}} \subset \mathfrak{G}_{\boldsymbol{K}}$$

is actually a subgroup of $\mathfrak{G}_{\boldsymbol{K}}$. We have $\pi(\mathfrak{G}^1_{\boldsymbol{K}}) = \boldsymbol{H}_{\boldsymbol{K}}$. Therefore, if $\tilde{E}$ is a commutative subgroup in $\mathfrak{G}^1_{\boldsymbol{K}}$, then the index of $\pi(\tilde{E})$ in $\boldsymbol{H}_K = \pi(\mathfrak{G}^1_{\boldsymbol{K}})$ is greater than or equal to $N = N_{\boldsymbol{K}}$. This implies that index of $\tilde{E}$ in $\mathfrak{G}^1_{\boldsymbol{K}}$ is also greater than or equal to $N = N_{\boldsymbol{K}}$.

4. Mumford's theta groups

We keep all the notation and assumptions of § 2.

We denote by $\mathcal{M}_X$ the constant sheaf (of rational functions) on the abelian variety X with respect to Zariski topology, which assigns to every non-empty open subset U of X its field of rational functions $k(U) = k(X)$. For every $f \in k(X)^*$ let us consider the sheaf (auto)morphism

$$[f]\colon \mathcal{M}_X \to \mathcal{M}_X$$

that is induced by multiplication by f in $k(X)$. If $y \in X(k)$, then $T_y^*\mathcal{M}_X = \mathcal{M}_X$ and the induced (by functoriality) sheaf (auto)morphism $[f]\colon T_y^*[f]\colon T_y^*\mathcal{M}_X \to T_y^*\mathcal{M}_X$ coincides (after the identification of $T_y^*\mathcal{M}_X$ and $\mathcal{M}_X$) with

$$[T_y^* f]\colon \mathcal{M}_X \to \mathcal{M}_X.$$

If D is a divisor on X, then we view the invertible sheaf $\mathcal{O}_X(D)$ as a certain subsheaf of $\mathcal{M}_X$ (see [**4**, Chapter 6, § 1]). Note that, for all $y \in X(k)$,

$$T_y^*\mathcal{O}_X(D) = \mathcal{O}_X(T_y^* D).$$

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If D_1 and D_2 are linearly equivalent divisors on X, then isomorphisms of invertible sheaves $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ are exactly (all the) morphisms of the form

$$[f]\colon \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$$

with $\operatorname{div}(f) = D_1 - D_2$. In particular, this set of isomorphisms is a k^*-torsor, since $\operatorname{div}(f)$ determines the rational function f up to multiplication by a non-zero constant.

If $[f]\colon \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ is an isomorphism of invertible sheaves and $y \in X(k)$, then the induced (by functoriality) isomorphism of invertible sheaves $T_y^*[f]\colon T_y^*\mathcal{O}_X(D_1) \cong T_y^*\mathcal{O}_X(D_2)$ coincides with

$$[T_y^* f]\colon \mathcal{O}_X(T_y^* D_1) \cong \mathcal{O}_X(T_y^* D_2).$$

Now let us choose an ample divisor on X (e.g. a hyperplane section) and set $L = \mathcal{O}_X(D)$. Then L is an ample invertible sheaf. Let us consider the (finite) commutative group

$$H(L) = \{x \in X(k) \mid L \cong T_x^* L\}.$$

Remark 4.1. Let n be a positive integer. Then nD remains ample, $\mathcal{O}_X(nD) = L^n$ and

$$H(L^n) = \{x \in X(k) \mid nx \in H(L)\}$$

(see [**1**, § 1, Proposition 4]). In particular, $H(L^n)$ contains the group X_n of all points of order n on X. Since the order of X_n is $n^{2\dim(X)}$ [**2**, Chapter 2, § 6], the order of $H(L^n)$ is divisible by $n^{2\dim(X)}$.

Following [**1**, § 1], let us consider the *theta group* $\mathfrak{G}(L)$ that consists of all pairs (x, ϕ), where $x \in H(L)$ and ϕ is an isomorphism of invertible sheaves $L \cong T_x^* L$. The group law on $\mathfrak{G}(L)$ is defined as follows. If $(x, \phi\colon L \cong T_x^* L) \in \mathfrak{G}(L)$ and $(y, \psi\colon L \cong T_y^* L) \in \mathfrak{G}(L)$, then its composition $(y, \psi)(x, \phi)$ is defined as

$$(x + y, T_x^* \phi\psi\colon L \cong T_y^* L \cong T_x^*(T_y^* L) = T_{x+y}^* L).$$

Taking into account our considerations in the beginning of this section and the equality $L = \mathcal{O}_X(D)$, we conclude that $H(L)$ coincides with the set of $x \in X(k)$ such that D is linearly equivalent to $T_x^* D$; the theta group $\mathfrak{G}(L)$ is the set of all pairs $(x, [f])$, where $x \in H(L)$ and f is a non-zero rational function on X such that $\operatorname{div}(f) = D - T_x^* D$. In addition, if $(y, [h]) \in \mathfrak{G}(L)$, then

$$(x, [f])(y, [h]) = (x + y, [T_x^* h \cdot f]) \in \mathfrak{G}(L).$$

Remark 4.2. It is known [**1**, § 1, Corollary of Theorem 1] that there exists a finite sequence of positive integers (elementary divisors) $\delta = (d_1, \dots, d_r)$ such that $d_{i+1} | d_i$ and the finite commutative group $K(\delta) = \bigoplus_{i=1}^r \mathbb{Z}/d_i\mathbb{Z}$ enjoys the following properties:

(i) $H(L)$ is isomorphic to $\boldsymbol{H}_{K(\delta)}$;

(ii) the groups $\mathfrak{G}_{K(\delta)}$ and $\mathfrak{G}(L)$ are isomorphic.

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Applying the results of § 3, we conclude that $\mathfrak{G}(L)$ contains a finite subgroup G that enjoys the following property: every commutative subgroup in G has an index that is greater than or equal to $\#(K(\delta)) = \sqrt{\#(H(L))}$.

Proof of Theorem 2.1. Comparing the multiplication formulae for $(x, [f])$ and $A(y, f)$ (see § 2), we conclude that the embedding

$$\mathfrak{G}(L) \hookrightarrow \mathrm{Bir}_1(X \times \mathbb{A}^1), (y, [h]) \mapsto A(y, h)$$

is actually a group homomorphism. So $\mathfrak{G}(L)$ is isomorphic to a subgroup of $\mathrm{Bir}_1(X \times \mathbb{A}^1)$. Applying this assertion to all ample divisors nD and invertible sheaves $L^n = \mathcal{O}_X(nD)$ (where n is a positive integer) and combining it with Remarks 4.1 and 4.2, we conclude that for every positive integer n there exists a finite subgroup

$$G \subset \mathfrak{G}(L^n) \hookrightarrow \mathrm{Bir}_1(X \times \mathbb{A}^1)$$

that enjoys the following property: every commutative subgroup in G has an index that is greater than or equal to $(n^{2\dim(X)})^{1/2} = n^{\dim(X)}$; in particular, this index is greater than or equal to n. This proves that $\mathrm{Bir}_1(X \times \mathbb{A}^1)$ is *not* a Jordan group. □

Acknowledgements. The author is grateful to Volodya Popov for a stimulating question, his interest in this paper and useful discussions. This work was partly supported by Grant 246625 from the Simons Foundation.

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