THE WORD PROBLEM FOR LATTICE ORDERED GROUPS

by A. M. W. GLASS†
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The purpose of this note is to show the existence of a lattice ordered group G with a finite set of generators and a recursively enumerable set of defining relations such that there is no decision procedure to determine whether or not an arbitrary word in the generators reduces to the identity in G. In addition to the usual group-theoretic words, we may also use the two lattice operations $\vee$ and $\wedge$; for example, $a^{-1}(b \vee c)$ is a word in the generators a, b and c. At first sight it might appear that since we have an even greater harvest of words than in group theory and there exist finitely presented groups H (H has a finite number of generators and defining relations) with an insoluble word problem (no decision procedure to determine whether an arbitrary word in the generators reduces to the identity)—see (**1**), (**2**), (**4**) or (**6**)—the same would be true of lattice ordered groups. Unfortunately, such a naïve approach overlooks two salient points. First, the class of lattice ordered groups is strictly smaller than the class of all groups; second, there are certain relations connecting the lattice operations with the group operations which hold true for all lattice ordered groups. For example, $a(b \vee c) = ab \vee ac$ and $(a \vee b)^{-1} = a^{-1} \wedge b^{-1}$.

The class of lattice ordered groups is a non-trivial variety and so there exist free lattice ordered groups (**3**, Chapter 4). As in group theory, we will use $G = (x_i,\ i \in I\colon\ r_j = 1,\ j \in J)$ to denote the lattice ordered group on the set of generators $\{x_i\colon\ i \in I\}$ with relations $r_j = 1$ for all $j \in J$. If I is finite and J is recursively enumerable, G is said to be a *recursively presented lattice ordered group*. G is the quotient of the free lattice ordered group F on the set of generators $\{x_i\colon\ i \in I\}$ by the l-ideal R of F generated by the set $\{r_j\colon\ j \in J\}$ (an l-ideal is a convex normal l-subgroup); i.e. $G = F/R$.

Let X be a set of positive integers with $1 \notin X$ and

$$G(X) = (a,\ b,\ c\colon\ b^{-n}ab^n \wedge c^{-n}ac^n = 1,\ n \in X).$$

We wish to show that if $m \notin X$, then $b^{-m}ab^m \wedge c^{-m}ac^m$ does not reduce to the identity in $G(X)$. In order to do this, it is enough to exhibit some lattice ordered group K on the three generators a, b and c such that $b^{-n}ab^n \wedge c^{-n}ac^n$ reduces to the identity in K if and only if $n \in X$.

For each $n \notin X$ such that $n-1 \in X$, let k_n be that integer with

$$\{n-1,\ \ldots,\ n-k_n\} \subseteq X \quad \text{and} \quad n-k_n-1 \notin X.$$

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Let $\mathbb{R}$ be the real line; $f \in \mathrm{Aut}(\langle \mathbb{R}, \leqq \rangle)$ if f is a map of $\mathbb{R}$ (as a set) on to $\mathbb{R}$ such that $f(r) < f(s)$ whenever $r < s$ ($r, s \in \mathbb{R}$). $\mathrm{Aut}(\langle \mathbb{R}, \leqq \rangle)$ is a lattice ordered group when $f \leqq g$ if and only if $f(r) \leqq g(r)$ for all $r \in \mathbb{R}$ (5). Let ϕ be any one-to-one order-preserving map of $[0, 4]$ on to itself such that $\phi(2) = 1$ and $\phi(3) = 2$. Let

$$a(y) = \begin{cases} y^{\frac{1}{2}} & \text{if } y \in [0, 1] \\ y & \text{otherwise} \end{cases}$$

$b(y) = y - 4$ and

$$c(y) = \begin{cases} y-4 & \text{if } n, n-1 \notin X \text{ and } y \in [4n-2, 4n+2] \\ \phi(y-4n+2)+4n-6 & \text{if } n \in X \text{ and } y \in [4n-2, 4n+2] \\ \phi^{-k_n}(y-4n+2)+4n-6 & \text{if } n \notin X \textit{ and } n-1 \in X \text{ and } y \in [4n-2, 4n+2]. \end{cases}$$

Observe that support $(b^{-n}ab^n) \cap$ support $(c^{-n}ac^n) = \varnothing$ if $n \in X$ and

$$b^{-n}ab^n = c^{-n}ac^n \quad \text{if} \quad n \notin X.$$

Now let K be the l-subgroup of $\mathrm{Aut}(\langle \mathbb{R}, \leqq \rangle)$ generated by a, b and c. Then $b^{-n}ab^n \wedge c^{-n}ac^n$ reduces to the identity in K if and only if $n \in X$, as required. Hence $G(X) = (a, b, c \colon b^{-n}ab^n \wedge c^{-n}ac^n = 1)$ if and only if $n \in X$. Letting X be any infinite subset of the positive integers such that $1 \notin X$, we see that there exist finitely generated lattice ordered groups which cannot be finitely presented.

Now let X be a recursively enumerable subset of the positive integers which is not recursive. If $G(X)$ had a soluble word problem, then there would be a decision procedure to determine whether or not an arbitrary positive integer belongs to X. This is impossible. Consequently:

Theorem. *There exist recursively presented lattice ordered groups with an insoluble word problem.*

The proof of this theorem was inspired by G. Higman's example and paper (4). In (4), it was shown that every recursively presented group can be embedded in a finitely presented group. Such a result for lattice ordered groups would ensure that there exist finitely presented lattice ordered groups with an insoluble word problem. G. Higman's proof used free products of groups with amalgamations. The class of lattice ordered groups does not satisfy the amalgamation property (7) so the corresponding result for lattice ordered groups does not follow from (4); even if it is true, I suspect its proof will be difficult but well worthwhile.

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Bowling Green State University
Bowling Green
Ohio, 43403
U.S.A.

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