# ONE-SIDED IDEALS IN NEAR-RINGS OF TRANSFORMATIONS

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## 1. Introduction

Let $(G, +)$ be an arbitrary group and let $T_0(G) = \{f \in \mathrm{Map}(G, G) : 0f = 0\}$; the system composed of $T_0(G)$ and the operations of pointwise addition and composition of functions form a (left) near-ring. Berman and Silverman, in their investigation of near-rings of transformations [3], found that for every group $G$ the associated near-ring of transformations $T_0(G)$ has no proper ideals. In the present paper left and right ideals of $T_0(G)$ are considered.

D. W. Blackett [4] investigated one-sided ideals in $T_0(G)$ for $G$ finite. Blackett's results appear as consequences of the more general study in this paper. We show that for $G$ finite $T_0(G)$ is the direct sum of minimal right ideals, but this does not hold for $G$ infinite. Minimal one-sided ideals will be characterized completely. Maximal one-sided ideals are considered; every right (left) ideal is contained in a maximal right (left) ideal. A correspondence between normal subgroups of $G$ and left ideals in $T_0(G)$ is given. The center of $(T_0(G), +)$ is found to be a left ideal.

## 2. Preliminaries on near-rings

An algebraic system $(N, +, \cdot)$ is a *(left) near-ring* if

(1) $(N, +)$ is a group, not necessarily abelian,

(2) $(N, \cdot)$ is a semigroup,

(3) $c(a+b) = ca+cb$ for each $a, b, c \in N$.

In this paper it shall further be required that zero is a two-sided annihilator. This is not a consequence of (1)–(3).

$T_0(G)$ serves as a motivational example of a near-ring. Furthermore every near-ring can be embedded in a $T_0(G)$ for some $G$ [5].

A group $G$ together with a mapping $(g, r) \to gr$ of $G \times N$ into $G$ such that for $x \in G$ and $n, m \in N$

(1) $x(n+m) = xn+xm$

(2) $x(nm) = (xn)m$



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is a *near-ring module* over the near-ring $N$. Note, it follows that if $0 \in G$, $a \in N$, then $0a = 0$.

Near-ring homomorphism and $N$-homomorphism are defined as usual. Ideals (submodules) are exactly the kernels of homomorphisms ($N$-homomorphisms). It is well known that an ideal of $N$ is a subset $M$ such that

(1) $(M, +)$ is a normal subgroup of $(N, +)$
(2) $nm \in M$ for each $n \in N, m \in M$
(3) $(n_1+m)n_2-n_1n_2 \in M$ for each $m \in M, n_1, n_2 \in N$.

Subsets satisfying (1) and (2) are *left ideals* of $N$ and subsets satisfying (1) and (3) are right ideals of $N$. A subgroup $B$ of $(N, +)$ is an $N$-subgroup (left $N$-subgroup) if $b \in B$ and $n \in N$ implies $bn \in B(nb \in B)$. Every right ideal is a $T_0(G)$-subgroup, but the converse does not hold.

## 3. Right ideals in $T_0(G)$

Let $S$ be a non-empty subset of $G$ and define $A(S) = \{\alpha \in T_0(G) : S\alpha = 0\}$. These annihilating sets, $A(S)$, are right ideals of $T_0(G)$. For convenience let $P_x = A(G-\{x\}), x \neq 0$. $P_x$ may also be described as $\{\alpha(x, y) : y \in G\}$, where $t\alpha(x, y) = 0$ if $t \neq x$ and $x\alpha(x, y) = y$. The following properties of the $\alpha(x, y)$ follow immediately:

PROPERTY A. *If* $y \neq s \neq 0$, *then* $\alpha(x, y)\alpha(s, t) = \alpha(x, 0) = 0$ *and* $\alpha(x, s)\alpha(s, t) = \alpha(x, t)$.

PROPERTY B. *The set* $\{\alpha(x, x) : x \in G-\{0\}\}$ *is a collection of pairwise orthogonal idempotents.*

PROPERTY C. 1) $\alpha(x, y)+\alpha(x, v) = \alpha(x, y+v)$
2) $-\alpha(x, y) = \alpha(x, -y)$
3) $\alpha(x, y)-\alpha(x, v) = \alpha(x, y-v)$.

PROPERTY D. *For* $\beta \in T_0(G)$, $\alpha(x, y)\beta = \alpha(x, y\beta)$.

For any $\alpha(x, y) \in P_x$ we have

$$\alpha(x, x)\alpha(x, y) = \alpha(x, y)$$

and hence $P_x$ is generated by the idempotent element $\alpha(x, x)$. If $y \neq 0$, then

$$\alpha(x, y)\alpha(y, x) = \alpha(x, x)$$

and hence each $\alpha(x, y)$ generates $P_x$. We have established

THEOREM 1. *For each* $x \neq 0$, $P_x$ *is a minimal right ideal and a minimal* $T_0(G)$-*subgroup.* $P_x$ *is generated by the idempotent* $\alpha(x, x)$ *which acts as a left identity for* $P_x$.

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It is easy to identify the groups $(P_x, +)$. The mapping $g \to \alpha(x, g)$ is a group isomorphism, hence $P_x \cong G$.

Let $x \in G - \{0\}$. Because of Theorem 1 $P_x \cap \sum_y P_y = (0)$, where $y \in G - \{0, x\}$. Thus the sum of the $P_x$ is a (group) direct sum. For convenience denote this direct sum by $P$.

THEOREM 2. *$P$ is a right ideal of $T_0(G)$.*
This theorem is a consequence of

LEMMA 0. *If $F$ is a family of right ideals from a near-ring $N$, then $\Sigma M$, $M \in F$, is a right ideal of $N$.*

PROOF. It is well known that $\Sigma M$ is a normal subgroup of $(N, +)$. Let $m = \sum_{i=1}^{k} m_i$, where $m_i \in M_i \in F$, be any element in $\Sigma M$. The proof that

$$(n_1 + m)n_2 - n_1 n_2 \in \Sigma M,$$

for each $n_1, n_2 \in N$, is by induction on $k$. The case $k = 1$ being obvious we proceed to the inductive step. Let $c_j = \sum_{i=1}^{j} m_i$. Note that

$$((n_1 + c_{k-1}) + m_k)n_2 - (n_1 + c_{k-1})n_2 \in M_k$$

and that $(n_1 + c_{k-1})n_2 - n_1 n_2 \in \Sigma M$ by the induction hypothesis. So $(n_1 + m)n_2 - n_1 n_2$ is in $\Sigma M$.

Note that if $G$ is finite, say $|G| = n$, then $T_0(G)$ and $P$ have the same cardinality, $n^{n-1}$. A consequence of these remarks is the following theorem originally arrived at by Blackett [4, p. 39] in a different fashion.

THEOREM 3. *If $G$ is finite, then $T_0(G)$ is the direct sum of the minimal right ideals.*

This theorem is not true for $G$ infinite. This can be seen in a variety of ways, one of the simpler being to examine cardinality: $|T_0(G)| = |G|^{|G|}$, while $|P| = |G|$.

The following four lemmas lead a complete classification of minimal right ideals in $T_0(G)$. In the following $M$ is a non-zero right ideal of $T_0(G)$.

LEMMA 1. *For each $x \in G$, there exists $\beta \in M$, $t \in G$, such that $t\beta = x$.*

LEMMA 2. *There exists non-zero $p \in M$, $t \in G$ such that $tp = 0$.*

PROOF. Let $\operatorname{rng} \beta = \{x\beta : x \in G\}$. We consider two cases.

CASE 1. There exists $\beta \in M$ such that $|\operatorname{rng} \beta| > 2$. Then there exists non-zero $x, t \in G$, $x \neq t$, such that

$$t\beta = y \neq z = x\beta \neq 0.$$

Choose $\gamma_1, \gamma_2 \in T_0(G)$ so that

$$(t)\gamma_1 = -y, \quad (x)\gamma_1 = -z, \quad (-z)\gamma_2 = -z, \quad (-y)\gamma_2 = 0.$$

Let $p = (\gamma_1 + \beta)\gamma_2 - \gamma_1 \gamma_2$. Then $xp = z \neq 0$ and $tp = 0$.

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Before turning to the case where $|\mathrm{rng}\,\beta| = 2$ we dispose of the special situation $|G| < 4$. Since $M$ is non-zero, $|G| = 1$ is not under consideration. For $|G| = 2$, $T_0(G)$ itself is the only minimal right ideal. Finally for $|G| = 3$ the minimal right ideals are:

$$\{(0,0,0),\ (0,0,1),\ (0,0,2)\} \text{ and } \{(0,0,0),\ (0,1,0),\ (0,2,0)\},$$

where $(x_0, x_1, x_2)$ represents the function on $C_3$ that takes $i$ into $x_i$. In each of these two cases the lemma is easily seen to be satisfied. For the remainder of the proof take $|G| \geqq 4$.

CASE 2. For each non-zero $\beta \in M$, $|\mathrm{rng}\,\beta| = 2$. Let $\beta$ be a non-zero element of $M$. Then there is a non-zero element $x$ of $G$ such that $(x)\beta = y \neq 0$. Let $t$ be a second non-zero element of $G$. Choose $\gamma_1, \gamma_2 \in T_0(G)$ such that

$$(x)\gamma_1 = -y,\ (t)\gamma_1 = z \text{ where } z \text{ is not } 0,\ -y, \text{ nor } -y-y,$$
$$(-y)\gamma_2 = -y,\ z\gamma_2 = 0, \text{ and } (z+y)\gamma_2 = (z+t\beta)\gamma_2 = 0.$$

Then $(\gamma_1+\beta)\gamma_2 - \gamma_1\gamma_2$ is the desired mapping $p$.

LEMMA 3. *If $M$ is a minimal right ideal, then $G = G_1 \cup G_2$, where*

$$G_1 \cap G_2 = \emptyset,\ \ 0 \in G_1,\ G_2 \neq \emptyset,\ \ G_1 M = \{0\},$$

*and for each $x \in G_2$, $x\beta \neq 0$ for each non-zero $\beta \in M$.*

PROOF. There exists non-zero $x \in G$, $\beta \in M$ such that $x\beta = 0$. The set $\{\gamma \in M : x\gamma = 0\}$ is a non-zero right ideal of $T_0(G)$ and hence is equal to $M$. Let

$$G_1 = \{t \in G : tM = \{0\}\} \text{ and } G_2 = G - G_1.$$

It is easily seen that $G_1$ and $G_2$ are the required sets.

LEMMA 4. *If $M$ is a minimal right ideal, then $|\mathrm{rng}\,\beta| = 2$ for every non-zero $\beta \in M$. Hence $G_2$ is a singleton.*

PROOF. Suppose $\beta \in M$ and $0, y_1$, and $y_2$ are distinct elements of the range of $\beta$. Choose $\gamma \in T_0(G)$ so that $t\gamma = 0$ if $t \neq y_2$, $y_2\gamma = y_2$. For some $x_1 \in G$, $x_1\beta = y_1$, hence $x_1\beta\gamma = 0$. Since there exists $x_2 \in G$ such that $x_2\beta = y_2$ and therefore $x_2\beta\gamma = y_2$ we have that $\beta\gamma$ is a non-zero element of $M$ that takes $x_1$ into zero, a contradiction to $x_1\beta \neq 0$ and Lemma 3.

In the proof of Lemma 2 the case $|G| < 4$ was completely investigated so we may take $|G| \geqq 4$.

Suppose $|G_2| > 1$. Then there exists distinct elements $x, t \in G_2$. Choose a non-zero mapping $f_c \in M$ and let $\mathrm{rng} f_c = \{0, c\}$. Choose $\gamma_1, \gamma_2 \in T_0(G)$ such that $x\gamma_1 = -c$, $t\gamma_1 = b \neq c$, where $b$ is not $0$ or $-c-c$, $(-c)\gamma_2 = -c$, $b\gamma_2 = 0$. Note that $b+c$ is not $b$, $c$, or $-c$ so we can define $(b+c)\gamma_2 = 0$. Then

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$$(x)[(\gamma_1+f_c)\gamma_2-\gamma_1\gamma_2] = c \text{ and } (t)[(\gamma_1+f_c)\gamma_2-\gamma_1\gamma_2] = 0.$$

But $(\gamma_1+f_c)\gamma_2-\gamma_1\gamma_2 \in M$ so $t \in G_1$, a contradiction.

Thus every element in a minimal right ideal is of the form $\alpha(x, y)$ for some fixed non-zero $x$. We thus have

THEOREM 4. *The only minimal right ideals of $T_0(G)$ are the $P_x$.*

A submodule $N$ of the $R$-module $M_R$ is called a simple submodule of $M_R$ if it contains no proper (nontrivial) submodules. $P$ is a $T_0(G)$-module and because of Theorem 1, $P_x$ is a simple submodule of $P$. The following lemma enables us to characterize all right ideals of $T_0(G)$ which are contained in $P$.

LEMMA 5. (Beidleman [1, p. 60]). *If $M_R$ is a near-ring module, then the following are equivalent:*

(1) *Every submodule is a sum of simple submodules of $M_R$,*

(2) *$M_R$ is a sum of simple submodules,*

(3) *$M_R$ is a direct sum of simple submodules,*

(4) *Every submodule of $M_R$ is a direct summand.*

THEOREM 5. *Every right ideal of $T_0(G)$ which is also contained in $P$ is of the form*

$$\sum_x \oplus P_x, \text{ where } x \in H \subseteq G-\{0\}.$$

The proof follows from Theorems 1 and 4, Lemma 5, and

$$P = \sum \oplus P_x, x \in G-\{0\}.$$

The above are 'internal' facts about $P$. The following are "external" facts.

$P$ is contained in a maximal right ideal. In fact a standard Zorn's Lemma argument shows that every right ideal is contained in a maximal right ideal. What maximal right ideals are available? For any non-zero $x \in G$ let $A(x) = A(\{x\})$.

THEOREM 6. *Every $A(x), x \neq 0$, is a maximal right ideal and a maximal $T_0(G)$-subgroup.*

PROOF. Since every right ideal of $T_0(G)$ is also a $T_0(G)$-subgroup, it suffices to show that $A(x)$ is a maximal $T_0(G)$-subgroup of $T_0(G)$. Let $B$ be a $T_0(G)$-subgroup of $T_0(G)$ which properly contains $A(x)$. Choose $\beta \in B$ such that $(x)\beta = b \neq 0$. Take $\eta \in T_0(G)$ such that $(b)\eta = x$ and $(t)\eta = 0$ for each $t \neq b$. Then $\beta\eta \in B$ and $(x)\beta\eta = x$. Hence, we can assume that $x\beta = x$. Choose $\Phi \in A(x)$ such that $(t)\Phi = -(t)\beta+t$, if $t \neq x$. Then $(t)[\beta+\Phi] = t$ if $t \neq x$ and $(x)[\beta+\Phi] = x$, hence $\beta+\Phi$ is the identity mapping on $G$. Since $\beta+\Phi-\beta \in A(x)$, it follows that $\beta+\Phi \in B$. This shows that $B = T_0(G)$ and the theorem follows.

Beidleman [2] defines the *radical* $J(M)$ of a near-ring module $M_N$ as the intersection of all submodules which are maximal as $N$-subgroups. As a consequence

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of Theorem 6, $J(T_0(G)) = 0$. Note that every right ideal of $T_0(G)$ is a $T_0(G)$-submodule and conversely.

It is easy to see that $P$ is not contained in any $A(x)$. In fact $P_x \cap A(x) = (0)$. By Theorem 6 it follows that $T_0(G) = P_x \oplus A(x)$. Note that by Theorem 5 $A(x) \cap P = \Sigma \oplus P_t$, $t \neq 0$, $x$ and hence (by the 2nd isomorphism theorem for near-ring modules)

$$T_0(G)/P = P + A(x)/P \cong A(x)/P \cap A(x).$$

There is an ascending sequence of right ideals containing $P$, however. Let

$$T_\lambda = \{f \in T_0(G) : |\text{support} f| < \aleph_\lambda\},$$

for each ordinal $\lambda$. Here support $f = \{x \in G : xf \neq 0\}$. Each $T_\lambda$ is a right ideal and since $P$ is exactly those functions of finite support we have $P = T_0$. Also for $\aleph_\tau = |G|$

$$P = T_0 \subset T_1 \subset \cdots \subset T_\tau \subset T_0(G),$$

where $T_\tau$ is contained in a maximal right ideal; the question arises: *is $T_\tau$ maximal*?

The right ideals $A_n = A(\{x_1, \cdots, x_n\})$, where the $x_n$ are distinct non-zero elements of $G$, gives rise to an infinite descending chain of right ideals in the case where $G$ is infinite. So $T_0(G)$ is not right Artinian for $G$ infinite. The question of whether $T_0(G)$ is right Noetherian is open, even for $G$ countable.

The following general lemma will prove useful in the next theorem.

LEMMA 6. (Beidleman [1, p. 54]). *If the $N$-module $M_N$ is the direct sum of submodules $M_\lambda$, $\lambda \in \Lambda$, then for each $m = m_{\lambda_1} + \cdots + m_{\lambda_n}$ in $M$, where $m_{\lambda_i} \in M_{\lambda_i}$, and for each $r \in N$,*

$$mr = m_{\lambda_1} r + \cdots + m_{\lambda_n} r.$$

THEOREM 7. *If there exists a right ideal $V$ of $T_0(G)$ such that $P \oplus V = T_0(G)$, then $G$ is finite and $V = (0)$, i.e. $P$ cannot be a proper direct summand.*

PROOF. If $G$ is finite, then $P = T_0(G)$ because of Theorems 3 and 4. Hence, assume $G$ is infinite and $V \neq (0)$. Then

$$1 = \alpha(x_1, y_1) + \cdots + \alpha(x_n, y_n) + \rho, \ \rho \in V.$$

This yields $-y_i + x_i = x_i\rho$ for $i = 1, 2, \cdots, n$ and for $t \neq x_i$, $t = t\rho$. If for some $j$, $y_j \neq x_j$, then we arrive at a contradiction as follows. Choose $\beta_x \in T_0(G)$ such that $t\beta_x = x$, if $t \neq 0$, where $x \notin \{x_1, \cdots, x_n, 0\}$. Then using Lemma 6, and property $D$ we obtain

$$\beta_x = 1\beta_x = \sum_{i=1}^{n} \alpha(x_i, x) + \rho\beta_x$$

and $x = x_j\beta_x = x + (-y_j + x_j)\beta_x = x + x$, or $x = 0$.

Hence

$$1 = \alpha(x_1, x_1) + \cdots + \alpha(x_n, x_n) + \rho;$$

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so again using Lemma 6 we have $\alpha(x, x) = \rho\alpha(x, x)$ for each $x \neq x_i$, $i = 1, \cdots, n$. But $\rho\alpha(x, x) \in V$, so $0 \neq \rho\alpha(x, x) \in P \cap V$, contrary to $P \cap V = (0)$.

## 4. Left ideals

Let $C^+$ be the center of $(T_0(G), +)$ and let $C(G)$ be the center of $G$.

LEMMA 7. *An element $f$ is in $C^+$ if and only if the image of $f$, written* rng $f$, *is in $C(G)$.*

PROOF. Let $f \in C^+$. Consider $(x)f$, for any non-zero $x \in G$. For each $y \in G$ there exists $h \in T_0(G)$ such that $(x)h = y$. So

$$(x)f + y = (x)[f+h] = (x)[h+f] = y + (x)f$$

and hence $(x)f \in C(G)$.

The converse follows immediately.

LEMMA 8. *$C^+$ is a left ideal of $T_0(G)$.*

PROOF. Of course $C^+$ is a normal subgroup of $(T_0(G), +)$. Since for each $g \in T_0(G)$, $f \in C^+$:

$$\operatorname{rng} gf \subseteq \operatorname{rng} f \subseteq C(G),$$

it follows by Lemma 7 that $gf \in C^+$.

NOTE. For any element $c \in C(G)$ the mapping $f_c \in T_0(G)$ defined by $xf_c = c$, for $x \neq 0$, has its image in $C(G)$ and hence must be in $C^+$. So if $G$ has a non-trivial center then $C^+ \neq (0)$; moreover $(T_0(G), +)$ is centerless if and only if $G$ is centerless.

Next left ideals of $T_0(G)$ will be classified in terms of normal subgroups of $G$. Let $L$ be a left $T_0(G)$-subgroup of $T_0(G)$. Define $GL = \{xl | x \in G, l \in L\}$.

LEMMA 9. (a) *If $L$ is a left $T_0(G)$-subgroup of $T_0(G)$, then $GL$ is a subgroup of $G$.*

(b) *If $L$ is a left ideal of $T_0(G)$, then $GL$ is a normal subgroup of $G$.*

PROOF. (a) Let $x_i l_i \in GL$, $i = 1, 2$. Then there exists $f \in T_0(G)$ such that $x_2 f = x_1$. Thus

$$(x_1)l_1 - (x_2)l_2 = (x_2)fl_1 - (x_2)l_2 = (x_2)[fl_1 - l_2].$$

Since $fl_1 \in L$ it follows that $fl_1 - l_2 \in L$, hence $(x_1)l_1 - (x_2)l_2 \in GL$. This shows $GL$ is a subgroup of $G$.

(b) Let $(x)l \in GL$ and $g \in G$. Consider $g + (x)l - g$. There exists $h \in T_0(G)$ such that $(x)h = g$, hence

$$g + (x)l - g = (x)h + (x)l - (x)h = (x)[h + l - h].$$

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Now $h+l-h \in L$, hence $GL$ is a normal subgroup of $G$.

To each left ideal $L$ there corresponds a normal subgroup $GL$. This correspondence need not be one-to-one. Note that $GL = (0)$ if and only if $L = (0)$ and $GT_0(G) = G$. It is possible for $GL = G$ and yet $L$ be a proper left ideal, as the following example illustrates.

EXAMPLE. Let $G$ be infinite and define

$$B_\lambda = \{f \in T_0(G) : |\mathrm{rng} f| < \aleph_\lambda\}.$$

For each ordinal $\lambda$, $B_\lambda$ is a left $T_0(G)$-subgroup. Consider $G$ abelian and $\aleph_\lambda \leqq |G|$; then $B_\lambda$ is a proper left ideal of $T_0(G)$ and satisfies $GB_\lambda = G$. The latter is clear since for each non-zero $a \in G$ the function $xf_a = 0$ if $x \neq a$, $af_a = a$ is in $B_\lambda$. For $\aleph_\alpha = |G|$ we have the proper ascending chain of left ideals.

$$(0) \subset B_0 \subset B_1 \subset \cdots \subset B_\alpha \subset T_0(G).$$

For any group $G$ if $z \in GL$, then for each non-zero $x \in G$, $\alpha(x, z) \in L$. Some consequences and their immediate corollaries are given in the following

THEOREM 8. *Let $L$ be a left $T_0(G)$-subgroup.*

(1) *If $GL = G$, then $P \subseteq L$,*

(2) *If $GL = G$ and $G$ is finite, then $L = T_0(G)$,*

(3) *If $G$ is a finite simple group, then $T_0(G)$ has no proper left ideals,*

(4) *If $L$ is also a right $T_0(G)$-subgroup, then $P \subseteq L$ and if in addition $G$ is finite, then $L = T_0(G)$,*

(5) *If $G$ is finite, then $T_0(G)$ is a simple near-ring,*

(6) *If $G$ is a simple group, then $P$ is contained in every non-zero left ideal of $T_0(G)$.*

To each normal subgroup $N$ of $G$ there corresponds a left ideal of $T_0(G)$. Consider

$$L_N = \{l \in T_0(G) : Gl \subseteq N\}.$$

LEMMA 10. *$L_N$ is a left ideal of $T_0(G)$.*

LEMMA 11. *If $M$ and $N$ are normal subgroups of $G$ and $M \subseteq N$, then $L_M \subseteq L_N$. If $L_1$ and $L_2$ are left ideals of $T_0(G)$ and $L_1 \subseteq L_2$, then $GL_1 \subseteq GL_2$.*

The proof of each of these lemmas follows directly from the definitions.

The inter-relationship between the normal subgroups given by Lemma 9 and the left ideals given by Lemma 10 is investigated next.

Starting with a left ideal $L$ we pass to a normal subgroup $GL$ and then a left ideal $L_{GL}$. We have $L \subseteq L_{GL}$. Again using Lemma 9 we obtain the normal subgroups $GL$ and $GL_{GL}$ where $GL \subseteq GL_{GL}$ by Lemma 11. But $GL_{GL} \subseteq GL$ by definition so $GL = GL_{GL}$.

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Next start with any normal subgroup $N$ of $G$, then pass to the left ideal $L_N$ and the normal subgroup $GL_N$. As in the previous paragraph, but using $L_N$ as the left ideal, this yields $L_N \subseteq L_M$, where $M = GL_N$. But $M = GL_N \subseteq N$, so $L_M \subseteq L_N$ and hence $L_N = L_M$.

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