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ON UNITARY POLARITIES OF FINITE PROJECTIVE PLANES

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1. Introduction. A unitary polarity of a finite projective plane $\mathscr{P}$ of order q^2 is a polarity θ having $q^3 + 1$ absolute points and such that each non-absolute line contains precisely $q + 1$ absolute points. Let $G(\theta)$ be the group of collineations of $\mathscr{P}$ centralizing θ. In [**15**] and [**16**], A. Hoffer considered restrictions on $G(\theta)$ which force $\mathscr{P}$ to be desarguesian. The present paper is a continuation of Hoffer's work. The following are our main results.

THEOREM I. *Let θ be a unitary polarity of a finite projective plane $\mathscr{P}$ of order q^2. Suppose that Γ is a subgroup of $G(\theta)$ transitive on the pairs x, X, with x an absolute point and X a nonabsolute line containing x. Then $\mathscr{P}$ is desarguesian and Γ contains* PSU$(3, q)$.

THEOREM II. *Let θ be a unitary polarity of a finite projective plane $\mathscr{P}$ of order q^2. Suppose that there are at least three non-collinear absolute points x such that $G(\theta)$ contains q (x, x^θ)-elations. Then $\mathscr{P}$ is desarguesian.*

Our definition of a unitary polarity shows that the absolute points and nonabsolute lines form a $2 - (q^3 + 1, q + 1, 1)$ design $\mathscr{U}(\theta)$ [**7**, p. 155]. Theorem I then implies that $\mathscr{P}$ is desarguesian if $G(\theta)$ is flag-transitive on $\mathscr{U}(\theta)$. Theorem II is intended to parallel part of the Lenz-Barlotti classification of projective planes (see [**7**]). This result was proved independently by Hoffer, except when q is odd and $G(\theta)$ contains Baer involutions. See § 6 for further remarks in the direction of a Lenz-Barlotti type of classification of finite planes having a unitary polarity.

The proofs of these theorems depend heavily on recent classification theorems concerning finite simple groups and permutation groups [**1**; **2**; **4**; **5**; **11**; **13**; **14**; **18**; **20**; **27**; **28**; **29**]. For the most part, the proofs use standard ideas. The only interesting feature is the use of methods and results involving transfer and fusion in order to handle Baer involutions in planes of odd order. A great deal of control over Baer involutions is provided by a result of Seib [**26**]. Our arguments are, however, frequently too group theoretic, and more geometric methods would be desirable.

We will use standard geometric and group theoretic notation. If Δ is a subgroup of a finite group, $C(\Delta)$ and $N(\Delta)$ are its centralizer and normalizer, Δ' its commutator subgroup, and $O(\Delta)$ its largest normal subgroup of odd

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order. If $\alpha, \beta \in \Delta$ then $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$ and $\alpha^\beta = \beta^{-1}\alpha\beta$. If Δ acts on a set S of points, then Δ^S is the induced permutation group, while, if $|S| > 1$, $\Delta(S)$ denotes the pointwise stabilizer of S; if T is a subset of S, Δ_T is the global stabilizer of T. In particular, if Δ acts on a projective plane $\mathscr{P}$ and L is a line, $\Delta(L)$ is the group of perspectivities with axis L. If x is a point, $\Delta(x)$ denotes the linewise stabilizer of x. Thus, if also Δ centralizes a polarity θ of $\mathscr{P}$, then $\Delta(L)$ consists of (L^θ, L)-perspectivities, while $\Delta(x)$ consists of (x, x^θ)-perspectivities. If $\delta \in \Delta$ is planar, then $\mathscr{P}_\delta$ will denote the subplane consisting of its fixed points and lines. If S is a set of points of $\mathscr{P}$, then $\mathscr{P}_\delta \cap S$ is the set of points of S in $\mathscr{P}_\delta$.

For all the relevant geometric definitions, see [**7**], especially §§ 3.1 and 3.3.

2. Preliminary lemmas.

Let $\mathscr{P}$ be a projective plane of odd order q.

LEMMA (2.1). (Ostrom [**23**]; Lüneburg [**19**].) *Let σ be an involutory (x, X)-homology and τ an involutory (y, Y)-homology. If $X \neq Y$ and $\sigma\tau = \tau\sigma$ then*

(i) *$\sigma\tau$ is an involutory $(X \cap Y, xy)$-homology; and*

(ii) *σ is the only involutory (x, X)-homology.*

LEMMA (2.2). *There is no abelian collineation group of order 8 generated by three involutory homologies not all having the same axis.*

Proof. Otherwise, let ρ, σ, τ be the given homologies and $\Gamma = \langle \rho, \sigma, \tau \rangle$. We may assume that $\sigma \in \Gamma(x, X)$ and that $\tau \in \Gamma(y, Y)$, $X \neq Y$. Then ρ fixes $x, y, X \cap Y, X, Y$, and xy, so x, y or $X \cap Y$ is the centre and X, Y or xy the axis of ρ. This contradicts (2.1).

LEMMA (2.3). *Let B be an oval and $x, y \in B$, $x \neq y$. Then there is at most one nontrivial homology preserving B fixing x and y.*

Proof. Such a homology σ fixes the tangents X and Y to B at x and y, respectively. Then σ fixes $X \cap Y$, so σ is an $(X \cap Y, xy)$-homology. If $z \in B - \{x, y\}$, the group of $(X \cap Y, xy)$-homologies preserving B moves z to at most two points of B, proving the result.

LEMMA (2.4). *Let B be an oval and Γ be a collineation group of $\mathscr{P}$ preserving B and 2-transitive on B. If Γ contains an involutory homology, then $\mathscr{P}$ is desarguesian, B is a conic, and Γ has a normal subgroup acting faithfully on B as* PSL$(2, q)$ *in its usual representation.*

Proof. We may assume that no proper normal subgroup of Γ meets the stated requirements. By a result of Lüneburg [**7**, p. 186], it suffices to show that Γ acts on B as PSL$(2, q)$.

If no involution in Γ fixes a point of B, then Γ is either PSL$(2, q)$ or $q > 3$ and Γ has a normal elementary abelian subgroup of order $q + 1$ transitive on B [**4**]. By (2.2), only the former possibility can occur.

If no involution in Γ fixes more than two points, but some involution fixes

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two points of B, then Γ is $\mathrm{PSL}(2, q)$ or $q = 5$ and Γ is A_6 [**13**]. However, in the latter case Γ contains $\mathrm{PSL}(2, 5)$, so $\mathscr{P}$ is desarguesian and B is a conic [**7**, p. 186], and then A_6 cannot preserve B.

Finally, suppose that some involution α fixes more than two points of B. Then α is a Baer involution, so $q \equiv 1 \pmod 4$. Since an involutory homology fixing no points of B is an odd permutation of B, by (2.1) there is an involutory homology σ fixing two points $x, y \in B$, and by (2.2), no conjugate $\neq \sigma$ of σ fixes x and y. Once again, it follows that Γ has a normal subgroup containing σ acting on B as $\mathrm{PSL}(2, q)$ [**18**, Theorem B].

LEMMA (2.5). *Let Γ be a collineation group of $\mathscr{P}$ such that the following conditions hold:*

(a) *Γ contains involutory homologies having different centres and also involutory homologies having different axes;*

(b) *All involutions in Γ are homologies;*

(c) *Γ has no normal subgroup of index 2; and*

(d) $Z(\Gamma/O(\Gamma)) = 1$.

Then $\Gamma/O(\Gamma)$ is isomorphic to one of the following:

(i) *a subgroup of* $\mathrm{P\Gamma L}(2, m)$ *containing* $\mathrm{PSL}(2, m)$, *for some odd m;*

(ii) *a subgroup of* $\mathrm{P\Gamma L}(3, m)$ *containing* $\mathrm{PSL}(3, m)$, *for some odd m;*

(iii) *a subgroup of* $\mathrm{P\Gamma U}(3, m)$ *containing* $\mathrm{PSU}(3, m)$, *for some odd m;*

(iv) A_7;

(v) M_{11};

(vi) $\mathrm{PSU}(3, 4)$; *or*

(vii) *a subgroup of* $\mathrm{P\Gamma L}(2, 2^e)$, $\mathrm{Aut}\,\mathrm{Sz}(2^e)$, *or* $\mathrm{P\Gamma U}(3, 2^e)$ *containing* $\mathrm{PSL}(2, 2^e)$, $\mathrm{Sz}(2^e)$, *or* $\mathrm{PSU}(3, 2^e)$, *respectively, where* $2^e \geqq 8$. *Moreover, in this case, commuting involutions have the same centre and axis.*

Proof. Suppose that Γ contains no elementary abelian subgroup of order 8. By [**2**], one of (i)–(vi) must hold.

Now suppose that Γ has an elementary abelian subgroup Σ of order 8. By (2.2), $\Sigma \leqq \Gamma(x, X)$ for some x, X. Let $1 \neq \sigma \in \Sigma$. Set $S = x^\Gamma$.

We claim that $|\Gamma_{xy}|$ is odd if $y \in S - \{x\}$. For if $\tau \in \Gamma_{xy}$ is an involution, it centralizes some involution σ' having centre x and some involution σ'' having centre y. We may assume that x is not the centre of τ, replacing y by x if necessary. Then (2.1ii) implies that σ' is the only (x, X')-involution, where X' is the axis of σ'. If $X' = X$, this contradicts our choice of σ. If $X' \neq X$, there is a unique (x, X)-involution [**7**, p. 120], which is again a contradiction.

It follows that $|\Gamma(S)|$ is odd and that Γ^S is one of the groups in (vii) [**5**], except that 2^e might be 4. However, $\mathrm{PSL}(2, 4) \approx \mathrm{PSL}(2, 5)$ appears in (i), while $\mathrm{PSU}(3, 4)$ appears in (vi).

3. Unitary polarities. Let θ be a unitary polarity of a projective plane $\mathscr{P}$ of order q^2. Its set of absolute points is denoted A.

LEMMA (3.1). (Seib [**26**].) *Let α be a Baer involution in $G(\theta)$. Then θ induces an orthogonal polarity on $\mathscr{P}_\alpha$. In particular,*

(i) *if q is even, then $\mathscr{P}_\alpha \cap A = L \cap A$, for some nonabsolute line L; and*

(ii) *if q is odd, then $\mathscr{P}_\alpha \cap A$ is an oval in $\mathscr{P}_\alpha$, and α fixes some nonabsolute line meeting $\mathscr{P}_\alpha \cap A$ and some nonabsolute line not meeting $\mathscr{P}_\alpha \cap A$.*

LEMMA (3.2). *If $\alpha \in G(\theta)$ is a Baer involution and $q \equiv 3 \pmod 4$, then α induces an odd permutation on the set of nonabsolute lines.*

Proof. By (3.1), α fixes q^2 of the $q^2(q^2 - q + 1)$ nonabsolute lines, and $q^2(q^2 - q + 1) - q^2 \equiv 2 \pmod 4$.

LEMMA (3.3). *If q is odd and $\alpha \in G(\theta)$ is a Baer involution, then $\langle\alpha\rangle$ is the pointwise stabilizer Φ of $\mathscr{P}_\alpha \cap A$ in $G(\theta)$.*

Proof. By (3.1), Φ fixes $\mathscr{P}_\alpha$ pointwise and fixes some nonabsolute lines L and L' such that $\mathscr{P}_\alpha \cap A \cap L \neq \emptyset = \mathscr{P}_\alpha \cap A \cap L'$. Since Φ acts regularly on $L - (\mathscr{P}_\alpha \cap A \cap L)$ and $L' - (\mathscr{P}_\alpha \cap A \cap L')$, it follows that $|\Phi| \mid (q - 1, q + 1) = 2$.

LEMMA (3.4). *If q is even, $x \in A$, and there is a Baer involution $\alpha \in G(\theta)$ fixing x, then all involutory elations in $G(\theta)$ with centre x commute.*

Proof. Choose L as in (3.1i). Let σ and τ be involutory (x, x^θ)-elations in $G(\theta)$. Then $[\alpha, \sigma]$ is an elation fixing $L \cap A$ pointwise, so $\alpha\sigma$ is a Baer involution. Similarly, $[\alpha\sigma, \tau] = 1$. Thus, $\alpha\sigma\tau = \tau\alpha\sigma = \alpha\tau\sigma$.

LEMMA (3.5). *If q is even, $\Gamma \leqq G(\theta)$ is transitive on A and Γ contains an involutory elation σ, then either*

(i) *$O(\Gamma)\langle\sigma\rangle \trianglelefteq \Gamma$ and $O(\Gamma)$ is transitive on A; or*

(ii) *$\mathscr{P}$ is desarguesian and, if $q > 2$, Γ contains* PSU$(3, q)$.

Proof. Suppose that (i) does not hold and that Γ has no proper normal subgroup satisfying the conditions on Γ. If Γ contains a Baer involution then, by (3.4), for each $x \in A$ there is a nontrivial normal 2-subgroup of $\Gamma(x)$. By a theorem of Shult [**27**], Γ has a normal subgroup satisfying our conditions but containing no Baer involution.

Thus, Γ has no Baer involutions. By a theorem of Bender [**5**], Γ acts on A as PSL$(2, q^3)$, Sz$(q^{3/2})$, or PSU$(3, q)$ in its usual representation. In the first two cases, we have $|\Gamma(x)| \geqq q^3$ or $q^{3/2}$, respectively, which is absurd. In the last case, (ii) holds by a result of Hoffer [**16**].

We note that if $q = 2$, there is a group of order 18 meeting the requirements of (3.5).

LEMMA (3.6). *Suppose that q is even, $\Gamma \leqq G(\theta)$ is transitive on A, and Γ has no nontrivial normal subgroup intransitive on A. If $O(\Gamma) \neq 1$, then $q = 2$.*

Proof. By the Feit-Thompson Theorem [**9**], $q^3 + 1 = p^e$ for some prime p. Thus, $q + 1 = p = 3$.

LEMMA (3.7). *Let A_0 be a subset of A containing at least three non-collinear points. Suppose that whenever $x, y \in A$, $x \neq y$, we have $xy \cap A \subseteq A_0$. Then $A_0 = A$.*

Proof Let $z \in A - A_0$. Each of the nonabsolute lines through z contains at most one point of A_0. Thus, $q^2 \geqq |A_0|$. However, A_0 is the set of points of a design with $k = q + 1$ and $\lambda = 1$. Thus, $|A_0| \geqq 1 + k(k - 1) = q^2 + q + 1$, which is a contradiction.

We conclude this section with a general result concerning designs with $\lambda = 1$.

LEMMA (3.8). *Let $\mathscr{D}$ be a design with $\lambda = 1$ and Γ an automorphism group transitive on non-incident point-line pairs. Then Γ is 2-transitive on points.*

Proof. [**24**; **17**]. For each point x, Γ_x is transitive on the blocks not on x. Let Γ_x have t orbits of points $\neq x$ and t' orbits of blocks on x. As each point-orbit determines such a block-orbit, we have $t' \leqq t$. Also, $t + 1 \leqq t' + 1$, by [**7**, p. 78]. Thus, if $x \in X$ then Γ_{xX} is transitive on $X - \{x\}$. Now Γ_X is 2-transitive on X, and the block-transitivity of Γ yields (3.8).

4. Preliminaries for Theorem I.

Let $\mathscr{P}$, q, θ and Γ be as in Theorem I. We may assume that Γ has no proper normal subgroup flag-transitive on $\mathscr{U}(\theta)$. By a result of Higman and McLaughlin [**7**, pp. 79–80], Γ is primitive on the set A of absolute points.

Let $x, y \in A$, $x \neq y$, so $L = xy \notin A^\theta$. Let Σ be a Sylow 2-subgroup of Γ.

LEMMA (4.1). *If q is even, then Theorem I holds.*

Proof. We may assume that Σ fixes x. If a central involution σ of Σ is an elation, the result follows from (3.5) and (3.6). Suppose that σ is a Baer involution and that $\mathscr{P}_\sigma \cap A = L \cap A$ (see (3.1)). Then Σ fixes L. Since Γ_x is transitive on the q^2 nonabsolute lines through x, this is impossible.

We may now assume in §§ 4 and 5 that q is odd. From now on, we will also assume that Theorem I is false for our $\mathscr{P}$ and Γ.

LEMMA (4.2). *$O(\Gamma) = 1$, Γ has no central involution, and Γ has no normal subgroup of index 2.*

Proof. Since $|A|$ is even and Γ is primitive on A, the first statement holds. If $\Gamma \triangleright \Gamma^*$ and $|\Gamma : \Gamma^*| = 2$, then Γ^* is transitive on A. Also, $\Gamma_x{}^*$ is transitive on the q^2 absolute lines on x, since q^2 is odd. This contradicts our minimal choice of Γ.

LEMMA (4.3). *Suppose that Γ contains an involutory homology. Then Γ is generated by its involutory homologies. There are $q^2(q^2 - q + 1)$ involutory homologies, and they are conjugate. If σ is an involutory homology with axis L, then $\sigma \in Z(\Gamma(L))$ and $C(\sigma) - \langle\sigma\rangle$ contains $q^2 - q$ involutory homologies.*

Proof. If Γ^* is the subgroup generated by the involutory homologies of Γ, then Gleason's lemma [**7**, p. 191] implies that Γ_x^* is transitive on the non-absolute lines on x. Since Γ^* is transitive on A, we must have $\Gamma^* = \Gamma$ by the minimality of Γ.

By (2.1), $\sigma \in Z(\Gamma(L))$ and all involutory homologies are conjugate. There are $q^2(q^2 - q + 1)$ conjugates. If $z \in A - A \cap L$, then σ centralizes the unique involution in $\Gamma(zz^\sigma)$. Thus, σ centralizes $(q^3 - q)/(q + 1) = q^2 - q$ involutory homologies other than σ.

LEMMA (4.4). Γ *contains Baer involutions.*

Proof. Suppose that all involutions are homologies. By (2.2), Γ has no elementary abelian subgroup of order 8. By (4.2) and (2.5), Γ is isomorphic to one of the following groups: $\mathrm{PSL}(2, m)$, A_7, $\mathrm{PSL}(3, m)$, $\mathrm{PSU}(3, m)$, M_{11}, or $\mathrm{PSU}(3, 4)$; here, m is an odd prime power. By (4.3), the ordered pair $(q^2(q^2 - q + 1), q^2 - q)$ must be $(m(m + 1)/2, \ (m \mp 1)/2)$, $(15, 8)$, $(m^2(m^2 + m + 1), m^2 + m)$, $(m^2(m^2 - m + 1), m^2 - m)$, $(11 \cdot 5 \cdot 3, 12)$, or $(65 \cdot 3, 2)$. It is immediate that Γ is $\mathrm{PSU}(3, q)$, and the lemma follows from a result of Hoffer [**16**].

LEMMA (4.5). $q \equiv 1 \pmod 4$, *and we may assume that* $\Sigma \leqq \Gamma_{\{x,y\}} \leqq \Gamma_L$.

Proof. $q \equiv 1 \pmod 4$ by (4.4), (4.2), and (3.2), and the second statement follows from $q^3 + 1 \equiv 2 \pmod 4$.

The remainder of the proof of Theorem I will now be divided into two cases: (A) Γ contains involutory homologies, and (B) Γ contains no involutory homologies.

5. Proof of Theorem I.

Case A. Γ *contains involutory homologies.* In this case, Γ is generated by its involutory homologies, and these are all conjugate by (4.3). $\Gamma(L)$ contains a unique involutory homology σ.

LEMMA (5.1). *Let* α *be a Baer involution and set* $B = \mathscr{P}_\alpha \cap A$. *Then* $C(\alpha)^B$ *is* 3-*transitive and acts as a subgroup of* $\mathrm{P\Gamma L}(2, q)$ *containing* $\mathrm{PGL}(2, q)$. *Moreover, the subgroup* $C_1(\alpha)$ *of* $C(\alpha)$ *generated by its involutory homologies is isomorphic to* $\mathrm{PGL}(2, q)$, *and all its involutions are homologies.*

Proof. B is an oval in $\mathscr{P}_\alpha$ by (3.1). Let $x_0, y_0 \in B$, $x_0 \neq y_0$. Then α centralizes the unique involution $\sigma_0 \in \Gamma(x_0, y_0)$. Here, σ_0 fixes just two points of B. By Gleason's lemma [**7**, p. 191], $C(\alpha)^B$ is 2-transitive. By (2.4), $C(\alpha)^B$ contains a normal subgroup acting on B as $\mathrm{PSL}(2, q)$ in its usual representation.

By (3.1), α fixes a line L' not meeting B. The involution in $\Gamma(L')$ centralizes α but fixes no point of B. Thus, $C_1(\alpha)^B$ is $\mathrm{PGL}(2, q)$.

It remains to show that $C_1(\alpha)$ contains no nontrivial element fixing B pointwise. If this were not so, then $\alpha \in C_1(\alpha)$ by (3.3). Let Σ_1 be a Sylow 2-subgroup of $C_1(\alpha)$. Let σ_1 and σ_2 be involutory homologies in Σ_1 such that $\langle\sigma_1, \sigma_2\rangle^B = \Sigma_1{}^B$. By (3.1), $\langle\sigma_1, \sigma_2\rangle < \Sigma_1$, and α is conjugate to no element of $\langle\sigma_1, \sigma_2\rangle$. Since $|\Sigma_1{:}\langle\sigma_1, \sigma_2\rangle| = 2$ by (3.3), Thompson's transfer lemma [**14**, Lemma 2.3] implies that $C_1(\alpha)$ has a normal subgroup of index 2 not containing α. This contradicts the definition of $C_1(\alpha)$.

LEMMA (5.2). *Let Σ, x, y and L be as in Lemma* (4.5). *Then $\sigma \in Z(\Sigma)$, and Σ has a normal Klein subgroup $\langle\sigma, \alpha\rangle$ with α a Baer involution.*

Proof. Suppose first that Σ has no normal Klein subgroup. Then Σ is dihedral or quasidihedral [**10**, p. 199], and by (4.2) has a single class of involutions [**10**, pp. 260, 265], which is not the case.

We may thus assume that Σ has a normal Klein subgroup of the form $\langle\sigma, \varphi\rangle$. Here, $|\Sigma{:}C_\Sigma(\varphi)| \leqq 2$.

Suppose that φ is a homology with axis L'. Then $C_\Sigma(\varphi)$ fixes $\{x, y\}$ and $\{x', y'\}$, where $x', y' \in A \cap L'$, $x' \neq y'$. Set $\Lambda = C_\Sigma(\varphi)_{xyx'y'}$. Then $|C_\Sigma(\varphi){:}\Lambda| \leqq 4$, so $|\Lambda| \geqq |\Sigma|/8$. Since x, y, x', y' are noncollinear, Λ fixes pointwise a subplane of $\mathscr{P}$. Let $\alpha \in Z(\Lambda)$ be an involution. We use the notation of (5.1). Λ^B fixes more than two points.

Let Σ_1 be a Sylow 2-subgroup of $C_1(\alpha)$ normalized by Λ. By (5.1), $\Sigma_1 \cap \Lambda \leqq C_1(\alpha) \cap \Lambda = 1$. Consequently, $|\Sigma_1\Lambda| = |\Sigma_1||\Lambda| \geqq 8|\Lambda|$, since Σ_1 is dihedral of order $\geqq 8$. However, $8|\Lambda| \geqq |\Sigma| \geqq |\Sigma_1\Lambda|$. Thus, $\Sigma_1\Lambda$ is a Sylow 2-subgroup of Γ and $|\Sigma_1| = 8$. In particular, q is not a square, so $\Lambda^B = 1$. Now $\Sigma_1\Lambda = \Sigma_1\langle\alpha\rangle$ and α is conjugate to no element of Σ_1. By Thompson's transfer lemma [**14**, Lemma 2.3], Γ has a normal subgroup of index 2. This contradicts (4.2).

LEMMA (5.3). *If an involutory homology ρ centralizes a 2-group Δ, all of whose involutions are homologies, then either $\rho \in \Delta$ or $|\Delta| \leqq 2$.*

Proof. Suppose that $\rho \notin \Delta$, and let M be the axis of ρ. Then no nontrivial element of Δ fixes a point of $M \cap A$. Since $q + 1 \equiv 2 (\mathrm{mod}\, 4)$, we have $|\Delta| \leqq 2$.

LEMMA (5.4). *$Z(\Sigma)$ contains no Baer involution.*

Proof. Let $\alpha \in Z(\Sigma)$ be a Baer involution. Set $\Sigma_1 = \Sigma \cap C_1(\alpha)$. Then $\Sigma_1 \lhd \Sigma$ and $\Sigma = \Sigma_1\Lambda$, where $\alpha \in \Lambda$ and Λ^B fixes more than 2 points. Here, $\Lambda/\langle\alpha\rangle$ is cyclic, so Λ is either cyclic or the direct product of $\langle\alpha\rangle$ with a cyclic group. By (5.1) and Thompson's transfer lemma [**14**, Lemma 2.3], Λ is not cyclic.

Let β be an involution in $\Lambda - \langle\alpha\rangle$. Set $\Sigma_1 = \langle\mu, \tau\rangle$, with τ an involutory homology, $\mu^\tau = \mu^{-1}$ and $[\Lambda, \tau] = 1$. Since q is a square, $|\mu| \geqq 8$.

Note that $\mu^\beta = \mu\sigma$ or $\mu^{-1}\sigma$. For, μ fixes just two points of B. In view of the action of β on B, β acts on $\langle\mu\rangle$ as the involutory automorphism of $\mathrm{GF}(q)$ acts on $GF(q)^*$. Consequently, $\mu^\beta = \mu\sigma$ if $\sqrt{q} \equiv 1 \pmod 4$, and $\mu^\beta = \mu^{-1}\sigma$ if $\sqrt{q} \equiv 3 \pmod 4$.

Set $\beta^* = \beta$ or $\beta\tau$, so $\mu^{\beta^*} = \mu\sigma$ and β^* is a Baer involution. Clearly, $C_\Sigma(\beta^*) = \langle \mu^2, \tau, \Lambda\rangle$ has index 2 in Σ. By (5.1), $C(\beta^*)$ contains a Sylow 2-subgroup $\Sigma^* > C_\Sigma(\beta^*)$ of Γ, and since $|\mu^2| \geqq 4$, we have $\sigma \in Z(\Sigma^*)$.

On the other hand, $\langle \sigma, \alpha\rangle$ is the only Klein group in $Z(\Sigma)$. By a lemma of Burnside [**12**, p. 203], α and $\sigma\alpha$ are not conjugate in Γ. However, $\langle \sigma, \alpha\rangle$ and $\langle \sigma, \beta^*\rangle$ are conjugate in Γ and $\beta^{*\mu} = \sigma\beta^*$, which is a contradiction.

LEMMA (5.5). *Σ has a normal subgroup $C_\Sigma(\alpha) = \Sigma_1\Lambda$ of index 2. Here, $\Sigma_1 = \langle \mu, \tau\rangle \lhd \Sigma$ is a Sylow 2-subgroup of $C_1(\alpha)$, where $\mu^\tau = \mu^{-1}$, τ is an involutory homology, and $\langle\sigma\rangle = Z(\Sigma_1)$. Also, $\alpha \in \Lambda$ and Λ is either a cyclic or Klein group, and $[\Lambda, \tau] = 1$. If $B = \mathscr{P}_\alpha \cap A$, then Λ^B fixes more than two points. Each element of $\Sigma - \Sigma_1\Lambda$ conjugates α to $\sigma\alpha$.*

Proof. By (5.2) and (5.4), $|\Sigma : C_\Sigma(\alpha)| = 2$ and $C_\Sigma(\alpha)$ is a Sylow 2-subgroup of $C(\alpha)$. Then $\Sigma_1 = \Sigma \cap C_1(\alpha)$ is a Sylow 2-subgroup of $C_1(\alpha)$ and is dihedral of order at least 8, by (5.1). Also, $\langle\sigma\rangle = Z(\Sigma_1)$.

By (5.1), $C_\Sigma(\alpha) = \Sigma_1\Lambda$, where $\alpha \in \Lambda$ and Λ^B fixes more than 2 points. Clearly, Λ centralizes an involution τ in $\Sigma_1 - \langle\sigma\rangle$. Since Λ^B is cyclic, by (3.3) Λ is cyclic or the direct product of $\langle\alpha\rangle$ with a cyclic group $\langle\gamma\rangle \neq 1$. If β is the involution in $\langle\gamma\rangle$ and $\gamma \neq \beta$, then $\langle \alpha, \gamma\rangle$ induces a group on $\mathscr{P}_\beta \cap A$ containing a Klein group and fixing more than 2 points, which contradicts (5.1).

Since Σ_1 is generated by the involutory homologies in $\Sigma_1\Lambda$, it is normal in Σ. Finally, the last assertion follows from the fact that $\langle \sigma, \alpha\rangle \lhd \Sigma$.

LEMMA (5.6). *Λ is cyclic.*

Proof. Assume that $\Lambda = \langle \alpha, \beta^*\rangle$ is a Klein group. Since q is a square, $|\mu| \geqq 8$. As in the proof of (5.4), we can set $\beta = \beta^*$ or $\tau\beta^*$, so β is a Baer involution such that $\mu^\beta = \mu\sigma$. Here, $\Sigma_1\Lambda = \Sigma_1\langle \alpha, \beta\rangle$.

Since $\mu^\tau = \mu^{-1}$ and $\mu^\beta = \mu\sigma$, we can find $\varphi \in \Sigma - \Sigma_1\langle\alpha, \beta\rangle$ such that $\mu^\varphi = \mu$. Then $\varphi^2 \in C(\mu) \cap \Sigma_1\langle \alpha, \beta\rangle = \langle \mu, \alpha\rangle$ and $\alpha^\varphi = \sigma\alpha$, so $\varphi^2 \in \langle\mu\rangle$. By considering the abelian group $\langle \mu, \varphi\rangle$, we may assume that $\varphi^2 = \mu$ or $\varphi^2 = 1$. Also, since β^φ centralizes μ^2, it belongs to $\langle \mu, \alpha, \beta\rangle$ and hence even to $\langle \sigma, \alpha, \beta\rangle$. Clearly, $\beta^\varphi \notin \langle \sigma, \alpha\rangle$.

Suppose that $\varphi^2 = \mu$. Set $\tau^\varphi = \mu^i\tau$. Then $\mu^{-2}\tau = \tau^{\varphi^2} = \mu^i\mu^i\tau$ and i is odd. However, $\beta^\varphi \in \langle \sigma, \alpha, \beta\rangle - \langle \sigma, \alpha\rangle$ centralizes τ^φ, whereas β does not centralize μ^i.

Thus, $\varphi^2 = 1$. Then $\beta^\varphi \in \langle \sigma, \beta\rangle$, as, otherwise, $(\beta\varphi)^2 = \beta\beta^\varphi = \alpha$ or $\sigma\alpha$, whereas $\beta\varphi \notin C_\Sigma(\alpha)$. If $\beta^\varphi = \sigma\beta$, then $(\alpha\beta)^\varphi = \sigma\alpha\sigma\beta = \alpha\beta$. We can replace β by $\alpha\beta$, if necessary, in order to obtain $\beta^\varphi = \beta$. Now $|\Sigma : C_\Sigma(\beta)| = 2$. By (5.1) and (5.4), $C_\Sigma(\beta)$ is a Sylow 2-subgroup of $C(\beta)$. Then $\Sigma \cap C_1(\beta)$ contains a subgroup isomorphic to Σ_1 and, hence, contains an involutory homology $\rho \notin \Sigma_1$. By (5.3), $C_{\Sigma_1}(\rho) = \langle\sigma\rangle$. Consequently, $\Sigma_1\langle\rho\rangle$ contains an element whose square is μ, and this leads to a contradiction as before.

LEMMA (5.7). *If* $|\Lambda| = 2$, *then* τ *and* $\mu\tau$ *are conjugate in* Σ.

Proof. Let $\varphi \in \Sigma - \Sigma_1\langle\alpha\rangle$. Since φ normalizes $\langle\mu\rangle$ and $\mu^\tau = \mu^{-1}$, we can choose φ so that either $\mu^\varphi = \mu$ or $|\mu| \geqq 8$ and $\mu^\varphi = \sigma\mu$. Let $\tau^\varphi = \mu^i\tau$, with i an integer.

Suppose that it is impossible to choose such an element φ of order 2. If $\mu^\varphi = \mu$, then, by considering the abelian group $\langle\mu, \varphi\rangle$, we may assume that $\varphi^2 = \mu$. Then $\mu^{-2}\tau = \tau^{\varphi^2} = \mu^i\mu^i\tau$, so i is odd and (5.7) holds. Let $\mu^\varphi = \sigma\mu$, $|\mu| \geqq 8$. Then $\varphi^2 \in C(\mu^2) \cap \Sigma_1\langle\alpha\rangle = \langle\mu, \alpha\rangle$, so, since $\alpha^\varphi = \sigma\alpha$, we may assume that $\varphi^2 = \mu\alpha$ or $\varphi^2 = \mu^2$. If $\varphi^2 = \mu\alpha$, then $\mu^{-2}\tau = \tau^{\varphi^2} = (\mu\sigma)^i\mu^i\tau$, i is odd, and (5.7) holds. Let $\varphi^2 = \mu^2$ and let k be an integer such that $\mu^{2k} = \sigma\mu^{-2}$. Here, k is odd. Then $(\varphi\mu^k)^2 = \varphi^2(\mu^k)^\varphi\mu^k = \mu^2\mu^k\sigma\mu^k = 1$, which contradicts our assumption.

Now suppose that we can choose φ of order 2. Then $\tau = \tau^{\varphi^2} = (\mu\sigma^j)^i\mu^i\tau$, where $j = 0$ or 1, so $1 = \sigma^{ji}\mu^{2i}$. Since $|\mu| \geqq 8$ if $j = 1$, it follows that i is even and hence that $\mu^i = 1$ or σ. In particular, τ and $\mu\tau$ are not conjugate in Σ. Also, $\varphi^\mu = \sigma^j\varphi$, $\varphi^\tau = \mu^i\varphi$, and $\varphi^\alpha = \sigma\alpha$, so $\langle\sigma, \varphi\rangle \lhd \Sigma$. Since φ is a Baer involution by (5.3), by (5.4), $C_\Sigma(\varphi)$ contains a Sylow 2-subgroup of $C(\varphi)$.

If $\tau^\varphi = \tau$ then $\tau^{\nu\alpha\varphi} = \sigma\tau$, where ν is an element of $\langle\mu\rangle$ of order 4. Here, $\mu^{\nu\alpha\varphi} = \mu^\varphi$ and $(\nu\alpha\varphi)^2 = \nu(\varphi\nu)^\alpha\varphi = \sigma\sigma = 1$. We may assume that $\tau^\varphi = \sigma\tau$.

By (5.1), $C_1(\varphi) \cap \Sigma$ is dihedral of order $|\Sigma_1|$. Since $\tau \notin C_1(\varphi) \cap \Sigma$, there is an involutory homology $\rho \in \Sigma - C_\Sigma(\alpha)$. By (5.3), $C_{\Sigma_1}(\rho) = \langle\sigma\rangle$. It follows that τ and $\mu\tau$ are conjugate in Σ, which is not the case.

LEMMA (5.8). *If* $|\Lambda| \geqq 4$, *then* τ *and* $\mu\tau$ *are conjugate in* Σ.

Proof. We have $\mu^\tau = \mu^{-1}$. As in the proof of (5.4), an element of Λ of order 4 conjugates μ to $\mu\sigma$ or $\mu^{-1}\sigma$. We can thus find $\varphi \in \Sigma - \Sigma_1\Lambda$ such that $\mu^\varphi = \mu$.

If $\langle\varphi^2\rangle = \langle\mu\rangle$, then (5.8) holds as in (5.7). By considering $\langle\mu, \varphi\rangle$, we may thus assume that $\varphi^2 = 1$. If $\tau^\varphi = \mu^i\tau$, then, as before, $\mu^i = 1$ or σ. If $\nu \in \langle\mu\rangle$ has order 4, then $\tau^{\nu\alpha\varphi} = \sigma\tau^\varphi$, $\mu^{\nu\alpha\varphi} = \mu$ and $(\nu\alpha\varphi)^2 = 1$. We may thus assume that $\tau^\varphi = \tau$. Now, $C_\Sigma(\Sigma_1) \geqq \langle\sigma, \alpha, \varphi\rangle$.

In fact, $C_\Sigma(\Sigma_1) = \langle\alpha, \sigma, \varphi\rangle = \langle\alpha, \varphi\rangle$. For otherwise, $|\Sigma_1 C_{\Sigma_1}(\Sigma) \cap \Lambda| \geqq 4$. However, if $\lambda \in \Lambda$ has order 4 and $\gamma \in \Sigma_1$, then $\mu^{\gamma\lambda} = (\mu^{\pm 1})^\lambda \neq \mu$.

Since Σ normalizes $\langle\alpha, \varphi\rangle$ and $\langle\alpha, \sigma\rangle$, it follows that Σ normalizes the second Klein subgroup $\langle\sigma, \varphi\rangle$ of the dihedral group $\langle\alpha, \varphi\rangle$ of order 8. Now Λ acts on $\langle\sigma, \varphi\rangle$ and $|\Lambda| \geqq 4$, so α centralizes φ, which is not the case.

LEMMA (5.9). Γ *is 2-transitive on* A.

Proof. By (4.3), (5.7), and (5.8), $C(\sigma) = \Gamma_L$ is transitive on the $q^2 - q$ involutory homologies $\tau \in C(\sigma) - \langle\sigma\rangle$. Also, Γ_L is transitive on $L \cap A$, where $|L \cap A| = q + 1$. Thus, each orbit of $C(\langle\sigma, \tau\rangle)$ on $L \cap A$ has length divisible by $(q + 1)/(q + 1, q(q - 1)) = (q + 1)/2$. Here, $(q + 1)/2$ is odd and τ fixes no points of $L \cap A$. Thus, $C(\langle\sigma, \tau\rangle)$ is transitive on $L \cap A$.

Let M be the axis of τ. Then $C(\langle\sigma, \tau\rangle)$ is transitive on $M \cap A$. Since $C(\sigma)$ is transitive on the $q^2 - q$ axes of involutory homologies in $C(\sigma) - \langle\sigma\rangle$, and the union of these lines contains $A - L \cap A$, it follows that $C(\sigma)$ is transitive on $A - L \cap A$. (5.9) now follows from (3.8).

LEMMA (5.10). *Each Baer involution in Γ_{xy} centralizes every homology interchanging x and y.*

Proof. The number of involutory homologies moving x is $q^2(q^2 - q + 1) - q^2$. By (5.9), $q^2(q^2 - q)/q^3 = q - 1$ of these interchange x and y.

Let β be any Baer involution in Γ_{xy}. By (5.1), $C_1(\beta)$ has $q - 1$ homologies interchanging x and y. This proves (5.10).

LEMMA (5.11). *The Klein group $\langle \sigma, \alpha \rangle$ of (5.2) can be assumed to fix x and y. No element $\neq \alpha^{L \cap A}$ of $\Sigma_{xy}{}^{L \cap A}$ is conjugate to $\alpha^{L \cap A}$ in $\Gamma_x{}^{L \cap A}$.*

Proof. Σ fixes $\mathscr{P}_\alpha \cap L \cap A$. If $\mathscr{P}_\alpha \cap L \cap A \neq \emptyset$, we may assume that this set is $\{x, y\}$. If $\mathscr{P}_\alpha \cap L \cap A = \emptyset$, then $\langle \mu \rangle$ acts on the $q + 1$ points of $\mathscr{P}_\alpha \cap A$ and σ induces an odd permutation, which is clearly impossible.

By (5.2), $\alpha^{L \cap A} \in Z(\Sigma^{L \cap A})$. Let $\beta = \alpha^\epsilon \in \Sigma$ be a Baer involution conjugate to α by an element ϵ of Γ_x, and suppose that $\beta^{L \cap A} \neq \alpha^{L \cap A}$.

Since $\beta \in \Sigma \leqq \Gamma_{\{x,y\}}$, β fixes x and y.

If $\beta \in C_\Sigma(\alpha)$, then $\beta \in (\Sigma_1 \Lambda)_{xy} = \langle \mu, \Lambda \rangle$, so $\beta \in \langle \sigma, \alpha \rangle$. This is not the case. Thus, $\Sigma = (\Sigma_1 \Lambda)\langle \beta \rangle$, $\alpha^\beta = \sigma\alpha$, and $(\alpha\beta)^2 = \sigma$. Moreover, $\Sigma_{xy} = \langle \mu, \Lambda, \beta \rangle$.

Let $\mu_1 \in \langle \mu \rangle$ have order 4. By (5.10), $(\mu_1 \alpha\beta)^2 = \mu_1{}^2 (\alpha\beta)^2 = \sigma\sigma = 1$. Here, $\mu_1 \alpha \beta \in \Sigma_{xy}$. By (5.10), β and $\mu_1 \alpha\beta$ centralize Σ_1, so μ_1 centralizes Σ_1, which is not the case.

We can now complete Case A. By (5.9), (5.10) and [**18**, Theorem D], $\Gamma_L{}^{L \cap A}$ has a normal subgroup acting as $PSL(2, q)$. Using $\tau^{L \cap A}$, we then find that $\Gamma_L{}^{L \cap A}$ even contains $PGL(2, q)$.

By (3.1), $\Sigma(L \cap A)$ is cyclic or generalized quaternion. By the Frattini argument, $\Gamma_L = \Gamma(L) N(\Sigma(L \cap A))$. Also, $N(\Sigma(L \cap A))/C(\Sigma(L \cap A)) \cdot \Sigma(L \cap A)$ is isomorphic to a group of outer automorphisms of $\Sigma(L \cap A)$. Since $q \equiv 1 (\text{mod } 4)$, $PSL(2, q)$ is simple, and it follows that $C(\Sigma(L \cap A))^{L \cap A}$ contains $PSL(2, q)$. Let Π be the preimage of $PSL(2, q)$ in $C(\Sigma(L \cap A))$. Then $\Sigma \cap \Pi$ is a Sylow 2-subgroup of Π. Also, $\Sigma(L \cap A) \cap \Pi \leqq Z(\Pi)$, so $\Pi(L \cap A) = (\Sigma(L \cap A) \cap \Pi) \times O(\Pi)$.

Now $\Pi/O(\Pi)$ is a central extension of $PSL(2, q)$ by the 2-group $\Sigma(L \cap A) \cdot O(\Pi)/O(\Pi)$. By a result of Schur [**25**], $\Pi/O(\Pi)$ has a unique normal subgroup $\Delta/O(\Pi)$ isomorphic to $PSL(2, q)$ or $SL(2, q)$. Since $\Sigma \rhd \Sigma(L \cap A)$, $\Sigma \rhd \Sigma \cap \Delta$.

If $\Delta/O(\Pi)$ is $PSL(2, q)$ then $\Sigma \cap \Delta$ does not contain σ. However, $(\Sigma \cap \Delta) \cap Z(\Sigma) \neq 1$, and this contradicts (5.4).

Thus, $\Sigma \cap \Delta$ is a generalized quaternion group. τ acts on $\Sigma \cap \Delta$. By (5.3), $C_{\Sigma \cap \Delta}(\tau) = \langle \sigma \rangle$ and $(\Sigma \cap \Delta)\langle \tau \rangle$ is quasidihedral.

Let 2^e be the largest power of 2 dividing $q - 1$. Then $(\Sigma \cap \Delta)\langle \tau \rangle$ contains 2^e homologies interchanging x and y. These are all in $C(\alpha) \cap \Sigma$ and, hence, in Σ_1. However, Σ_1 has only 2^e homologies interchanging x and y. Thus,

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$(\Sigma \cap \Delta)\langle\tau\rangle$ contains Σ_1 as a subgroup of index 2. Clearly, $(\Sigma \cap \Delta)\langle\tau\rangle = (\Sigma \cap \Delta)\Sigma_1 \lhd \Sigma$. Also, $\Sigma \cap \Delta$ is not in $C_\Sigma(\alpha)$, so $\Sigma = (\Sigma \cap \Delta)\Sigma_1 \cdot \Lambda$. The involution $\alpha \in \Lambda$ is conjugate to no involution in $(\Sigma \cap \Delta)\Sigma_1$. We can now apply Thompson's transfer lemma in order to contradict (4.2). Consequently, Case A cannot occur.

Remark. Once (5.5) and (5.6) are known, it is possible to eliminate Case A solely by group theoretic methods. Dr. Anne MacWilliams has done this using ingenuity and a transfer theorem of Grün. We have chosen a slightly shorter approach, involving the comparatively difficult result [**18**, Theorem D] in order to exhibit more of the geometric nature of our situation. In particular, it should be clear that Case A would be greatly simplified if we had started with a group 2-transitive on A.

Case B. All involutions are Baer involutions. By (3.1), each involution in $\Gamma_L{}^{L\cap A}$ fixes 0 or 2 points, and some involution β fixes 0 points, while some other involution fixes 2 points. Here, β is an odd permutation of $L \cap A$. Let Π be the subgroup of Γ_L consisting of even permutations on $L \cap A$, so that $|\Gamma_L : \Pi| = 2$.

Let Σ, x and y be as in (4.5). We may assume that $\beta \in \Sigma - \Sigma_{xy}$.

LEMMA (5.12). *One of the following holds:*

(i) Σ *is dihedral or quasidihedral; or*

(ii) $\Sigma_{xy} \leqq \Pi$, Π *has two orbits on* $L \cap A$ *of length* $\frac{1}{2}(q+1)$, *and* Π *acts on each of these as a* 2*-transitive group containing a normal subgroup acting as* $\mathrm{PSL}(2, 2^e)$, $\mathrm{Sz}(2^e)$, *or* $\mathrm{PSU}(3, 2^e)$, $2^e \geqq 4$, *in its usual representation.*

Proof. By the preceding remarks, if (i) does not hold, then [**13**, Hilfsatz 6] implies that $\Sigma_{xy} \leqq \Pi$, Π has two orbits on $L \cap A$ of length $\frac{1}{2}(q+1)$, and Π acts on each of these orbits as a transitive group in which each involution fixes a single point. By Bender's theorem [**5**], either (ii) holds or $\Sigma \cap \Pi$ is cyclic or generalized quaternion. In the latter case, Klein groups fix no absolute points. β acts on $\Sigma \cap \Pi$ and $C_{\Sigma\cap\Pi}(\beta)$ acts on $\mathscr{P}_\beta \cap A$. Since α acts as an odd permutation of $\mathscr{P}_\beta \cap A$, $C_{\Sigma\cap\Pi}(\beta) = \langle\alpha\rangle$. Thus, (i) again holds in this case.

We now consider (i) and (ii) separately.

(i) Recall that $O(\Gamma) = 1$ and that Γ has no normal subgroup of index 2. Γ_L is a subgroup of Γ containing a Sylow 2-subgroup Σ and having a normal subgroup Π of index 2. Here, $|\Pi|$ is even.

Suppose first that Σ is dihedral. By the Gorenstein-Walter Theorem [**11**], Γ is isomorphic to A_7 or has a normal subgroup $\Gamma^* \approx \mathrm{PSL}(2, m)$, for some odd m, where $|\Gamma : \Gamma^*|$ is odd. If $\Gamma \approx A_7$, it is easy to use the property $|\Gamma_L : \Pi| = 2$ to check that $|\Gamma : \Gamma_L| = 7$, $7\binom{6}{2}$ or $7 \cdot 5 \cdot 9$, whereas none of these numbers has the form $q^2(q^2 - q + 1)$. Next, consider the group $\Gamma^* \approx \mathrm{PSL}(2, m)$. Set $\Pi^* = \Pi \cap \Gamma^*$, so that $|\Gamma^*{}_L : \Pi^*| = 2$. By [**8**, pp. 285–286], $\Gamma^*{}_L$ either centralizes an involution in Π^* or is isomorphic to S_4. Then Π^* has a non-

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trivial characteristic elementary abelian 2-subgroup Σ^*. Here, $\Sigma^* \trianglelefteq \Gamma_L$. Since $\{x, y\}$ is fixed by Σ, if $\{x^\gamma, y^\gamma\} \neq \{x, y\}$, with $\gamma \in \Gamma_L$, then Σ^* fixes both $\{x, y\}$ and $\{x^\gamma, y^\gamma\}$. By (3.1), either $|\Sigma^*| = 2$ or $q + 1 = 4$. If $\Sigma^* = \langle\alpha\rangle$, with $\alpha \in \Pi$ a Baer involution, then Γ_L fixes $\mathscr{P}_\alpha \cap A \cap L$, which is not the case. If $|\Sigma^*| > 2$ and $q + 1 = 4$, then some involution in Σ^* fixes no point of $L \cap A = \{x, y, x^\gamma, y^\gamma\}$, which contradicts (3.1).

Thus, Σ must be quasidihedral. By a result of Alperin, Brauer, and Gorenstein [**1**], Γ is isomorphic to M_{11} or has a normal subgroup $\Gamma^* \approx \mathrm{PSL}(3, m)$ or $\mathrm{PSU}(3, m)$, for some odd m. If $\Gamma \approx M_{11}$, it is easy to use the property $|\Gamma_L:\Pi| = 2$ to check that $|\Gamma:\Gamma_L| = 11$, $\binom{11}{2}$, $11\binom{10}{2}$, or $\binom{11}{3}$, whereas none of these numbers has the form $q^2(q^2 - q + 1)$. Next, consider the group $\Gamma^* \approx \mathrm{PSL}(3, m)$ or $\mathrm{PSU}(3, m)$. Since Γ^* contains all involutions of Γ, we have $|\Gamma^*{}_L:\Pi^*| = 2$, where $\Pi^* = \Pi \cap \Gamma^*$. Using results of Mitchell [**21**] or Bloom [**6**] we find that the subgroup $\Gamma^*{}_L$ of $\mathrm{PSL}(3, m)$ or $\mathrm{PSU}(3, m)$ containing a Sylow 2-subgroup and having a normal subgroup of index 2 has one of the following forms: $\Gamma_L{}^*$ centralizes a unique involution; $\Pi^* \approx A_6$ but $\Gamma^*{}_L \not\approx S_6$, where $\Gamma^* \approx \mathrm{PSU}(3, 5)$; or $\Gamma^* \approx \mathrm{PSL}(3, m)$ and $\Gamma^*{}_L$ is in the normalizer of an elementary abelian subgroup M_0 of order m^2, all of whose elements are elations of $\mathrm{PG}(2, m)$.

The first possibility is eliminated as in the dihedral case. Suppose that $\Pi^* \approx A_6$. Let E be an orbit of Π^* on $L \cap A$ such that some involution in Π^* fixes a point of E. Since A_6 has no transitive representation in which all involutions fix a single point, E is unique. Then $\Pi^* \triangleleft \Gamma_L$ implies that $E = L \cap A$. Also, $\Pi^*{}_x$ is a self-normalizing subgroup of Π^*. Since $q + 1 \equiv 2 \pmod 4$, $\Pi^*{}_x$ contains either a cyclic group of order 4 or a Klein group. It follows that $q + 1 = 6$ or 10. However, $\Gamma^*{}_L$ is not isomorphic to S_6, so $q = 9$. Thus, $|\Gamma^*|$ divides $q^2(q^2 - q + 1) \cdot 2|A_6| = 9^2 \cdot 73 \cdot 2|A_6|$, which is not the case.

Now assume that $\Gamma_L{}^*$ is in the normalizer of M_0 in $\Gamma^* \approx \mathrm{PSL}(3, m)$. Since $N_{\Gamma^*}(M_0)/M_0$ has a central involution, so does Π^*/M, where $M = \Pi^* \cap M_0 = \Gamma^*{}_L \cap M_0$. Thus, $M\langle\alpha\rangle \trianglelefteq \Gamma^*{}_L$ for some involution α, so $\Gamma^*{}_L = M(\Gamma^*{}_L \cap C(\alpha))$ and $\Pi^* = MC_{\Pi^*}(\alpha)$. Since $\Gamma^*{}_L \geqq \Sigma$, $M\langle\alpha\rangle$ is characteristic in $\Gamma^*{}_L$. Then $M\langle\alpha\rangle \trianglelefteq \Gamma_L$. Each point of $L \cap A$ is fixed by a conjugate of α in Γ_L, and all such involutions are even conjugate in $M\langle\alpha\rangle$. Consequently, $|L \cap A \cap \mathscr{P}_\alpha| = 2$ implies that M has just two orbits on $L \cap A$. From the transitivity of Γ_L on $L \cap A$, it follows that $q + 1 = 2p^i$ for some i, where p is the prime dividing m.

Clearly, $\Sigma \cap \Pi$ is a maximal subgroup of the quasidihedral group Σ such that $\Sigma - \Sigma \cap \Pi$ contains an involution. Consequently, $\Sigma \cap \Pi$ is a cyclic or generalized quaternion group. In particular, Π_x contains no Klein group. Then neither Γ_{xL} nor Γ_x can contain a Klein group. It follows that $\Gamma^*{}_L$ has a single class of involutions. This implies that $C_{\Gamma^*}(\alpha)$ is transitive on $B = \mathscr{P}_\alpha \cap A$.

By (3.3), $C_{\Gamma^*}(\alpha)/\langle\alpha\rangle$ acts faithfully on B. Clearly,

$$C_{\Gamma^*}(\alpha)/Z(C_{\Gamma^*}(\alpha)) \approx \mathrm{PGL}(2, m),$$

where $Z(C_{\Gamma^*}(\alpha))$ has order $2t = (m-1)/(m-1, 3)$. The transitivity of $C_{\Gamma^*}(\alpha)^B$ implies that all orbits of $Z(C_{\Gamma^*}(\alpha))^B$ have length t. Thus, $t|q + 1 = 2p^i$, so $t = 1$ or 2, and $m = 3, 5, 7$, or 13 is a prime. Since $q + 1| \, |C_{\Gamma^*}(\alpha)|$, it follows that $q + 1 = 2m$. In each case, $q^2 \nmid |\mathrm{P\Gamma L}(3, m)|$. This is a contradiction.

(ii) This situation is simpler than that of (i). Here, $\frac{1}{2}(q + 1) = 2^{ei} + 1$, with $i = 1, 2$, or 3, so $q = 2^{ei-1} + 1$ is a square only if $q = 9$, $e = 2$, and $i = 1$.

Suppose that q is not a square. Let α be an involution in $Z(\Sigma_{xy})$. By (3.1) and (3.3), $\Sigma_{xy}/\langle\alpha\rangle$ acts faithfully on the oval $\mathscr{P}_\alpha \cap A$ and each of its involutions induces a homology of $\mathscr{P}_\alpha$. Since $\Sigma_{xy}/\langle\alpha\rangle$ contains a Klein group, this contradicts (2.3).

Thus, $q = 9$, $|\Sigma_{xy}| = 4$, and $|\Sigma| = 8$. Since Σ is not dihedral (by (i)) and contains a Klein group, Σ is abelian. Since $\beta \in \Sigma - \Sigma_{xy}$, Σ is elementary abelian. By (4.2), all involutions in Σ are conjugate, and there is a 7-element $\varphi \in N(\Sigma) - C(\Sigma)$ [**12**, pp. 203, 215]. Let $\Sigma = \langle\alpha, \beta, \gamma\rangle$.

Suppose that φ fixes some nonabsolute line L'. Since $7 \nmid |\Gamma_{L'}{}^{L' \cap A}|$, $\varphi \in \Gamma(L' \cap A)$. Also, φ fixes one of the $q^2 - q = 9 \cdot 8$ nonabsolute lines on L'^θ, say M. As before, $\varphi \in \Gamma(M \cap A)$, so φ is planar. However, φ fixes at least 11 points of L', namely, M^θ and the points of $L' \cap A$, which is impossible.

Consequently, the number of fixed nonabsolute lines of Σ is divisible by 7.

If β induces a Baer involution on $\mathscr{P}_\alpha$, then γ acts on the plane $\mathscr{P}_\alpha \cap \mathscr{P}_\beta$ of order 3 and fixes a nonabsolute line. Now φ acts on the set of lines of $\mathscr{P}_\alpha \cap \mathscr{P}_\beta \cap \mathscr{P}_\gamma$ and, hence, fixes one of them, which is not the case.

Thus, β and γ induce homologies on $\mathscr{P}_\alpha$, and Σ fixes their axes. By (2.1), Σ fixes just three nonabsolute lines.

This contradiction completes the proof of Theorem I.

6. Elations. We next turn to unitary polarities θ centralized by many perspectivities of the plane $\mathscr{P}$ of order q^2. If there are enough homologies in $G(\theta)$, then $\mathscr{P}$ is desarguesian (see (6.5)). However, if, for example, $|\Gamma(L)| = q + 1$ for only a few nonabsolute lines, it seems difficult to obtain information concerning $G(\theta)$ or $\mathscr{P}$.

Consequently, we shall consider elations. Let $\Gamma \leqq G(\theta)$ and set

$$A_0 = \{x \in A \,|\, |\Gamma(x)| = q\}.$$

Conceivably, $|A_0| \leqq 1$. Theorem II concerns the case $\Gamma = G(\theta)$ and $|A_0| > q + 1$. We first consider what happens when $1 < |A_0| \leqq q + 1$.

THEOREM III. *Let θ be a unitary polarity of a projective plane of order q^2. Set $\Gamma = G(\theta)$, and let A be the set of absolute points of θ and L a nonabsolute line.*

Suppose that $|\Gamma(x)| = q$ *for at least two points* x *of* $L \cap A$. *Then one of the following holds for* $\Pi = \langle \Gamma(x) | x \in L \cap A \rangle$:

(a) $\Pi \approx \mathrm{SL}(2, q)$ *and* Π *acts on* $L \cap A$ *as* $\mathrm{PSL}(2, q)$ *in its usual* 2-*transitive representation; or*

(b) $q = 2^{3e}$ *with* $e \geqq 2$ *and* Π *acts on* $L \cap A$ *as* $\mathrm{PSU}(3, 2^e)$ *in its usual* 2-*transitive representation.*

We have not been able to eliminate (b). However, Theorem III will be used in the proof of Theorem II only when q is odd. We also do not know whether $\mathscr{P}$ must be desarguesian when (a) holds. This is quite a difficult question, as is shown by the fact that in [**22**], O'Nan implicitly had much more information than this and yet needed to use a very long and ingenious argument in his situation.

Proof of Theorem III. Clearly, $\Pi^{L \cap A}$ is 2-transitive and satisfies the hypotheses of the theorems of Shult [**28**] and Hering, Kantor, and Seitz [**14**]. Since $[\Pi(x), \Pi(L \cap A)] = 1$, $\Pi(L \cap A) \leqq Z(\Pi)$. Thus, Π is a central extension of $\Pi^{L \cap A}$. We may assume that $\Pi^{L \cap A}$ is not of the form (b).

LEMMA (6.1). *Either* Π *is as in Theorem* III *or one of the following holds:*

(i) $\Pi \approx \mathrm{PSL}(2, q)$ *and* q *is odd;*

(ii) $\Pi^{L \cap A}$ *is sharply* 2-*transitive,* $q \neq 3$, *and either* $\Pi \approx \Pi^{L \cap A}$ *or* $q + 1 = p^e$, *for an odd prime* p, *and* $\Pi(L \cap A)$ *is a* p-*group;*

(iii) q *is odd,* $|\Pi(L \cap A)|$ *is odd, and* $\Pi^{L \cap A}$ *is* $\mathrm{PSU}(3, q^{1/3})$ *or a group of Ree type; or*

(iv) q *is a power of* 2 *and* $\Pi \approx \mathrm{Sz}(q^{1/2})$.

Proof. The following argument is that of [**14**, Lemma 4.4]. $\Pi^{L \cap A}$ is sharply 2-transitive, $\mathrm{PSL}(2, q)$, $\mathrm{Sz}(q^{1/2})$, $\mathrm{PSU}(3, q^{1/3})$, or a group of Ree type [**28**; **14**].

If $\Pi^{L \cap A}$ is $\mathrm{PSL}(2, q)$, $q > 3$, then $\Pi(x) \leqq \Pi'$, so $\Pi = \Pi'$. By results of Schur [**25**], Π is $\mathrm{PSL}(2, q)$, $\mathrm{SL}(2, q)$, or is a homomorphic image of the covering group of $\mathrm{PSL}(2, 4)$ or $\mathrm{PSL}(2, 9)$. In the latter cases, if Σ is a Sylow 2-subgroup or 3-subgroup of Π_x, then $\Sigma = \Pi(x) \times \Sigma(L \cap A)$, so a result of Gaschütz [**12**, p. 246] implies that Π splits over $\Sigma(L \cap A)$, which is not the case.

If $\Pi^{L \cap A}$ is unitary or of Ree type, then (iii) holds by [**14**, Lemmas 3.2 (ix) and 3.3 (x)]. If $\Pi^{L \cap A}$ is $\mathrm{Sz}(q^{1/2})$, then, by [**3**], (iv) holds unless $q = 64$ and $\Pi(L \cap A)$ is a 2-group. As in the preceding paragraph, the latter situation cannot occur.

Finally, suppose that $\Pi^{L \cap A}$ is sharply 2-transitive of degree $q + 1 = p^e$, with p a prime. There is a unique Sylow p-subgroup Σ in Π, and $\Pi = \Sigma\Gamma(x)$. If p is odd, then (ii) holds. Let $p = 2$. Then Π is generated by elements of odd order and, hence, has no normal subgroup of index 2. By [**14**, Lemma 2.7], (ii) holds if $q + 1 \neq 4$, while if $q + 1 = 4$, then Π is $\mathrm{PSL}(2, 3)$ or $\mathrm{SL}(2, 3)$. This proves (6.1).

We now treat these possibilities separately.

(i) Suppose first that each involution in Π is a homology. Let $\langle\sigma, \tau\rangle$ be a Klein group, and suppose that σ and τ have different axes. Since $\langle\sigma, \tau\rangle$ fixes L, L must be the axis of σ, τ, or $\sigma\tau$, which is not the case. Thus, if σ and τ are commuting involutions then they have the same axis. It follows that $q \neq 5$ and all involutions have the same axis L'. Consequently, $\Pi = \Pi(L')$ has order dividing $q + 1$, which is not the case.

Thus, all involutions in Π are Baer involutions, and by (3.2), $q \equiv 1 \pmod 4$.

Let $w \in L - L \cap A$ and set $w' = L \cap w^\theta$. Then $\frac{1}{2}(q^2 - q) \geqq |\Pi : \Pi_{\{w,w'\}}|$ implies that $|\Pi_{\{w,w'\}}| \geqq q + 1$. Clearly, $(q, |\Pi_{\{w,w'\}}|) = 1$. By [**8**, pp. 285–286], either $\Pi_{\{w,w'\}}$ is dihedral of order $q + 1$ or is isomorphic to A_4, S_4, or A_5.

Suppose that $\Pi_{\{w,w'\}}$ is dihedral of order $q + 1$. Then a Klein group $\langle\alpha, \beta\rangle$ in Π fixes no points of L, since $\frac{1}{2}(q + 1)$ is odd. However, β acts on $\mathscr{P}_\alpha$ as an involution, so α and β have a common fixed point $z \not\in L^\theta$, and then $\langle\alpha, \beta\rangle$ fixes $zL^\theta \cap L$, which is a contradiction.

Thus, each $\Pi_{\{w,w'\}}$ is A_4, S_4, or A_5. There are integers a, b, c such that

$$\tfrac{1}{2}(q^2 - q) = a \cdot \tfrac{1}{2}q(q^2 - 1)/12 + b \cdot \tfrac{1}{2}q(q^2 - 1)/24 + c \cdot \tfrac{1}{2}q(q^2 - 1)/60.$$

Then $(q + 1)|120$. Since $q \equiv 1 \pmod 4$ and $(q, |\Pi_{\{w,w'\}}|) = 1$, $q = 5$ or 29 is prime. Each $\Pi_{\{w,w'\}}$ contains a subgroup A_4 fixing w and w'. Also, Π is not transitive on the pairs $\{w, w'\}$; this is clear if $q = 5$, while, if $q = 29$, we have $|\Pi_{\{w,w'\}}| \geqq 30$, and hence $a = b = 0$ and $c = 2$. Consequently, a Klein group $\langle\alpha, \beta\rangle$ fixes more than 2 points of L. Since β induces a homology on $\mathscr{P}_\alpha$, L must be its axis. In particular, β fixes $(\mathscr{P}_\alpha \cap L) \cap A$ pointwise. However, this set is nonempty since $q \equiv 1 \pmod 4$, and this contradicts the fact that Klein groups in $\Pi^{L \cap A}$ fix no points.

(ii) Now suppose that $|\Pi^{L\cap A}| = (q + 1)q$, with $q + 1 = p^e$. Let Σ be the Sylow p-subgroup of Π. Let $p = 2$, so $|\Sigma| = 2^e$ by (6.1). If an involution $\sigma \in \Sigma$ is a Baer involution, then, since $q + 1 \equiv 0 \pmod 4$, Π has a normal subgroup of index 2 by (3.2), which is not the case. Thus, σ is a homology fixing L. Since $\Pi(x)$ moves each line not on x, Σ must contain two involutions σ, τ with different axes. Since $\langle\sigma, \tau\rangle$ fixes L, L must be the axis of σ, τ, or $\sigma\tau$, which is again a contradiction.

Consequently, $p > 2$. Since Σ acts on the $q^2 - q$ points of $L - L \cap A$, it fixes one of them, say M^θ. Using $\Pi(x)$ we find that Σ fixes qd points of $L - L \cap A$, where $d \geqq 1$.

Let $z \in M \cap A$ and suppose that $1 \neq \epsilon \in \Pi_z$. Then $\epsilon \in \Sigma$. Clearly, ϵ is planar. $\mathscr{P}_\epsilon$ has order $m \geqq qd - 1$. By Bruck's lemma [**7**, p. 145], $m = q - 1$ or q. Then ϵ fixes no absolute points on L, but fixes an even number of non-absolute points of L [**7**, p. 153], so $m + 1$ is even and, hence, $m = q$. Then ϵ acts on the $(q^2 + 1) - (q + 1)$ lines on L^θ not in $\mathscr{P}_\epsilon$ and fixes at least one of them, which is a contradiction.

Thus, Σ is faithful and sharply transitive on $M \cap A$. It follows that $|\Sigma| = q + 1$ and $|z^\Pi| = (q + 1)q$. Let $z \neq z^\gamma \in A - A \cap L$, $\gamma \in \Pi$. If

Σ fixes zz^{γ}, then zz^{γ} does not meet $L \cap A$ and $\gamma \in \Sigma$. If Σ does not fix zz^{γ}, then zz^{γ} must meet $L \cap A$ in the centre of γ. In either case, $A \cap zz^{\gamma} \subset A_1 = (L \cap A) \cup z^{\Pi}$. By (3.7) we must have $A_1 = A$, $(q+1)q = q^3 - q$, $q = 2$, and Theorem III holds in this case.

(iii) Here the involutions in Π fix $q^{1/3} + 1$ points of $L \cap A$, contradicting (3.1).

(iv) The case of $\mathrm{Sz}(q^{1/2})$ is handled as in (i) using Suzuki's list of subgroups [**29**]. We omit the proof as it is straightforward.

Note that when $\Pi^{L \cap A} \approx \mathrm{PSU}(3, 2^e)$ it is still conceivable that $\Pi(L \cap A) \neq 1$. It can be shown that $\Pi(L \cap A) = \Pi(L)$ has odd order, and the action of Π on $L - L \cap A$ can be completely determined.

Proof of Theorem II. We may assume that $q > 2$ and that Γ is generated by the given elations.

LEMMA (6.2). Γ *is transitive on* A, *and Theorem* III *applies to each nonabsolute line.*

Proof. Let $x \in A$ with $|\Gamma(x)| = q$. Let $A_0 = x^{\Gamma}$. If $y, z \in A_0$, $y \neq z$, then using $\langle \Gamma(y), \Gamma(z) \rangle$ we find that $yz \cap A \subseteq A_0$. By (3.7), $A_0 = A$, and (6.2) holds.

LEMMA (6.3). *Each nontrivial normal subgroup of* Γ *is transitive on* A. *Moreover*, $O(\Gamma) = 1$ *if* q *is even.*

Proof. Let $1 \neq \Delta \lhd \Gamma$ and suppose that Δ has an orbit A_0 on A, with $1 < |A_0| < |A|$. If $x \in A_0$, then $\Gamma(x)$ fixes A_0. Thus, if $x, y \in A_0$, $x \neq y$, then $xy \cap A \subseteq A_0$. By (3.7), $A_0 = A \cap L$ for some nonabsolute line L. Then Δ fixes each of the $(q^3 + 1)/(q + 1) > 3$ images of L under Γ. Since Γ leaves invariant the set $\mathscr{P}_{\Delta}$ of fixed points and lines of Δ, while Γ fixes no point or line of $\mathscr{P}$, it follows that $\mathscr{P}_{\Delta}$ is a subplane. θ induces a polarity $\bar{\theta}$ on $\mathscr{P}_{\Delta}$. However, Δ fixes no point of A. Thus, $\bar{\theta}$ has no absolute points, contradicting a theorem of Baer [**7**, p. 152].

The last assertion follows from (3.6).

LEMMA (6.4). *We may assume that* q *is odd. Then* Γ *is flag-transitive on* $\mathscr{U}(\theta)$.

Proof. If q is even, the theorem follows from (6.3) and (3.5). Assume that q is odd. By (6.1) and Theorem III, each nonabsolute line is the axis of an involutory homology. Then (6.4) follows, as in (4.3), from Gleason's lemma.

Theorem II now follows from Theorem I.

We note that most of the difficulties in the proof of Theorem I could have been eliminated if we had only proved Theorem II; the latter theorem is much simpler than the former. Also, an alternative approach, avoiding Theorem III, can be obtained by using the argument of § 5, Case B, keeping in mind that Γ_L is 2-transitive on $L \cap A$.

We conclude by pointing out a corollary of Theorem I.

COROLLARY (6.5). *If $\mathscr{P}$ is a finite projective plane, θ is a unitary polarity of $\mathscr{P}$, p is a prime, and each nonabsolute line is the axis of a homology in $G(\theta)$ of order p, then $\mathscr{P}$ is desarguesian.*

This is again proved as in (4.3). In particular, if $q > 2$ and each nonabsolute line is the axis of $(q+1)/(q+1, 3)$ or $q+1$ homologies in $G(\theta)$, then the plane is desarguesian.

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