A NOTE ON THE POSITIVE SCHUR PROPERTY

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The purpose of this note is to characterize those Banach lattices $(E, \|\cdot\|)$ which have the property:

an operator $T: E \to c_0$ is a Dunford–Pettis operator if and only if T is regular $\quad (*)$

(i.e., T is the difference of two positive operators). Our characterization generalizes a theorem recently proved by Holub [6] and Gretsky and Ostroy [4], who have remarked that the space $L^1[0, 1]$ has the property $(*)$. The main result presented here is the following theorem.

THEOREM. *For a Banach lattice* $(E, \|\cdot\|)$ *the following statements are equivalent.*

(i) *E is σ-Dedekind complete and* $(E, \|\cdot\|)$ *has the property* $(*)$.

(ii) *E has the positive Schur property, i.e.,* $(x_n) \subset E_+$ *and* $x_n \to 0$ *weakly imply* $\|x_n\| \to 0$.

All notions concerning Banach lattices and not explained here can be found in [2] and [1]. Let us only recall that a Riesz space E is called *σ-Dedekind complete* if every countable subset of E order bounded from above has a supremum, and a subset A of a Banach lattice $(E, \|\cdot\|)$ is called *almost order bounded,* if for every $r > 0$ there exists $u \in E_+$ such that $A \subset [-u, u] + rB(1)$, where $B(1)$ denotes the unit ball (compare [9, p. 501]).

The proof of our Theorem will be preceded by some lemmas.

LEMMA 1. *Let* $(E, \|\cdot\|)$ *denote a σ-Dedekind complete Banach lattice. The following statements are equivalent:*

(i) *Every Dunford–Pettis operator* $T: E \to c_0$ *is regular.*

(ii) *The norm* $\|\cdot\|$ *is order continuous.*

Proof. (i)$\Rightarrow$(ii). If $\|\cdot\|$ were not order continuous then $(E, \|.\|)$ would contain a positively complemented closed Riesz subspace order and topologically isomorphic to ℓ^∞. Let $P: E \to \ell^\infty$ be a positive projection and let $T: \ell^\infty \to c_0$ be a weakly compact operator which is non-compact. The space ℓ^∞ has the Dunford–Pettis property, and so T is a Dunford–Pettis operator. Therefore the composition $T \circ P: E \to c_0$ is a Dunford–Pettis operator too. The assumption implies that $T \circ P = T_1 - T_2$, where $T_i: E \to c_0$ $(i = 1, 2)$ are positive. If $y \in \ell^\infty$, then $Ty = T \circ Py = T_1 y - T_2 y$, i.e., $T = S_1 - S_2$, where S_i denotes the restriction of T_i to ℓ^∞. Operators S_i, as positive operators, are compact because they map the unit ball (which is an order interval), into an interval in c_0, i.e., into a compact set. Hence T is a compact operator and we have a contradiction.

(ii)$\Rightarrow$(i) Order bounded subsets of E are weakly compact and so they are mapped by a Dunford–Pettis operator into norm compact subsets which are order bounded in c_0. Thus every Dunford–Pettis operator from E into c_0 is regular.

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Remark. In a class $\mathscr{A}$ of Banach lattices with order continuous norms the following subclasses coincide: the subclass of Banach lattices with the Schur property (i.e., weak null and norm null sequences are the same) and the subclass consisting of those Banach lattices $E \in \mathscr{A}$ such that every continuous linear operator $T: E \to c_0$ is a Dunford–Pettis operator. Indeed, Banach lattices with order continuous norms are Gelfand–Phillips spaces (i.e., a subset F of such a space is conditionally compact if and only if every weak-* null sequence of functionals converges uniformly on F)—see [3]. Therefore, if E contains a weakly null sequence (x_n) with $\inf_n \|x_n\| > 0$, then it is possible to find a weak-* null sequence (f_k) of functionals such that $\sup_n |f_k(x_n)| \not\to 0$ because $\{x_n : n \in \mathbb{N}\}$ is not conditionally compact. The operator $T: E \to c_0$ defined by the equality $Tx = (f_k(x))$ is continuous and it is not a Dunford–Pettis operator.

In particular every separable Banach space is a Gelfand-Phillips space and therefore for every separable Banach space E without the Schur property there exists a continuous linear operator $T: E \to c_0$ which is not a Dunford–Pettis operator.

The next lemma tells about properties of Banach lattices satisfying (*).

Lemma 2. *If $(E, \|\cdot\|)$ is a Banach lattice having the property (*) and with E being σ-Dedekind complete, then the following statements hold.*

(i) *$(E, \|\cdot\|)$ is a KB-space.*

(ii) *If $(x_n) \subset E_+$ is a sequence of norm one pairwise disjoint elements, then no subsequence of (x_n) is a weak Cauchy sequence.*

Proof. (i) Suppose that $(E, \|\cdot\|)$ is not a KB-space. Therefore E contains a closed Riesz subspace order and topologically isomorphic to c_0. The norm $\|\cdot\|$ is order continuous in view of Lemma 1, and so by the famous Meyer-Nieberg result [7] there exists a positive projection P from E onto c_0. The property (*) implies that P is a Dunford–Pettis operator which is impossible because the sequence (e_n) of unit vectors in c_0 is a sequence of norm one elements weakly converging to zero and $e_n = Pe_n$.

(ii) Let (x_n) be a sequence of elements of E_+ with properties mentioned in (ii) and let (y_k) be an arbitrary subsequence of (x_n). If (y_k) were a weak Cauchy sequence, then (y_k) would converge weakly to some element $y \in E$ because, by (i), $(E, \|\cdot\|)$ is a KB-space and so $(E, \|\cdot\|)$ is weakly sequentially complete. There are no difficulties in verifying that $y = 0$. Choose a sequence (f_i) of positive linear functionals defined on E with the properties $f_i(y_i) = 1 = \|f_i\|$. If P_i denotes the band projection onto the band generated by the one-point set $\{y_i\}$, then the functionals $F_i = f_i \circ P_i$ are positive pairwise disjoint and their norms are equal to one. We have $F_i \to 0$ in the weak-* topology by the order continuity of the norm $\|\cdot\|$ (compare Exercise 7 of [2, p. 245]). Thus the equality $Tx = (F_i x)$ defines a positive (and hence continuous) linear operator from E into c_0. Since the space E has the property (*) the operator T is a Dunford–Pettis operator. Hence $(\|Ty_k\|)$ tends to zero and we have a contradiction because $\|Ty_k\| = 1$.

Remark. We have assumed E is σ-Dedekind complete in Lemmas 1 and 2. This assumption is essential—the Banach lattice c of all convergent real sequences is not

σ-Dedekind complete and has the property (*) (this fact is a consequence of the following equivalences holding for every linear operator $T: c \to c_0$: T is regular $\Leftrightarrow T$ is compact $\Leftrightarrow T$ is a Dunford–Pettis operator). However c has neither the properties mentioned in Lemma 2 nor the property (ii) of Lemma 1.

The last Lemma will combine the property described in Lemma 2 (ii) with the positive Schur property.

Lemma 3. *If $(E, \|\cdot\|)$ is a Banach lattice such that no sequence of norm one positive pairwise disjoint elements of E converges weakly to zero, then $0 \leq x_n \to 0$ weakly implies that $\|x_n\| \to 0$.*

Proof. Suppose that $x_n \to 0$ weakly and $\inf_n \|x_n\| = c > 0$ for some sequence $(x_n) \subset E_+$. Putting $y_n = c^{-1} x_n$ and using [**5**, Corollary 5] we find a subsequence (n_k), a constant $d > 0$, and a sequence (z_k) of pairwise disjoint elements such that $0 < z_k \leq y_{n_k}$ and $\|z_k\| \geq d$. It is clear that $(z_k/\|z_k\|)$ tends weakly to zero but this fact contradicts the assumption.

Now it is possible to present the proof of our main result.

Proof of Theorem. The implication (i)$\Rightarrow$(ii) follows from Lemmas 2 and 3. Suppose that (ii) holds. It is obvious that a Banach lattice having the positive Schur property cannot contain a closed Riesz subspace order isomorphic to c_0. Hence, the norm $\|\cdot\|$ has to be order continuous (moreover, $(E, \|\cdot\|)$ is a KB-space), and so every Dunford–Pettis operator $T: E \to c_0$ is regular in view of Lemma 1.

Let $T: E \to c_0$ be a positive linear operator and A a relatively weakly compact subset of E. Since $(E, \|\cdot\|)$ is a KB-space the set sol A (the solid hull of A) is also relatively weakly compact ([**2**, Theorem 13.8]). If (x_n) is a sequence of pairwise disjoint elements belonging to sol A, then we have $|x_n| \to 0$ weakly ([**2**, Theorem 13.3]) and hence $\|x_n\| \to 0$ by the positive Schur property. Therefore $\|Tx_n\| \to 0$ and, according to [**2**, Theorem 13.5], for each number $r > 0$ there exists some $u \in E_+$ such that $\|T((|x| - u)^+)\| < r$ holds for all $x \in \text{sol}\, A$. However $|Tx| - Tu \leq T(|x| - u) \leq T((|x| - u)^+)$ so $(|Tx| - Tu)^+ \leq T((|x| - u)^+)$. Thus $\|(|Tx| - Tu)^+\| < r$. In other words, $T(\text{sol}\, A)$ is almost order bounded ([**9**, Theorem 122.1]). Since every almost order bounded subset of c_0 is norm totally bounded we have $T(A)$ is conditionally compact; i.e., T is a Dunford–Pettis operator. Finally, every regular operator $T: E \to c_0$ is a Dunford–Pettis operator.

An immediate consequence of the Theorem is the following result.

Corollary. *(See [**6**]) An operator $T: L^1(\mu) \to c_0$ is a Dunford–Pettis operator if and only if T is regular.*

Remark. For discrete Banach lattices the positive Schur property and the Schur property coincide. Indeed, it is sufficient to show that if $(E, \|\cdot\|)$ is a discrete Banach lattice without the Schur property, then there exists a sequence (x_n) consisting of positive disjoint elements which is weakly null and $\inf_n \|x_n\| > 0$.

Suppose E is discrete and without the Schur property. If E contains a closed Riesz

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subspace order isomorphic to c_0, then unit vectors satisfy our requirements. Therefore, we can assume $(E, \|\cdot\|)$ is a KB-space. Fix a weakly null sequence $(y_n) \subset E$ with $\|y_n\| \geqslant 1$ for all n, and put $A = \mathrm{sol}\{y_n : n \in \mathbb{N}\}$ (i.e., A is the solid hull of $\{y_n : n \in \mathbb{N}\}$). According to [**2**, Theorem 13.8] the set A is relatively weakly compact. Since A is norm bounded and A is not conditionally compact, then by [**1**, Theorem 21.15] we find a sequence $(x_k) \subset A$ of pairwise disjoint elements with $\inf_k \|x_k\| > 0$. To end the proof it is sufficient to prove that $|x_k| \to 0$ weakly.

We have $|x_k| \leqslant |y_{n_k}|$ for some subsequence (n_k) and the sequence $(|y_{n_k}|)$ has infinitely many distinct terms (otherwise the sequence $(|x_k|)$ would be order bounded and thus, in view of the order continuity of the norm, norm convergent to zero). The absolute weak topology (which is finer than the weak topology), has the same convergent sequences as the weak topology for discrete Banach lattices ([**1**, Theorem 21.4]). Therefore $y_{n_k} \to 0$ in the absolute weak topology. Since this topology is locally solid we obtain $|y_{n_k}| \to 0$ and $|x_k| \to 0$ in the absolute weak topology. Hence $|x_k| \to 0$ weakly.

The fact, that a discrete Banach lattice without the Schur property contains a suitable sequence consisting of positive disjoint elements, was noticed by Varshavskaya and Chuchaev [**8**] but they did not quote any proof.

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