A CUSP—COUNTING FORMULA FOR CAUSTICS DUE TO MULTIPLANE GRAVITATIONAL LENSING

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1. Main Theorem

Consider a gravitational lens system with k planes. If light rays are traced back from the observer to the light source plane, then the points on the first lens plane where a light ray either terminates, or, passes through and terminates before reaching the light source plane, are "obstruction points." More precisely, tracing rays back to the source plane induces a *k-plane lensing map* $\eta : U \subseteq \mathbf{R}^2 \to \mathbf{R}^2$ of the form $\eta(\mathbf{x}_1) = \mathbf{x}_1 - \sum_{i=1}^{k} \alpha_i(\mathbf{x}_i(\mathbf{x}_1))$. We then define an *obstruction point* of η to be a point $\mathbf{a}$ of U where $\lim_{\mathbf{x}_1 \to \mathbf{a}} |\alpha_i(\mathbf{x}_i(\mathbf{x}_1))| = \infty$ for some "deflection angle" α_i.

Without any additional assumptions on the lensing map η, one readily finds that cusp-counting becomes a hopeless task. In order to have some control on this problem, *we assume that η obeys the following mathematical constraints.* First, η has finitely many obstruction points and is smooth everywhere except at such points. Second, the domain U of η is an open disc punctured by mutually disjoint (small) closed discs that are centered at the obstruction points and disjoint from the critical curves of η. Third, all lenses are isolated, i.e., $\lim_{|\mathbf{x}_i| \to \infty} |\alpha_i(\mathbf{x}_i)| = 0$.

Notation:

(1) If $c(s)$ is a caustic that has no cusps and is parametrized by arc length s, and if $c(0)$ is the initial point of c, then $c(0)$ is an *outside starting point* when there is a line of support at $c(0)$ (i.e., a straight line L through $c(0)$ such that either L coincides with the path traced out by c or the path lies on one side of L). Henceforth, assume each caustic c (possibly with cusps) is piecewise parametrized according to arc length and has a specified outside starting point.

C. S. Kochanek and J. N. Hewitt (eds), Astrophysical Applications of Gravitational Lensing, 281–282.

(2) If c is a caustic with an initial point on a fold, and if by a rigid motion the path traced out by c is positioned such that the x-axis is a line of support at $c(0)$, and the path lies on the side of the positive y-direction, then c is called *positive* (resp., *negative*) if the initial velocity vector $\dot{c}(0)$ is in the positive (resp., negative) x-direction. When the initial point $c(0)$ is a cusp and c is positioned through a rigid motion such that the x-axis is a line of support at $c(0)$ and the path lies on the side of the positive y-direction, then c is *positive*. Let $N^+_{caustics}$ and $N^-_{caustics}$ be, respectively, the total number of positive and negative caustics of η.
(3) If $c(s_1) = c(s_2)$, where $s_1 < s_2$, is a normal self-intersection of c, and if the ordered pair $\{\dot{c}(s_1), \dot{c}(s_2)\}$ is opposite (resp., identical) to the standard orientation of the plane, then the self-intersection is called *positive* (resp., *negative*). Let $N^\pm_{self}(c)$ be, respectively, the number of positive and negative self-intersections of a caustic c. Set $N^\pm_{self} = \sum_c N^\pm_{self}(c)$ and $N_{cusps} = \sum_c N_{cusps}(c)$, where each sum runs over all caustics.

The following theorem applies to generic multiplane gravitational lens systems. It expresses the total number of cusps in terms of the number of light path obstruction points and an algebraic number of caustics and caustic self-intersections.

Theorem 1 *Let η be a locally stable k-plane lensing map. Then the total number of cusps of η is given as follows:*

$$N_{cusps} = 2\left[g_\eta + (N^+_{caustics} - N^-_{caustics}) + (N^+_{self} - N^-_{self})\right].$$

A proof and detailed discussion of Theorem 1 will appear in a forthcoming paper on the global geometry of caustics (Petters 1995). In the special case of caustic networks with no self-intersections, Theorem 1 reduces to a formula of Levine, Petters & Wambsganss (1993).

Corollary 2 *Let η be a locally stable k-plane lensing map. Then:*

$$0 \leq N_{cusps} \leq 2[g_\eta + N_{caustics} + N_{self}],$$

where $N_{caustics}$ and N_{self} are, respectively, the total number of caustics and caustic self-intersections (excluding crossings of different caustics).

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References

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Levine, H., Petters, A.O., & Wambsganss, J., 1993, J Math Phys, 34(10), 4781