Appendix A

Analysis

In Chapters 3 and 4, we use a number of facts of analysis, and especially complex analysis, which are not necessarily included in most introductory graduate courses. We review them here and give some details of the proofs (when they are sufficiently elementary and enlightening) or detailed references.

A.1 Summation by Parts

Analytic number theory makes very frequent use of "summation by parts," which is a discrete form of integration by parts. We state the version that we use.

Lemma A.1.1 (Summation by parts) *Let* $(a_n)_{n\geqslant 1}$ *be a sequence of complex numbers and* $f\colon [0, +\infty[\to \mathbf{C}$ *a function of class* C^1*. For all* $x \geqslant 0$*, define*

$$\mathrm{M}_a(x) = \sum_{1\leqslant n\leqslant x} a_n.$$

For $x \geqslant 0$*, we then have*

$$\sum_{1\leqslant n\leqslant x} a_n f(n) = \mathrm{M}_a(x) f(x) - \int_1^x \mathrm{M}_a(t) f'(t)dt. \qquad \text{(A.1)}$$

If $\mathrm{M}_a(x) f(x)$ *tends to* 0 *as* $x \to +\infty$*, then we have*

$$\sum_{n\geqslant 1} a_n f(n) = -\int_1^{+\infty} \mathrm{M}_a(t) f'(t)dt,$$

provided either the series or the integral converges absolutely, in which case both of them do.

Using this formula, one can exploit known information (upper bounds or asymptotic formulas) concerning the summation function M_a, typically when

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the sequence (a_n) is irregular, in order to understand the summation function for $a_n f(n)$ for many sufficiently regular functions f.

The reader should attempt to write a proof of this lemma, but we give the details for completeness.

Proof Let $\mathrm{N} \geqslant 0$ be the integer such that $\mathrm{N} \leqslant x < \mathrm{N}+1$. We have

$$\sum_{1\leqslant n\leqslant x} a_n f(n) = \sum_{1\leqslant n\leqslant \mathrm{N}} a_n f(n).$$

By the usual integration by parts formula, we then note that

$$\mathrm{M}_a(x)f(x) - \int_1^x \mathrm{M}_a(t)f'(t)dt = \mathrm{M}_a(\mathrm{N})f(\mathrm{N}) - \int_1^{\mathrm{N}} \mathrm{M}_a(t)f'(t)dt$$

(because M_a is constant on the interval $\mathrm{N} \leqslant t \leqslant x$). We therefore reduce to the case $x = \mathrm{N}$. We then have

$$\begin{aligned}
\sum_{n\leqslant \mathrm{N}} a_n f(n) &= \sum_{1\leqslant n\leqslant \mathrm{N}} (\mathrm{M}_a(n) - \mathrm{M}_a(n-1))f(n) \\
&= \sum_{1\leqslant n\leqslant \mathrm{N}} \mathrm{M}_a(n)f(n) - \sum_{0\leqslant n\leqslant \mathrm{N}-1} \mathrm{M}_a(n)f(n+1) \\
&= \mathrm{M}_a(\mathrm{N})f(\mathrm{N}) + \sum_{1\leqslant n\leqslant \mathrm{N}-1} \mathrm{M}_a(n)(f(n) - f(n+1)) \\
&= \mathrm{M}_a(\mathrm{N})f(\mathrm{N}) - \sum_{1\leqslant n\leqslant \mathrm{N}-1} \mathrm{M}_a(n)\int_n^{n+1} f'(t)dt \\
&= \mathrm{M}_a(\mathrm{N})f(\mathrm{N}) - \int_1^{\mathrm{N}} \mathrm{M}_a(t)f'(t)dt,
\end{aligned}$$

which concludes the first part of the lemma. The last assertion follows immediately by letting $x \to +\infty$, in view of the assumption on the limit of $\mathrm{M}_a(x)f(x)$. ☐

A.2 The Logarithm

In Chapters 3 and 4, we sometimes use the logarithm for complex numbers. Since this is not a globally defined function on $\mathbf{C}^\times$, we clarify here what we mean.

Definition A.2.1 Let $z \in \mathbf{C}$ be a complex number with $|z| < 1$. We define

$$\log(1-z) = -\sum_{k\geqslant 1} \frac{z^k}{k}.$$

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Proposition A.2.2 (1) *For any complex number z such that $|z| < 1$, we have*

$$e^{\log(1-z)} = 1 - z.$$

(2) *Let $(z_n)_{n\geqslant 1}$ be a sequence of complex numbers such that $|z_n| < 1$. If*

$$\sum_n |z_n| < +\infty,$$

then

$$\prod_{n\geqslant 1}(1 - z_n) = \exp\Big(\sum_{n\geqslant 1}\log(1 - z_n)\Big).$$

(3) *For $|z| \leqslant \frac{1}{2}$, we have $|\log(1-z)| \leqslant 2|z|$.*

Proof Part (1) is standard since the series used in the definition is the Taylor series of the logarithm around 1 (evaluated at $-z$), and this power series has radius of convergence 1.

Part (2) is then simply a consequence of the continuity of the exponential and the fact that the product is convergent under the assumption on $(z_n)_{n\geqslant 1}$.

For (3), we note that for $|z| < 1$, we have

$$\log(1-z) = -z\left(1 + \frac{z}{2} + \frac{z^2}{3} + \cdots - \frac{z^{k-1}}{k} + \cdots\right)$$

so that if $|z| \leqslant \frac{1}{2}$, we get

$$|\log(1-z)| \leqslant |z|\left(1 + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k-1}} + \cdots\right) \leqslant 2|z|.$$

□

A.3 Mellin Transform

The Mellin transform is a multiplicative analogue of the Fourier transform, to which it can indeed in principle be reduced. We consider it only in simple cases. Let

$$\varphi\colon [0, +\infty[\longrightarrow \mathbf{C}$$

be a continuous function that decays faster than any polynomial at infinity (for instance, a function with compact support). Then the Mellin transform $\hat{\varphi}$ of φ is the holomorphic function defined by the integral

$$\hat{\varphi}(s) = \int_0^{+\infty} \varphi(x)x^s\,\frac{dx}{x},$$

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for all those $s \in \mathbf{C}$ for which the integral makes sense, which under our assumption includes all complex numbers with $\mathrm{Re}(s) > 0$.

The basic properties of the Mellin transform that are relevant for us are summarized in the next proposition:

Proposition A.3.1 *Let $\varphi\colon [0, +\infty[\longrightarrow \mathbf{C}$ be smooth and assume that φ and all its derivatives decay faster than any polynomial at infinity.*

(1) *The Mellin transform $\hat{\varphi}$ extends to a meromorphic function on $\mathrm{Re}(s) > -1$, with at most a simple pole at $s = 0$ with residue $\varphi(0)$.*

(2) *For any real numbers $-1 < \mathrm{A} < \mathrm{B}$, the Mellin transform has rapid decay in the strip $\mathrm{A} \leqslant \mathrm{Re}(s) \leqslant \mathrm{B}$, in the sense that for any integer $k \geqslant 1$, there exists a constant $\mathrm{C}_k \geqslant 0$ such that*

$$|\hat{\varphi}(s)| \leqslant \mathrm{C}_k(1 + |t|)^{-k}$$

for all $s = \sigma + it$ with $\mathrm{A} \leqslant \sigma \leqslant \mathrm{B}$ and $|t| \geqslant 1$.

(3) *For any $\sigma > 0$ and any $x \geqslant 0$, we have the Mellin inversion formula*

$$\varphi(x) = \frac{1}{2i\pi} \int_{(\sigma)} \hat{\varphi}(s) x^{-s} ds.$$

In the last formula, the notation $\int_{(\sigma)} (\cdots) ds$ refers to an integral over the vertical line $\mathrm{Re}(s) = \sigma$, oriented upward.

Proof (1) We integrate by parts in the definition of $\hat{\varphi}(s)$ for $\mathrm{Re}(s) > 0$ and obtain

$$\hat{\varphi}(s) = \left[\varphi(x)\frac{x^s}{s}\right]_0^{+\infty} - \frac{1}{s}\int_0^{+\infty} \varphi'(x)x^{s+1}\frac{dx}{x} = -\frac{1}{s}\int_0^{+\infty} \varphi'(x)x^{s+1}\frac{dx}{x}$$

since φ and φ' decay faster than any polynomial at ∞. It follows that $\psi(s) = s\hat{\varphi}(s)$ is holomorphic for $\mathrm{Re}(s) > -1$, and hence that $\hat{\varphi}(s)$ is meromorphic in this region. Since

$$\lim_{s \to 0} s\hat{\varphi}(s) = \psi(0) = -\int_0^{+\infty} \varphi'(x)dx = \varphi(0),$$

it follows that there is at most a simple pole with residue $\varphi(0)$ at $s = 0$.

(2) Iterating the integration by parts $k \geqslant 2$ times, we obtain for $\mathrm{Re}(s) > -1$ the relation

$$\hat{\varphi}(s) = \frac{(-1)^k}{s(s+1)\cdots(s+k)} \int_0^{+\infty} \varphi^{(k)}(x)x^{s+k}\frac{dx}{x}.$$

Hence, for $\mathrm{A} \leqslant \sigma \leqslant \mathrm{B}$ and $|t| \geqslant 1$, we obtain the bound

$$|\hat{\varphi}(s)| \ll \frac{1}{(1+|t|)^k} \int_0^{+\infty} |\varphi^{(k)}(x)|x^{\mathrm{B}+k}\frac{dx}{x} \ll \frac{1}{(1+|t|)^k}.$$

(3) We interpret $\hat{\varphi}(s)$, for $s = \sigma + it$ with $\sigma > 0$ fixed, as a Fourier transform; by means of the change of variable $x = e^y$, we have

$$\hat{\varphi}(s) = \int_0^{+\infty} \varphi(x) x^{\sigma} x^{it} \frac{dx}{x} = \int_{\mathbf{R}} \varphi(e^y) e^{\sigma y} e^{iyt} dy,$$

which shows that $t \mapsto \hat{\varphi}(\sigma + it)$ is the Fourier transform (with the above normalization) of the function $g(y) = \varphi(e^y)e^{\sigma y}$. Note that g is smooth and tends to zero very rapidly at infinity (for $y \to -\infty$, this is because φ is bounded close to 0, but $e^{\sigma y}$ then tends exponentially fast to 0). Therefore the Fourier inversion formula holds, and for any $y \in \mathbf{R}$, we obtain

$$\varphi(e^y) e^{\sigma y} = \frac{1}{2\pi} \int_{\mathbf{R}} \hat{\varphi}(\sigma + it) e^{-ity} dt.$$

Putting $x = e^y$, this translates to

$$\varphi(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \hat{\varphi}(\sigma + it) x^{-\sigma - it} dt = \frac{1}{2i\pi} \int_{(\sigma)} \hat{\varphi}(s) x^{-s} ds.$$

□

One of the most important functions of analysis is classically defined as a Mellin transform. This is the Gamma function of Euler, which is essentially the Mellin transform of the exponential function, or more precisely of $\exp(-x)$. In other words, we have

$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^s \frac{dx}{x}$$

for all complex numbers s such that $\mathrm{Re}(s) > 0$. Proposition A.3.1 shows that Γ extends to a meromorphic function on $\mathrm{Re}(s) > -1$, with a simple pole at $s = 0$ with residue 1. In fact, much more is true:

Proposition A.3.2 *The function $\Gamma(s)$ extends to a meromorphic function on $\mathbf{C}$ with only simple poles at $s = -k$ for all integers $k \geqslant 0$, with residue $(-1)^k/k!$. It satisfies*

$$\Gamma(s+1) = s\Gamma(s)$$

for all $s \in \mathbf{C}$, with the obvious meaning if s or $s+1$ is a pole, and in particular we have

$$\Gamma(n) = (n-1)!$$

for all integers $n \geqslant 0$.

Moreover, the function $1/\Gamma$ is entire.

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Proof It suffices to prove that $\Gamma(s+1) = s\Gamma(s)$ for $\mathrm{Re}(s) > 0$. Indeed, this formula proves, by induction on $k \geqslant 1$, that Γ has an analytic continuation to $\mathrm{Re}(s) > -k$, with a simple pole at $-k+1$, where the residue r_{-k+1} satisfies

$$r_{-k+1} = \frac{r_{-k+2}}{-k+1}.$$

This easily gives every statement in the proposition. And the formula we want is just a simple integration by parts away:

$$\Gamma(s+1) = \int_0^{+\infty} e^{-x} x^s dx = \left[-e^{-x} x^s\right]_0^{+\infty} + s \int_0^{+\infty} e^{-x} x^{s-1} dx = s\Gamma(s).$$

Since Γ is meromorphic, its inverse $1/\Gamma$ is also meromorphic; for the proof that $1/\Gamma$ is in fact entire (i.e., that $\Gamma(s) \neq 0$ for $s \in \mathbf{C}$), we refer to [116, p. 149] (it follows, e.g., from the formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

valid for all $s \in \mathbf{C}$, since the known poles of $\Gamma(1-s)$ are compensated by those of $1/\sin(\pi s)$). $\square$

An important feature of the Gamma function, which is often quite important, is that its asymptotic behavior in very wide ranges of the complex plane is very clearly understood. This is the so-called *Stirling formula*.

Proposition A.3.3 *Let $\alpha > 0$ be a real number and let X_α be either the set of $s \in \mathbf{C}$ such that $\mathrm{Re}(s) > \alpha$ or the set of $s \in \mathbf{C}$ such that $|\mathrm{Im}(s)| > \alpha$. We have*

$$\log \Gamma(s) = s \log s - s - \frac{1}{2} \log s + \frac{1}{2} \log 2\pi + \mathrm{O}(|s|^{-1}),$$

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + \mathrm{O}(s^{-2})$$

for any $s \in \mathrm{X}_\alpha$.

For a proof, see for instance [16, Ch. VII, Prop. 4].

A.4 Dirichlet Series

We present in this section some of the basic analytic properties of series of the type

$$\sum_{n \geqslant 1} a_n n^{-s},$$

where $a_n \in \mathbf{C}$ for $n \geqslant 1$. These are called *Dirichlet series*, and we refer to Titchmarsh's book [116, Ch. 9] for basic information about these functions.

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If $a_n = 0$ for n large enough (so that there are only finitely many terms), the series converges of course for all s, and the resulting function is called a *Dirichlet polynomial*.

Lemma A.4.1 *Let $(a_n)_{n\geqslant 1}$ be a sequence of complex numbers. Let $s_0 \in \mathbf{C}$. If the series*

$$\sum_{n\geqslant 1} a_n n^{-s_0}$$

converges, then the series

$$\sum_{n\geqslant 1} a_n n^{-s}$$

converges uniformly on compact subsets of $\mathrm{U} = \{s \in \mathbf{C} \mid \mathrm{Re}(s) > \mathrm{Re}(s_0)\}$. *In particular the function*

$$f(s) = \sum_{n\geqslant 1} a_n n^{-s}$$

is holomorphic on U.

Sketch of proof We may assume (by considering $a_n n^{-s_0}$ instead of a_n) that $s_0 = 0$. For any integers $\mathrm{N} < \mathrm{M}$, let

$$s_{\mathrm{N},\mathrm{M}} = a_{\mathrm{N}} + \cdots + a_{\mathrm{M}}.$$

By Cauchy's criterion, we have $s_{\mathrm{N},\mathrm{M}} \to 0$ as $\mathrm{N}, \mathrm{M} \to +\infty$. Suppose that $\sigma = \mathrm{Re}(s) > 0$. Let $\mathrm{N} < \mathrm{M}$ be integers. By the elementary summation by parts formula (Lemma A.1.1), we have

$$\sum_{\mathrm{N}\leqslant n\leqslant \mathrm{M}} a_n n^{-s} = a_{\mathrm{M}}\mathrm{M}^{-s} - \sum_{\mathrm{N}\leqslant n<\mathrm{M}} ((n+1)^{-s} - n^{-s}) s_{\mathrm{N},n}.$$

It is however also elementary that

$$|(n+1)^{-s} - n^{-s}| = \left| s\int_n^{n+1} x^{-s-1}dx \right| \leqslant \frac{|s|}{\sigma}(n^{-\sigma} - (n+1)^{-\sigma}). \quad \text{(A.2)}$$

Hence

$$\left| \sum_{\mathrm{N}\leqslant n\leqslant \mathrm{M}} a_n n^{-s} \right| \leqslant \frac{|s|}{\sigma} \max_{\mathrm{N}\leqslant n\leqslant \mathrm{M}} |s_{\mathrm{N},n}| \left(\frac{1}{\mathrm{N}^\sigma} - \frac{1}{\mathrm{M}^\sigma} \right).$$

It therefore follows by Cauchy's criterion that the Dirichlet series $f(s)$ converges uniformly in any region in $\mathbf{C}$ defined by the condition

$$\frac{|s|}{\sigma} \leqslant \mathrm{A}$$

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for some $A > 0$. This includes, for a suitable value of A, any compact subset of the half-plane $\{s \in \mathbf{C} \mid \sigma > 0\}$. ☐

In general, the convergence is *not* absolute. We can see in this lemma a first instance of a fairly general principle concerning Dirichlet series: if some particular property holds for some $s_0 \in \mathbf{C}$ (or for all s_0 with some fixed real part), then it holds – or even a stronger property holds – for any s with $\mathrm{Re}(s) > \mathrm{Re}(s_0)$.

This principle also applies in many cases to the possible analytic continuation of Dirichlet series beyond the region of convergence. The next proposition is another example, concerning the size of the Dirichlet series.

Proposition A.4.2 *Let $\sigma \in \mathbf{R}$ be a real number and let $(a_n)_{n\geqslant 1}$ be a bounded sequence of complex numbers such that the Dirichlet series*

$$f(s) = \sum_{n\geqslant 1} a_n n^{-s}$$

converges for $\mathrm{Re}(s) > \sigma$. *Then, for any $\sigma_1 > \sigma$, we have*

$$|f(s)| \ll 1 + |t|$$

uniformly for $\mathrm{Re}(s) \geqslant \sigma_1$.

Proof We may assume that $\sum a_n$ converges by replacing (a_n) by $(a_n n^{-\tau})$ for some $\tau > \sigma$. The partial sums

$$s_N = a_1 + \cdots + a_N$$

are then bounded. Let $s \in \mathbf{C}$ be such that $\sigma = \mathrm{Re}(s) > 0$. Then we have by partial summation

$$\sum_{n=1}^{N} \frac{a_n}{n^s} = \sum_{n=1}^{M} \frac{a_n}{n^s} + \sum_{n=M+1}^{N} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) s_n - \frac{s_M}{(M+1)^s} + \frac{s_N}{(N+1)^s}$$

for any integers $M \leqslant n$. Letting $N \to +\infty$, as we may, we get

$$f(s) = \sum_{n=1}^{M} \frac{a_n}{n^s} + \sum_{n>M} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) s_n - \frac{s_M}{(M+1)^s}.$$

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Applying (A.2), this leads to

$$|f(s)| \ll \sum_{n=1}^{M} \frac{1}{n^{\sigma}} + \frac{|s|}{\sigma} \sum_{n>M} \left(\frac{1}{n^{\sigma}} - \frac{1}{(n+1)^{\sigma}} \right) + \frac{1}{(M+1)^{\sigma}}$$
$$\ll M + tM^{-\sigma} + 1,$$

and the desired bounds follow by taking $M = \lceil \mathrm{Im}(s) \rceil$ (see also [116, 9.33]). □

In order to express in a practical manner a Dirichlet series outside of its region of convergence, one can use *smooth partial sums*, which exploit harmonic analysis.

Proposition A.4.3 *Let $\varphi\colon [0, +\infty[\longrightarrow [0,1]$ be a smooth function with compact support such that $\varphi(0) = 1$. Let $\hat{\varphi}$ denote its Mellin transform. Let $\sigma > 0$ be given with $0 < \sigma_0 < 1$, and let $(a_n)_{n\geqslant 1}$ be any sequence of complex numbers with $|a_n| \leqslant 1$ such that the Dirichlet series*

$$\sum_{n\geqslant 1} a_n n^{-s}$$

extends to a holomorphic function $f(s)$ in the region $\mathrm{Re}(s) > \sigma_0$ with at most a simple pole at $s = 1$ with residue $c \in \mathbf{C}$.

For $N \geqslant 1$, define

$$f_N(s) = \sum_{n\geqslant 1} a_n \varphi\left(\frac{n}{N}\right) n^{-s}.$$

Let σ be a real number such that $\sigma_0 < \sigma < 1$. Then we have

$$f(s) - f_N(s) = -\frac{1}{2i\pi} \int_{(-\delta)} f(s+w) N^{w} \hat{\varphi}(w)\,dw - cN^{1-s}\hat{\varphi}(1-s)$$

for any $s = \sigma + it$ and any $\delta > 0$ such that $-\delta + \sigma > \sigma_0$.

It is of course possible that $c = 0$, which corresponds to a Dirichlet series that is holomorphic for $\mathrm{Re}(s) > \sigma_0$.

This result gives a convergent approximation of $f(s)$, inside the strip $\mathrm{Re}(s) > \sigma_1$, using the *finite* sums $f_N(s)$ – the point is that $|N^{w}| = N^{-\delta}$, so that the polynomial growth of f on vertical lines combined with the fast decay of the Mellin transform $\hat{\varphi}$ shows that the integral on the right tends to 0 as $N \to +\infty$. Moreover, the shape of the formula makes it very accessible to further manipulations, as done in Chapter 3.

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Proof Fix $\alpha > 1$ such that the Dirichlet series $f(s)$ converges absolutely for $\mathrm{Re}(s) = \alpha$. By the Mellin inversion formula, followed by exchanging the order of the sum and integral, we have

$$\begin{aligned} f_{\mathrm{N}}(s) &= \sum_{n \geqslant 1} a_n n^{-s} \times \frac{1}{2i\pi} \int_{(\alpha)} \mathrm{N}^w n^{-w} \hat{\varphi}(w) dw \\ &= \frac{1}{2i\pi} \int_{(-\delta)} \Big(\sum_{n \geqslant 1} a_n n^{-s-w} \Big) \mathrm{N}^w \hat{\varphi}(w) dw \\ &= \frac{1}{2i\pi} \int_{(-\delta)} f(s+w) \mathrm{N}^w \hat{\varphi}(w) dw, \end{aligned}$$

where the absolute convergence justifies the exchange of sum and integral.

Now consider some $\mathrm{T} \geqslant 1$ and some δ such that $0 < \delta < 1$. Let $\mathcal{R}_{\mathrm{T}}$ be the rectangle in $\mathbf{C}$ with sides $[\alpha - i\mathrm{T}, \alpha + i\mathrm{T}]$, $[\alpha + i\mathrm{T}, -\delta + i\mathrm{T}]$, $[-\delta + i\mathrm{T}, -\delta - \mathrm{T}]$, $[-\delta - i\mathrm{T}, \alpha - i\mathrm{T}]$, oriented counterclockwise. Inside this rectangle, the function

$$w \mapsto f(s+w) \mathrm{N}^w \hat{\varphi}(w)$$

is meromorphic. It has a simple pole at $w = 0$, by our choice of δ and the properties of the Mellin transform of φ given by Proposition A.3.1, and the residue at $w = 0$ is $\varphi(0) f(s) = f(s)$, again by Proposition A.3.1. If $c \neq 0$, it may also have a simple pole at $w = 1 - s$, with residue equal to $c\mathrm{N}^{1-s} \hat{\varphi}(1-s)$.

Cauchy's theorem therefore implies that

$$f_{\mathrm{N}}(s) = f(s) + \frac{1}{2i\pi} \int_{\mathcal{R}_{\mathrm{T}}} f(s+w) \mathrm{N}^w \hat{\varphi}(w) dw + c\mathrm{N}^{1-s} \hat{\varphi}(1-s).$$

Now we let $\mathrm{T} \to +\infty$. Our assumptions imply that $w \mapsto f(s+w)$ has polynomial growth on the strip $-\delta \leqslant \mathrm{Re}(w) \leqslant \alpha$, and therefore the fast decay of $\hat{\varphi}$ (Proposition A.3.1 again) shows that the contribution of the two horizontal segments to the integral along $\mathcal{R}_{\mathrm{T}}$ tends to 0 as $\mathrm{T} \to +\infty$. Taking into account orientation, we get

$$f(s) - f_{\mathrm{N}}(s) = -\frac{1}{2i\pi} \int_{(-\delta)} f(s+w) \mathrm{N}^w \hat{\varphi}(w) dw - c\mathrm{N}^{1-s} \hat{\varphi}(1-s),$$

as claimed. □

We also recall the formula for the product of two Dirichlet series, which involves the so-called *Dirichlet convolution* (see also Section C.1 for more properties and examples of this operation).

Proposition A.4.4 *Let $(a(n))_{n \geqslant 1}$ and $(b(n))_{n \geqslant 1}$ be sequences of complex numbers. For any $s \in \mathbf{C}$ such that the Dirichlet series*

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$$A(s) = \sum_{n\geqslant 1} a(n)n^{-s} \quad \text{and} \quad B(s) = \sum_{n\geqslant 1} b(n)n^{-s},$$

converge absolutely, we have

$$A(s)B(s) = \sum_{n\geqslant 1} c(n)n^{-s},$$

where

$$c(n) = \sum_{\substack{d|n \\ d\geqslant 1}} a(d)b\left(\frac{n}{d}\right).$$

We will denote $c(n) = (a \star b)(n)$ and often abbreviate the definition by writing

$$(a \star b)(n) = \sum_{d|n} a(d)b\left(\frac{n}{d}\right) \quad \text{or} \quad (a \star b)(n) = \sum_{de=n} a(d)b(e).$$

Proof Formally, this is quite clear:

$$\begin{aligned} A(s)B(s) &= \left(\sum_{n\geqslant 1} a(n)n^{-s}\right)\left(\sum_{n\geqslant 1} b(m)m^{-s}\right) \\ &= \sum_{m,n\geqslant 1} a(n)b(m)(nm)^{-s} = \sum_{k\geqslant 1} k^{-s}\left(\sum_{mn=k} a(n)b(m)\right) = C(s). \end{aligned}$$

The assumptions are sufficient to allow us to rearrange the double series so that these manipulations are valid. □

A.5 Density of Certain Sets of Holomorphic Functions

Let D be a nonempty open disc in **C** and $\bar{D}$ its closure. We denote by H(D) the Banach space of all continuous functions $f : \bar{D} \longrightarrow \mathbf{C}$ that are holomorphic in D, with the norm

$$\|f\|_\infty = \sup_{z\in\bar{D}} |f(z)|.$$

We also denote by C(K) the Banach space of continuous functions on a compact space K, also with the norm

$$\|f\|_\infty = \sup_{x\in K} |f(x)|$$

(so that there is no risk of confusion if K = D and we apply this to a function that also belongs to H(D)). We denote by C(K)′ the dual of C(K), namely, the

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space of continuous linear functionals $\mathrm{C}(\mathrm{K}) \longrightarrow \mathbf{C}$. An element $\mu \in \mathrm{C}(\mathrm{K})'$ can also be interpreted as a *complex measure* on K (by the Riesz–Markov Theorem; see, e.g., [40, Th. 7.17]), and in this interpretation, one would write

$$\mu(f) = \int_{\mathrm{K}} f(x) d\mu(x)$$

for $f \in \mathrm{C}(\mathrm{K})$.

Theorem A.5.1 *Let* D *be as above. Let* $(f_n)_{n\geqslant 1}$ *be a sequence of elements of* H(D) *with*

$$\sum_{n\geqslant 1} \|f_n\|_\infty^2 < +\infty.$$

Let X *be the set of sequences* (α_n) *of complex numbers with* $|\alpha_n| = 1$ *such that the series*

$$\sum_{n\geqslant 1} \alpha_n f_n$$

converges in H(D).

Assume that X *is not empty and that, for any continuous linear functional* $\mu \in \mathrm{C}(\bar{\mathrm{D}})'$ *such that*

$$\sum_{n\geqslant 1} |\mu(f_n)| < +\infty, \tag{A.3}$$

the Laplace transform of μ *is identically* 0*. Then, for any* $\mathrm{N} \geqslant 1$*, the set of series*

$$\sum_{n\geqslant \mathrm{N}} \alpha_n f_n$$

for (α_n) *in* X *is* dense *in* H(D).

Here, the Laplace transform of μ is defined by

$$g(z) = \mu(w \mapsto e^{wz})$$

for $z \in \mathbf{C}$. In the interpretation of μ as a complex measure, which can be viewed as a complex measure on $\mathbf{C}$ that is supported on $\bar{\mathrm{D}}$, one would write

$$g(z) = \int_{\mathbf{C}} e^{wz} d\mu(w).$$

Proof This result is proved, for instance, in [4, Lemma 5.2.9], except that only the case $\mathrm{N} = 1$ is considered. However, if the assumptions hold for $(f_n)_{n\geqslant 1}$, they hold equally for $(f_n)_{n>\mathrm{N}}$, hence the general case follows. ☐

We will use the last part of the following lemma as a criterion to establish that the Laplace transform is zero in certain circumstances.

Lemma A.5.2 *Let* K *be a complex subset of* **C** *and* $\mu \in C(\mathrm{K})'$ *a continuous linear functional. Let*

$$g(z) = \int e^{wz} d\mu(z) = \mu(w \mapsto e^{wz})$$

be its Laplace transform.

(1) *The function* g *is an entire function on* **C**, *that is, it is holomorphic on* **C**.

(2) *We have*

$$\limsup_{|z| \to +\infty} \frac{\log |g(z)|}{|z|} < +\infty.$$

(3) *If* $g \neq 0$, *then*

$$\limsup_{r \to +\infty} \frac{\log |g(r)|}{r} \geqslant \inf_{z \in \mathrm{K}} \mathrm{Re}(z).$$

Proof (1) Let $z \in \mathbf{C}$ be fixed. For $h \neq 0$, we have

$$\frac{g(z+h) - g(z)}{h} = \mu(f_h),$$

where $f_h(w) = (e^{w(z+h)} - e^{wz})/h$. We have

$$f_h(w) \to we^{wz}$$

as $h \to 0$, and the convergence is uniform on K. Hence we get

$$\frac{g(z+h) - g(z)}{h} \longrightarrow \mu(w \mapsto we^{wz}),$$

which shows that g is holomorphic at z with derivative $\mu(w \mapsto we^{wz})$. Since z is arbitrary, this means that g is entire.

(2) We have

$$|g(z)| \leqslant \|\mu\| \, \|w \mapsto e^{wz}\|_\infty \leqslant \|\mu\| e^{|z|\mathrm{M}}$$

where $\mathrm{M} = \sup_{w \in \mathrm{K}} |w|$, and therefore

$$\limsup_{|z| \to +\infty} \frac{\log |g(z)|}{|z|} \leqslant \mathrm{M} < +\infty.$$

(3) This is proved, for instance, in [4, Lemma 5.2.2], using relatively elementary properties of entire functions satisfying growth conditions such as those in (2). ☐

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Finally, we will use the following theorem of Bernstein, extending a result of Pólya.

Theorem A.5.3 *Let* $g\colon \mathbf{C} \longrightarrow \mathbf{C}$ *be an entire function such that*

$$\limsup_{|z|\to+\infty} \frac{\log|g(z)|}{|z|} < +\infty.$$

Let (r_k) *be a sequence of positive real numbers, and let* α, β *be real numbers such that*

(1) *we have* $\alpha\beta < \pi$*;*

(2) *we have*

$$\limsup_{\substack{y\in\mathbf{R}\\ |y|\to+\infty}} \frac{\log|g(iy)|}{|y|} \leqslant \alpha;$$

(3) *we have* $|r_k - r_l| \gg |k-l|$ *for all* k, $l \geqslant 1$, *and* $r_k/k \to \beta$.

Then it follows that

$$\limsup_{k\to+\infty} \frac{\log|g(r_k)|}{r_k} = \limsup_{r\to+\infty} \frac{\log|g(r)|}{r}.$$

This is explained in Lemma [4, 5.2.3].

Example A.5.4 Taking $g(z) = \sin(\pi z)$, with $\alpha = 1$, $r_n = n\pi$ so that $\beta = \pi$, we see that the first condition is best possible.

We also use a relatively elementary lemma due to Hurwitz on zeros of holomorphic functions.

Lemma A.5.5 *Let* D *be a nonempty open disc in* $\mathbf{C}$. *Let* (f_n) *be a sequence of holomorphic functions in* H(D). *Assume* f_n *converges to* f *in* H(D). *If* $f_n(z) \neq 0$ *all* $n \geqslant 1$ *and* $z \in$ D, *then either* $f = 0$ *or* f *does not vanish on* D.

Proof We assume that f is not zero, and show that it has no zero in D. Let $z_0 \in$ D be fixed, and let C be a circle of radius $r > 0$ centered at z_0 such that C $\subset$ D and such that f has no zero, except possibly z_0, in the disc with boundary C. We have $\delta = \inf_{z\in \mathrm{C}} |f(z)| > 0$. For n large enough, we get

$$\sup_{z\in\mathrm{C}} |f(z) - f_n(z)| < \delta,$$

and then the relation $f = f - f_n + f_n$ combined with Rouché's Theorem (see, e.g., [116, 3.42]) shows that f has the same number of zeros as f_n in the disc bounded by C. This means that f has no zeros there and, in particular, that $f(z_0) \neq 0$. ☐