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# Rank-one non-singular actions of countable groups and their odometer factors

ALEXANDRE I. DANILENKO†‡ and MYKYTA I. VIEPRIK‡§

† *B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv 61164, Ukraine*
*e-mail: alexandre.danilenko@gmail.com*

‡ *Mathematical Institute of the Polish Academy of Sciences, ul. Śniadeckich 8, Warszawa 00-656, Poland*

§ *V. N. Karazin Kharkiv National University, 4 Svobody sq., Kharkiv 61077, Ukraine*
*(e-mail: nikita.veprik@gmail.com)*

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*Abstract.* For an arbitrary countable discrete infinite group $G$, non-singular rank-one actions are introduced. It is shown that the class of non-singular rank-one actions coincides with the class of non-singular $(C, F)$-actions. Given a decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups in $G$ with $\bigcap_{n=1}^{\infty} \bigcap_{g\in G} g\Gamma_n g^{-1} = \{1_G\}$, the projective limit of the homogeneous $G$-spaces $G/\Gamma_n$ as $n \to \infty$ is a $G$-space. Endowing this $G$-space with an ergodic non-singular non-atomic measure, we obtain a dynamical system which is called a non-singular odometer. Necessary and sufficient conditions are found for a rank-one non-singular $G$-action to have a finite factor and a non-singular odometer factor in terms of the underlying $(C, F)$-parameters. Similar conditions are also found for a rank-one non-singular $G$-action to be isomorphic to an odometer. Minimal Radon uniquely ergodic locally compact Cantor models are constructed for the non-singular rank-one extensions of odometers. Several concrete examples are constructed and several facts are proved that illustrate a sharp difference of the non-singular non-commutative case from the classical finite measure preserving one: odometer actions which are not of rank-one and factors of rank-one systems which are not of rank one; however, each probability preserving odometer is a factor of an infinite measure preserving rank-one system, etc.

Key words: rank-one transformation, odometer, factor
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## 1. *Introduction*

This work is motivated by a recent paper [**Fo–We**], where Foreman *et al* describe odometer factors of rank-one transformations in terms of the underlying cutting-and-stacking parameters. This description is considered as a step towards classification of the rank-one transformations up to isomorphism relation. Our purpose here is to generalize the main results of [**Fo–We**] in the following three directions:

- to consider actions of arbitrary countable infinite groups including non-amenable ones ([**Fo–We**] deals only with $\mathbb{Z}$-actions);
- to consider arbitrary non-singular group actions ([**Fo–We**] deals only with probability preserving actions); and
- to consider rank-one actions along an arbitrary sequence of shapes ([**Fo–We**] deals only with the classical rank one, that is, rank one along a sequence of intervals in $\mathbb{Z}$. In particular, our results hold for the *funny* rank-one probability preserving $\mathbb{Z}$-actions which were not studied in [**Fo–We**]).

We now briefly outline the content of the paper, which consists of six sections. Section 2 is divided into seven subsections. In §2.1, we define, for an arbitrary countable group $G$, non-singular $G$-actions of rank one. According to this definition, a non-singular $G$-action $T$ is of rank one if $T$ is free and $T$ admits a refining sequence of Rokhlin towers that approximate both the entire $\sigma$-algebra of Borel subsets and the $G$-orbits and, in addition, the Radon–Nikodym derivative of $T$ is constant on each transposition of the levels within every tower (see Definition 2.1). This extends the concept given in [**RuSi**] for $\mathbb{Z}$-actions. Definition 2.1 can be considered as an *abstract* definition of rank one. In the case of probability preserving $\mathbb{Z}$-actions, there exist several equivalent *constructive* definitions of this concept [**Fe**]. One of the most useful of these is the *cutting-and-stacking construction*, which explicitly associates a rank-one transformation to a sequence of integer-valued parameters. (Thus, the class of rank-one transformations is parametrized with a nice Polish space of integer parameters. However, different sequences of parameters can define isomorphic rank-one maps. A challenging open problem in this field is to find necessary and sufficient conditions for the parameters that determine isomorphic transformations.) This transformation is defined on the unit interval. It preserves the Lebesgue measure. A natural generalization of this construction for general countable groups was suggested in [**Da1**, **dJ2**] in similar but non-equivalent versions. We call it the $(C, F)$-construction. The most general version of the $(C, F)$-construction, including the versions from [**Da1**, **dJ2**] as particular cases, appeared in [**Da3**]. However, [**Da3**] deals only with *measure preserving* actions. In §§2.2–2.3 here, we define *non-singular* $(C, F)$-actions. Section 2.2 is preliminary: we define $(C, F)$-equivalence relations and related quasi-invariant $(C, F)$-measures. The non-singular $(C, F)$-actions related to the $(C, F)$-equivalence relations and $(C, F)$-measures appear in §2.3. They include all the non-singular rank-one transformations (and actions of Abelian groups) that have been studied earlier in the literature: see [**Aa**, **AdFrSi**, **Da1**, **Da2**, **DaSi**, **HaSi**] and references therein. The main result of §2 is the following (see Theorem 2.13).

THEOREM A. *Each non-singular $(C, F)$-action of $G$ is of rank one and each rank-one non-singular action of $G$ is isomorphic to a $(C, F)$-action.*

It is worth noting that if a probability preserving $G$-action is of rank one along a sequence $(F_n)_{n=1}^{\infty}$ of subsets in $G$, then $G$ is amenable and $(F_n)_{n=1}^{\infty}$ is left Følner (see Corollary 2.11(ii)).

Important concepts of telescoping and reduction for the parameters of $(C, F)$-actions are introduced in §2.4. They are used in §2.5 to construct continuous models of the non-singular $(C, F)$-actions. We remind that the famous Jewett–Krieger theorem provides strictly ergodic models for the ergodic probability preserving $\mathbb{Z}$-actions. In [**Yu**], an analogue of this theorem was proved for the *infinite* measure preserving ergodic transformations. In the present paper, we prove the existence of Radon uniquely ergodic minimal topological models for the *rank-one non-singular* actions (see Theorem 2.19).

THEOREM B. *If $(X, \mu, (T_g)_{g\in R})$ is a non-singular G-action of rank one, then there are a Radon uniquely ergodic minimal free continuous G-action $(R_g)_{g\in G}$ on a locally compact Cantor space Y, an R-quasi-invariant Radon measure $\nu$ on Y and a measure preserving isomorphism $\phi$ of $(X, \mu)$ onto $(Y, \nu)$ such that $\phi T_g \phi^{-1} = R_g$ almost everywhere and the Radon–Nikodym derivative $\rho_g := d\nu \circ R_g/d\nu$ is a continuous mapping from Y to $\mathbb{R}^*_+$ for each $g \in G$. Moreover, $\nu$ is the only (up to scaling) R-quasi-invariant Radon measure on Y whose Radon–Nikodym cocycle equals $(\rho_g)_{g\in G}$.*

We note that the continuity of the Radon–Nikodym derivatives was used essentially in [**DadJ**] for the almost continuous orbit classification of non-singular homeomorphisms of Krieger type *III*. Theorems A and B generalize respectively [**Da3**, Theorem 1.6 and Corollary 1.9], where only measure preserving systems were under consideration.

In §§2.6 and 2.7, we discuss the case of non-singular $\mathbb{Z}$-actions of rank one along intervals in more detail. It is shown in §2.6 that the $(C, F)$-construction in this case is equivalent to the classical cutting-and-stacking with a single tower at every step of the construction. However, in contrast with the measure preserving case, the towers are now divided into subtowers of different width. It is explained in §2.7 how the underlying $(C, F)$-parameters are used to present a rank-one non-singular transformation as a transformation built over a classical non-singular odometer of product type (called also Krieger's adding machine) and under a piecewise constant function.

Section 3 is devoted to the description of finite factors of rank-one non-singular actions. We remind that a *factor* of a dynamical system is an invariant sub-$\sigma$-algebra of measurable subsets. Equivalently, a factor of a system is a dynamical system which appears as the image of the original system under a non-singular equivariant mapping. Hence, the finite factors of an ergodic $G$-action correspond to the $G$-equivariant mappings onto homogeneous $G$-spaces $G/\Gamma$, where $\Gamma$ is a cofinite subgroup in $G$. Each non-singular $(C, F)$-action is parametrized by an underlying sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ of $(C, F)$-parameters, where $C_n$ and $F_{n-1}$ are finite subsets of $G$, $\kappa_n$ is a probability on $C_n$ and $\nu_{n-1}$ is measure on $F_{n-1}$ for each $n \in \mathbb{N}$. These parameters have to satisfy some conditions listed in §2.3. The following is the main result of §3 (a stronger version of it is proved as Theorem 3.3; see also Remark 3.4).

THEOREM C. *A non-singular $(C, F)$-action T of G has a finite factor $G/\Gamma$ if and only there is a telescoping $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ of the $(C, F)$-parameters of T and a coset $g\Gamma \in G/\Gamma$ such that*

$$\sum_{n=1}^{\infty} \kappa_n(\{c \in C_n \mid c \notin g\Gamma g^{-1}\}) < \infty.$$

*An explicit formula for the factor mapping is obtained.*

We note that if $G$ is Abelian, $\Gamma \subset G$ is a cofinite subgroup and the homogeneous $G$-space $G/\Gamma$ is a factor-space of an ergodic $G$-action $T$, then the corresponding factor-algebra of $T$ is defined uniquely. This is no longer true for non-Abelain $G$: we provide an example of a rank-one $G$-action $T$ and two $T$-invariant sub-$\sigma$-algebras $\mathfrak{F}_1 \neq \mathfrak{F}_2$ such that $T \restriction \mathfrak{F}_1$ and $T \restriction \mathfrak{F}_2$ are isomorphic $G$-actions on finite spaces (Example 3.5).

A criterion of total ergodicity for a non-singular $(C, F)$-action in terms of the underlying $(C, F)$-parameters is obtained as a corollary from Theorem C (see Corollary 3.6).

Starting from §4, we assume that $G$ is residually finite. Section 4 consists of two subsections. In §4.1, we consider *topological $G$-odometers* as the projective limits of homogeneous $G$-spaces $G/\Gamma_n$ for a decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups in $G$ such that $\bigcap_{n=1}^{\infty} \bigcap_{g\in G} g\Gamma_n g^{-1} = \{1_G\}$. By a *non-singular $G$-odometer*, we mean a topological $G$-odometer endowed with a $G$-quasi-invariant measure. Topological properties of odometers are not of our primary interest in the present work. Measure theoretical odometers (for general groups) were under study in [**DaLe**, **LiSaUg**], but only in the finite measure preserving case. In this paper, we study non-singular $G$-odometers. Some sufficient conditions for a non-singular odometer to be of rank one are found in Proposition 4.2. These conditions are satisfied for all known non-singular odometers (see Example 4.3). It is worth noting that there exist odometers which are not of rank one. Examples of non-rank-one probability preserving $G$-odometers for non-amenable $G$ are given in Example 4.4 and for amenable $G$ (including the Grigorchuk group) in Examples 4.5 and 4.6. However, each probability preserving $G$-odometer is a factor of an infinite measure preserving rank-one $G$-action (see Theorem 4.9 for a slightly stronger result):

THEOREM D. *For a topological $G$-odometer $O$ defined on a compact space $Y$, there exist:*

- *a rank-one measure preserving continuous $G$-action $T$ on a locally compact Cantor space $X$ equipped with a $\sigma$-finite measure $\mu$; and*
- *a $G$-equivariant continuous mapping $\pi : X \to Y$*

*such that $O$ is a factor of $T$ and the measure $\mu \circ \pi^{-1}$ is equivalent (that is, has the same ideal of subsets of zero measure) to the Haar measure on $Y$.*

Thus, a factor of a rank-one non-singular action is not necessarily of rank one. This is in contrast with the classical case of rank-one finite measure preserving $\mathbb{Z}$-actions [**Fe**]. Theorem 4.9 is about an interplay between odometer factors and an 'unordered' sequence of finite factors for an ergodic $G$-action. This theorem is trivial in the case where $G$ is Abelian.

Non-singular normal covers for non-singular odometers are introduced in §4.2. The existence of non-singular normal covers is proved in Proposition 4.11.

In §4, we study odometer factors of non-singular $(C, F)$-actions. The main result of the paper is the following (see Theorem 5.4).

THEOREM E. *Let $(X, \mu, T)$ be the non-singular $(C, F)$-action of $G$ associated with a sequence of $(C, F)$-parameters $\mathcal{T}$. Let $O$ be the topological $G$-odometer defined on the projective space $Y = \text{proj lim}_{n\to\infty} G/\Gamma_n$ corresponding to a nested sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups in $G$ such that $\bigcap_{n=1}^{\infty} \bigcap_{g\in G} g\Gamma_n g^{-1} = \{1_G\}$. A measurable $G$-equivariant mapping $\pi : X \to Y$ exists if and only if there are a telescoping $\mathcal{T}' = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ of $\mathcal{T}$ and an element $(g_n\Gamma_n)_{n=1}^{\infty} \in Y$ such that*

$$\sum_{n=1}^{\infty} \kappa_n(\{c \in C_n \mid c \notin g_n\Gamma_n g_n^{-1}\}) < \infty.$$

*An explicit formula for $\pi$ is obtained. Necessary and sufficient conditions for $\pi$ to be an isomorphism of $(X, \mu, T)$ onto $(Y, O, \mu \circ \pi^{-1})$ are given in terms of $\mathcal{T}'$.*

It is worth noting that each rank-one non-singular action $T$ is parametrized by the $(C, F)$-parameters $\mathcal{T}$ (see Theorem A) in a highly non-unique way. However, the properties of $\mathcal{T}$ specified in the statement of Theorem E (to determine an odometer factor $O$ or an isomorphism of $T$ with $O$) are independent on the choice of $\mathcal{T}$. Hence, Theorem E can be considered as a contribution to the classification problem for the rank-one non-singular systems.

As a corollary from Theorem E, criteria for a $(C, F)$-action to have an odometer factor or to be isomorphic to an odometer factor in terms of the underlying $(C, F)$-parameters are obtained in Corollaries 5.6 and 5.7, respectively. Corollary 5.8 provides minimal Radon uniquely ergodic models for the rank-one non-singular extensions of non-singular odometers. This corollary can be interpreted as a 'relative' counterpart of Theorem B.

THEOREM F. *Let $(X, \mu, T)$ be a rank-one non-singular $G$-action, $(Y, \nu, O)$ a non-singular $G$-odometer and $\pi : X \to Y$ a $G$-equivariant mapping with $\mu \circ \pi^{-1} = \nu$. Then there exist a locally compact Cantor space $\widetilde{X}$, a minimal Radon uniquely ergodic free continuous $G$-action $\widetilde{T}$ on $\widetilde{X}$, a continuous $G$-equivariant mapping $\widetilde{\pi} : X \to Y$ and a Borel isomorphism $R : X \to \widetilde{X}$ such that $\mu \circ R^{-1}$ is a Radon measure on $\widetilde{X}$, $RT_gR^{-1} = \widetilde{T}_g$ for each $g \in G$, the Radon–Nikodym derivative $\rho_g := d(\mu \circ R^{-1}) \circ \widetilde{T}_g / d(\mu \circ R^{-1})$ is a continuous mapping from $\widetilde{X}$ to $\mathbb{R}_+^*$ for each $g \in G$ and $\widetilde{\pi} \circ R = \pi$. Moreover, $\widetilde{T}$ is also Radon $(\rho_g)_{g\in G}$-uniquely ergodic.*

It follows from the Glimm–Effros theorem (see [**DaSi**, **Ef**]) that each topological odometer $(Y, O)$ (in fact, each topological $G$-action with a recurrent point) has uncountably many ergodic quasi-invariant measures. However, the space of these measures is huge and 'wild' to describe it in good parameters. Using Theorems E and F, we can isolate a good class of ergodic finite quasi-invariant measures that admits a good parametrization. This is the class of factor measures on $Y$ for all rank-one non-singular $G$-actions for which $Y$ is a factor. Every such measure can be parametrized by the $(C, F)$-parameters (see Corollary 5.9).

Section 6 is devoted completely to construction of five concrete rank-one actions with odometer factors and interesting properties. In §6.1, we continue studying the example of non-odometer rank-one probability preserving $\mathbb{Z}$-action $(X, \mu, T)$ from [**Fo–We**]. It was shown there that the maximal odometer factor $\mathfrak{F}$ of $T$ is non-trivial and isomorphic to the classical 2-adic odometer. We prove that $\mathfrak{F}$ is the Kronecker factor of $T$ and that $T$ is an uncountable-to-one extension of $\mathfrak{F}$. It follows, in particular, that the spectrum of $T$ has a continuous component. In §6.2, we consider non-singular counterparts of the aforementioned system $(X, \mu, T)$. In particular, for each $\lambda \in [0, 1]$, we construct a measure $\mu_\lambda$ on $X$ such that:

- the triple $(X, \mu_\lambda, T)$ is a rank-one non-singular system of Krieger type $III_\lambda$;
- $(X, \mu_\lambda, T)$ has a factor $\mathfrak{F}$ which is isomorphic to the probability preserving 2-adic odometer;
- $\mathfrak{F}$ is the maximal (in the class of non-singular odometers) factor of $(X, \mu_\lambda, T)$;
- the extension $X \to \mathfrak{F}$ is uncountable-to-one (mod $\mu_\lambda$).

In the $III_0$-case, we extend this result to systems whose associated flow is an arbitrary finitary AT in the sense of Connes and Woods [**CoWo**]. In §6.3, we provide an example of rank-one $\mathbb{Z}^2$-action $T = (T_g)_{g\in\mathbb{Z}^2}$ such that the generators $T_{(0,1)}$ and $T_{(1,0)}$ have $\mathbb{Z}$-odometer factors, but $T$ has no $\mathbb{Z}^2$-odometer factor. Another construction of such an action has appeared earlier in [**JoMc**, §6], but our example is much simpler. In §6.4, we construct a rank-one action $T$ of the Heisenberg group $H_3(\mathbb{Z})$ which has an odometer factor $\mathfrak{F}$, but which is not isomorphic to any odometer action. We show there that $\mathfrak{F}$ is the maximal odometer factor of $T$ and the extension $T \to \mathfrak{F}$ is uncountable-to-one. In §6.5, we provide an example of non-normal $H_3(\mathbb{Z})$-odometer which is canonically isomorphic to a normal odometer.

The final §7 is devoted to the article [**JoMc**] which appeared in the course of our work on the present paper. The purpose of [**JoMc**] is the same as ours: to generalize [**Fo–We**]. However, only finite measure preserving actions of amenable groups and only normal odometers are studied in [**JoMc**]. Therefore, in §7, we discuss the results of [**JoMc**] and compare them with results of the present paper.

## 2. *Rank-one non-singular actions of countable groups and $(C, F)$-construction*

2.1. *Non-singular actions of rank one.* Let $G$ be a discrete infinite countable group. Let $T = (T_g)_{g\in G}$ be a free non-singular action of $G$ on a standard $\sigma$-finite non-atomic measure space $(X, \mathfrak{B}, \mu)$. By a *Rokhlin tower for $T$*, we mean a pair $(B, F)$, where $B \in \mathfrak{B}$ with $0 < \mu(B) < \infty$ and $F$ is a finite subset of $G$ with $1_G \in F$ such that:

- the subsets $T_f B$, $f \in F$, are mutually disjoint;
- the Radon–Nikodym derivative $d\mu \circ T_f / d\mu$ is constant on $B$ for each $f \in F$.

Given a Rokhlin tower $(B, F)$, we let $X_{B,F} := \bigsqcup_{f\in F} T_f B \in \mathfrak{B}$. Of course, $\mu(X_{B,F}) < \infty$. By $\xi_{B,F}$, we mean the finite partition of $X_{B,F}$ into the subsets $T_f B$, $f \in F$. If $x \in T_f B$, then we set $O_{B,F}(x) := \{T_g x \mid g \in F f^{-1}\}$.

*Definition 2.1.* Let $\{1_G\} = F_0 \subset F_1 \subset F_2 \subset \cdots$ be an increasing sequence of finite subsets in $G$. We say that $T$ is of *rank-one along* $(F_n)_{n=0}^\infty$ if there is a decreasing sequence $B_0 \supset B_1 \supset \cdots$ of subsets of positive measure in $X$ such that $(B_n, F_n)$ is a Rokhlin tower for $T$ for each $n \in \mathbb{N}$ and:

(i) $\xi_{B_n,F_n} \prec \xi_{B_{n+1},F_{n+1}}$ for each $n \geq 0$ and $\bigvee_{n=0}^\infty \xi_{B_n,F_n}$ is the partition of $X$ into singletons (mod 0);

(ii) $\{T_g x \mid g \in G\} = \bigcup_{n=1}^\infty O_{B_n,F_n}(x)$ for almost every (a.e.) $x \in X$.

It follows from property (i) that $X_{B_0,F_0} \subset X_{B_1,F_1} \subset X_{B_2,F_2} \subset \cdots$ and $\bigcup_{n=0}^{\infty} X_{B_n,F_n} = X$. The piecewise constant property of the Radon–Nikodym derivative on the Rokhlin towers yields that:

(iii) if $T_c B_{n+1} \subset B_n$ for some $c \in F_{n+1}$, then

$$\frac{\mu(T_{fc}B_{n+1})}{\mu(T_f B_n)} = \frac{\mu(T_c B_{n+1})}{\mu(B_n)} \quad \text{for each } f \in F_n \text{ and } n \geq 0.$$

PROPOSITION 2.2. *Let $T$ satisfy condition (i) from Definition 2.1. Then $T$ is ergodic. In particular, every rank-one non-singular action is ergodic.*

*Proof.* Let two subsets $A_1, A_2 \in \mathfrak{B}$ be of positive measure. It follows from condition (i) that there are $n > 0$ and $f_1, f_2 \in F_n$ such that

$$\mu(A_1 \cap T_{f_1} B_n) > 0.9\mu(T_{f_1} B_n) \quad \text{and} \quad \mu(A_2 \cap T_{f_2} B_n) > 0.9\mu(T_{f_2} B_n).$$

As $(B_n, F_n)$ is a Rokhlin tower, $T_{f_2 f_1^{-1}} T_{f_1} B_n = T_{f_2} B_n$ and

$$\frac{d\mu \circ T_{f_2 f_1^{-1}}}{d\mu}(x) = \frac{\mu(T_{f_2} B_n)}{\mu(T_{f_1} B_n)} \quad \text{at a.e. } x \in A_1.$$

It follows that

$$\begin{aligned}\mu(T_{f_2 f_1^{-1}} A_1 \cap T_{f_2} B_n) &= \mu(T_{f_2 f_1^{-1}}(A_1 \cap T_{f_1} B_n)) \\ &= \mu(A_1 \cap T_{f_1} B_n) \frac{\mu(T_{f_2} B_n)}{\mu(T_{f_1} B_n)} \\ &> 0.9\mu(T_{f_2 B_n}).\end{aligned}$$

Therefore, $\mu(T_{f_2 f_1^{-1}} A_1 \cap T_{f_2} B_n \cap A_2) > 0.8\mu(T_{f_2} B_n)$. Hence, $\mu(T_{f_2 f_1^{-1}} A_1 \cap A_2) > 0$, as desired. □

2.2. *$(C, F)$-equivalence relations and non-singular $(C, F)$-measures.* Fix two sequences $(F_n)_{n \geq 0}$ and $(C_n)_{n \geq 1}$ of finite subsets in $G$ such that $F_0 = \{1_G\}$ and for each $n > 0$,

$$\begin{aligned}&1_G \in F_n \cap C_n, \ \#C_n > 1, \\ &F_n C_{n+1} \subset F_{n+1}, \\ &F_n c \cap F_n c' = \emptyset \quad \text{if } c, c' \in C_{n+1} \text{ and } c \neq c'.\end{aligned} \tag{2.1}$$

We let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ and endow this set with the infinite product topology. Then $X_n$ is a compact Cantor space. The mapping

$$X_n \ni (f_n, c_{n+1}, c_{n+2}, \ldots) \mapsto (f_n c_{n+1}, c_{n+2}, \ldots) \in X_{n+1}$$

is a continuous embedding of $X_n$ into $X_{n+1}$. Therefore, the topological inductive limit $X$ of the sequence $(X_n)_{n \geq 0}$ is well defined. Moreover, $X$ is a locally compact Cantor space. Given a subset $A \subset F_n$, we let

$$[A]_n := \{x = (f_n, c_{n+1}, \ldots) \in X_n, f_n \in A\}$$

and call this set an *n-cylinder* in $X$. It is open and compact in $X$. For brevity, we will write $[f]_n$ for $[\{f\}]_n$ for an element $f \in F_n$.

We remind that two points $x = (f_n, c_{n+1}, \ldots)$ and $x' = (f'_n, c'_{n+1}, \ldots)$ of $X_n$ are *tail equivalent* if there is $N > n$ such that $c_l = c'_l$ for each $l > N$. We thus obtain the tail equivalence relation on $X_n$.

*Definition 2.3.* [Da3] The $(C, F)$*-equivalence relation (or the tail equivalence relation)* $\mathcal{R}$ on $X$ is defined as follows: for each $n \geq 0$, the restriction of $\mathcal{R}$ to $X_n$ is the tail equivalence relation on $X_n$.

The following properties of $\mathcal{R}$ are easy to check:

- each $\mathcal{R}$-class is countable;
- $\mathcal{R}$ is *minimal*, that is, the $\mathcal{R}$-class of every point is dense in $X$;
- $\mathcal{R}$ is *hyperfinite*, that is, there is a sequence $(\mathcal{S}_n)_{n=1}^\infty$ of subrelations of $\mathcal{R}$ such that $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots$, $\bigcup_{n=1}^\infty \mathcal{S}_n = \mathcal{R}$ and $\#\mathcal{S}_n(x) < \infty$ for each $x \in X$ and $n > 0$. Indeed, we can define $\mathcal{S}_n$ by the following: $(x, y) \in \mathcal{S}_n$ if either $x, y \notin X_n$ and $x = y$ or $x = (f_n, c_{n+1}, \ldots) \in X_n$, $y = (f'_n, c'_{n+1}, \ldots) \in X_n$ and $c_m = c'_m$ for all $m > n$.

We recall that the *full group* $[\mathcal{R}]$ *of* $\mathcal{R}$ is the group of all Borel bijections $\gamma : X \to X$ such that $(x, \gamma x) \in \mathcal{R}$ for each $x \in X$. A Borel measure $\mu$ on $X$ is called $\mathcal{R}$-quasi-invariant if $\mu \circ \gamma \sim \mu$ for each $\gamma \in [\mathcal{R}]$. Then there is a Borel mapping $\rho_\mu : \mathcal{R} \to \mathbb{R}^*_+$ such that

$$\rho_\mu(x, y)\rho_\mu(y, z) = \rho_\mu(x, z) \quad \text{for all } (x, y), (y, z) \in \mathcal{R}$$

and $\rho_\mu(\gamma x, x) = (d\mu \circ \gamma / d\mu)(x)$ at a.e. $x \in X$ for each $\gamma \in [\mathcal{R}]$. The mapping $\rho_\mu$ is called *the Radon–Nikodym cocycle of* $(\mathcal{R}, \mu)$.

Suppose that for each $n \in \mathbb{N}$, a non-degenerated probability measure $\kappa_n$ on $C_n$ is given. We now let $\mu_0 := \nu_0 \otimes \kappa_1 \otimes \kappa_2 \otimes \cdots$, where $\nu_0$ is the Dirac measure supported at $1_G$. Then, $\mu_0$ is an $(\mathcal{R} \restriction X_0)$-quasi-invariant probability on $X_0$. Of course, $\mu_0$ is non-atomic if and only if

$$\prod_{n>0} \max_{c \in C_n} \kappa_n(c) = 0. \tag{2.2}$$

By the Kolmogorov 0-1 law, $(\mathcal{R} \restriction X_0)$ is ergodic on the probability space $(X_0, \mu_0)$. There are many ways to extend $\mu_0$ to an $\mathcal{R}$-quasi-invariant measure on $X$. However, all such measures will be mutually equivalent. Select for each $n \in \mathbb{N}$ a non-degenerated finite measure $\nu_n$ on $F_n$ such that

$$\nu_{n+1}(fc) = \nu_n(f)\kappa_{n+1}(c) \quad \text{for each } f \in F_n \text{ and } c \in C_{n+1}. \tag{2.3}$$

It is often convenient to consider $\nu_n$ and $\kappa_n$ as finite measures on $G$ supported on $F_n$ and $C_n$, respectively. Then equation (2.3) means that $\nu_{n+1} \restriction F_n C_{n+1} = \nu_n * \kappa_{n+1}$, where the symbol $*$ means the convolution. We now define a Borel measure $\mu$ on $X$ by setting

$$\mu([f]_n) := \nu_n(f) \quad \text{for each } g \in F_n \text{ and every } n \in \mathbb{N}.$$

It is straightforward to verify that $\mu$ is a well-defined $\sigma$-finite Radon measure. Moreover, $\mu$ is $\mathcal{R}$-quasi-invariant and

$$\rho_\mu(x, y) = \frac{\nu_n(f_n)}{\nu_n(f'_n)} \prod_{m>n} \frac{\kappa_m(c_m)}{\kappa_m(c'_m)},$$

whenever $x = (f_n, c_{n+1}, \ldots)$ and $y = (f'_n, c'_{n+1}, \ldots)$ are $\mathcal{R}$-equivalent points that belong to $X_n = F_n \times C_{n+1} \times C_{n+2} \times \cdots$ for some $n > 0$.

The following definition extends [**Da1**, Definition 4.2], where the case of Abelian $G$ was considered.

*Definition 2.4.* If equations (2.2) and (2.3) hold, then we call $\mu$ *the* $(C, F)$*-measure on* $X$ *determined by* $(\kappa_n)_{n=1}^\infty$ *and* $(\nu_n)_{n=0}^\infty$.

Consider another sequence $(\nu'_n)_{n=0}^\infty$ of non-degenerated measures on $(F_n)_{n=0}^\infty$ (in $n$) such that $\nu'_0$ is the Dirac measure supported at $1_G$ and $\nu'_{n+1}(fc) = \nu'_n(f)\kappa_{n+1}(c)$ for each $f \in F_n$ and $c \in C_{n+1}$ for each $n > 0$. Then, the $(C, F)$-measure $\mu'$ determined by $(\kappa_n)_{n=1}^\infty$, and $(\nu'_n)_{n=0}^\infty$ is equivalent to $\mu$ and

$$\frac{d\mu'}{d\mu}(x) = \frac{\nu'_n(f_n)}{\nu_n(f_n)} \quad \text{if } x = (f_n, \ldots) \in X_n.$$

Another useful observation is that given $(\kappa_n)_{n=1}^\infty$, we can always find $(\nu_n)_{n=0}^\infty$ satisfying equation (2.3). Thus, the equivalence class of a non-singular $(C, F)$-measure is completely determined by $(\kappa_n)_{n=1}^\infty$ alone. In particular, we may always replace a $\sigma$-finite non-singular $(C, F)$-measure with an equivalent finite non-singular $(C, F)$-measure.

*Remark 2.5.* We note that $\mathcal{R}$ is *Radon uniquely ergodic*, that is, there is a unique $\mathcal{R}$-invariant Radon measure $\xi$ on $X$ such that $\xi(X_0) = 1$. We call it the *Haar measure for* $\mathcal{R}$. It is $\sigma$-finite. Let $k_n$ be the equidistribution on $C_n$ and let $\nu_n(f) = \prod_{k=1}^n \kappa_k(1_G)$ for each $f \in F_n$ and $n \geq 0$. Then, equations (2.2) and (2.3) hold for $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$. Of course, the Haar measure for $\mathcal{R}$ is a $(C, F)$-measure determined by $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$. The Haar measure is finite if and only if

$$\prod_{n=1}^\infty \frac{\#F_{n+1}}{\#F_n \#C_{n+1}} < \infty.$$

It is easy to verify that $\mathcal{R}$ is conservative and ergodic on the $\sigma$-finite measure space $(X, \mu)$. This means that for each $\mathcal{R}$-invariant subset $A \subset X$, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Since the set of quasi-invariant probability measures with a fixed Radon–Nikodym derivative is a simplex [**GrSc**], it makes sense to introduce the following definition.

*Definition 2.6.* Let $\mathcal{S}$ be a Borel countable equivalence relation on a locally compact Polish space $Z$. Given a Borel cocycle $\rho : \mathcal{S} \to \mathbb{R}^*_+$, we say that $\mathcal{S}$ is *Radon* $\rho$*-uniquely ergodic* if there is a unique (up to scaling) Radon $\mathcal{S}$-quasi-invariant measure $\lambda$ on $Z$ such that $\rho_\lambda = \rho$.

PROPOSITION 2.7. *Let $\mu$ be a $(C, F)$-measure on X determined by two sequences $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$ of finite measures satisfying equations (2.2) and (2.3). Then, $\mathcal{R}$ is Radon $\rho_\mu$-uniquely ergodic.*

*Proof.* Let $\lambda$ be a Radon measure on $X$ such that $\rho_\lambda = \rho_\mu$ and $\lambda(X_0) = 1$. We will prove that $\lambda = \mu$. For that, it suffices to show that $\lambda([f]_n) = \mu([f]_n)$ for all $f \in F_n$ and $n \geq 0$. As

$$\mu([f]_n) = \frac{\nu_n(f)}{\nu_n(1_G)}\mu([1_G]_n) \quad \text{and} \quad \lambda([f]_n) = \frac{\nu_n(f)}{\nu_n(1_G)}\lambda([1_G]_n),$$

it is enough to prove that $\mu([1_G]_n) = \lambda([1_G]_n)$ for each $n \geq 0$. This will be done inductively. Of course, $\mu(X_0) = \mu([1_G]_0) = \lambda([1_G]_0) = \lambda(X_0) = 1$. Suppose that $\mu([1_G]_n) = \lambda([1_G]_n)$ for some $n$. Then, for each $c \in C_{n+1}$,

$$\lambda([c]_{n+1}) = \frac{\nu_n(c)}{\nu_n(1_G)}\lambda([1_G]_{n+1}) = \frac{\nu_n(1_G)\kappa_{n+1}(c)}{\nu_n(1_G)\kappa_{n+1}(1_G)}\lambda([1_G]_{n+1}).$$

Since $[1_G]_n = \bigsqcup_{c\in C_{n+1}}[c]_{n+1}$, we obtain that

$$\frac{\lambda([1_G]_n)}{\lambda([1_G]_{n+1})} = \frac{\sum_{c\in C_{n+1}}\lambda([c]_{n+1})}{\lambda([1_G]_{n+1})} = \frac{\sum_{c\in C_{n+1}}\kappa_{n+1}(c)}{\kappa_{n+1}(1_G)} = \frac{1}{\kappa_{n+1}(1_G)} = \frac{\mu([1_G]_n)}{\mu([1_G]_{n+1})}.$$

Hence, $\lambda([1_G]_{n+1}) = \mu([1_G]_{n+1})$, as desired. □

2.3. *Non-singular $(C, F)$-actions.* Non-singular $(C, F)$-actions were defined in [**Da1**, **Da2**] for Abelian groups only. We extend this definition to arbitrary countable groups. Given $g \in G$, let

$$X_n^g := \{(f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n \mid gf_n \in F_n\}.$$

Then, $X_n^g$ is a compact open subset of $X_n$ and $X_n^g \subset X_{n+1}^g$. Hence, the union $X^g := \bigcup_{n\geq 0} X_n^g$ is an open subset of $X$. Let $X^G := \bigcap_{g\in G} X^g$. Then, $X^G$ is a $G_\delta$-subset of $X$. Hence, $X^G$ is Polish and totally disconnected in the induced topology. Given $g \in G$ and $x \in X_G$, there is $n > 0$ such that $x = (f_n, c_{n+1}, \ldots) \in X_n$ and $gf_n \in F_n$. We now let $T_gx := (gf_n, c_{n+1}, \ldots) \in X_n \subset X$. It is straightforward to verify that:

(i) $T_gx \in X^G$;
(ii) the mapping $T_g : X^G \ni x \mapsto T_gx \in X^G$ is a homeomorphism of $X^G$; and
(iii) $T_gT_{g'} = T_{gg'}$ for all $g, g' \in G$.

Hence, $T := (T_g)_{g\in G}$ is a continuous $G$-action on $X^G$.

*Definition 2.8.* [**Da3**] The action $T$ is called *the topological $(C, F)$-action of G associated with $(C_n, F_{n-1})_{n=1}^\infty$.*

This action is free. The subset $X^G$ is $\mathcal{R}$-invariant. The $T$-orbit equivalence relation coincides with the restriction of $\mathcal{R}$ to $X^G$.

PROPOSITION 2.9. [**Da3**, Proposition 1.2] *$X^G = X$ if and only if for each $g \in G$ and $n > 0$, there is $m > n$ such that*

$$gF_nC_{n+1}C_{n+2}\cdots C_m \subset F_m. \tag{2.4}$$

Thus, if equation (2.4) holds, then $T$ is a minimal continuous $G$-action on a locally compact Cantor space $X$. Moreover, $T$ is *Radon uniquely ergodic*, that is, there exists a unique $T$-invariant Radon measure $\xi$ on $X$ such that $\xi(X_0) = 1$.

From now on, $T$ is a topological $(C, F)$-action of $G$ on $X^G$ and $\mu$ is the non-singular $(C, F)$-measure on $X$ determined by $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$ satisfying equations (2.2) and (2.3). Since $X^G$ is $\mathcal{R}$-invariant, we obtain that either $\mu(X^G) = 0$ or $\mu(X \setminus X^G) = 0$. In the latter case, $T$ is $\mu$-non-singular, conservative and ergodic.

PROPOSITION 2.10. *The following are equivalent.*

(i) $\mu(X \setminus X^G) = 0$.

(ii) *For each $g \in G$ and every $n \geq 0$,*

$$\lim_{m\to\infty} \nu_m((F_n C_{n+1} C_{n+2} \cdots C_m) \cap g^{-1} F_m) = \nu_n(F_n).$$

(iii) *For each $g \in G$,*

$$\lim_{m\to\infty} \kappa_1 * \cdots * \kappa_m(g^{-1} F_m) = 1.$$

*Proof.* (i)⇔(ii) Since $\mu(X \setminus X^G) = 0$ if and only if $\mu(X_n \cap X_m^g) \to \mu(X_n)$ as $m \to \infty$ for each $g \in G$ and $n \geq 0$, it suffices to note that

$$\begin{aligned}\mu(X_n \cap X_m^g) &= \mu([F_n]_n \cap [F_m \cap g^{-1} F_m]_m)\\ &= \mu([F_n C_{n+1} \cdots C_m]_m \cap [F_m \cap g^{-1} F_m]_m)\\ &= \mu([F_n C_{n+1} \cdots C_m \cap F_m \cap g^{-1} F_m]_m)\\ &= \nu_m((F_n C_{n+1} \cdots C_m) \cap g^{-1} F_m)\end{aligned}$$

and $\mu(X_n) = \mu([F_n]_n) = \nu_n(F_n)$.

(ii)⇒(iii) We set $\kappa_{1,m} := \kappa_1 * \cdots * \kappa_m$. Then,

$$\begin{aligned}\kappa_{1,m}((C_1 \cdots C_m) \setminus g^{-1} F_m) &= \nu_m((F_0 C_1 \cdots C_m) \setminus g^{-1} F_m)\\ &= \nu_m(F_0 C_1 \cdots C_m) - \nu_m((F_0 C_1 \cdots C_m) \cap g^{-1} F_m)\\ &= \nu_0(F_0) - \nu_m((F_0 C_1 \cdots C_m) \cap g^{-1} F_m).\end{aligned}$$

Hence, $\lim_{m\to\infty} \kappa_{1,m}((C_1 \cdots C_m) \setminus g^{-1} F_m) = 0$ according to item (ii). As $\kappa_{1,m}$ is supported on $C_1 \cdots C_m$, it follows that

$$\begin{aligned}\lim_{m\to\infty} \kappa_{1,m}(g^{-1} F_m) &= \lim_{m\to\infty} \kappa_{1,m}((C_1 \cdots C_m) \cap g^{-1} F_m)\\ &= \lim_{m\to\infty} \kappa_{1,m}(C_1 \cdots C_m) = 1,\end{aligned}$$

as desired.

(iii)⇒(i) Fix $g \in G$. Take arbitrary $n \geq 0$ and $f \in F_n$. Then, it follows from property (iii) that for $\mu$-a.e. $x = (1_G, c_{n+1}, c_{n+2}, \ldots) \in [1_G]_n$, there exists $m > 0$ such that $gfc_{n+1} \cdots c_m \in F_m$. This means that for $\mu$-a.e. $y = (f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n$,

$$g f_n c_{n+1} \cdots c_m \in F_m \text{ eventually in } m,$$

that is, $y \in X^g$. Hence, $\mu(X \setminus X^g) = 0$ and property (i) follows. □

In the case where $\mu$ is the Haar measure for $\mathcal{R}$, the equivalence (i)$\Leftrightarrow$(ii) of Proposition 2.10 was proved in [**Da3**].

COROLLARY 2.11.

(i) *If* $\mu(X \setminus X^G) = 0$ *and* $\mu(X) < \infty$, *then* $\nu_n(F_n \triangle gF_n) \to 0$ *as* $n \to \infty$ *for each* $g \in G$.

(ii) *If* $\mu(X \setminus X^G) = 0$, $\mu(X) < \infty$ *and* $\mu$ *is the Haar measure for* $\mathcal{R}$, *then* $G$ *is amenable and* $(F_n)_{n=1}^\infty$ *is a left Følner sequence in G.*

(iii) *If* $\mu(X \setminus X^G) = 0$, $\mu(X) < \infty$, $\mu$ *is the Haar measure for* $\mathcal{R}$ *and there exists a subgroup H of G such that* $C_n \subset H$ *eventually in n, then H is of finite index in G.*

*Proof.* (i) We note that $\nu_n(F_n) = \mu([F_n]_n) = \mu(X_n) \to \mu(X)$ as $n \to \infty$. Hence, it follows from Proposition 2.10(ii) that for each $\epsilon > 0$, there is $n > 0$ such that if $m > n$, then

$$\nu_m(F_n C_{n+1} C_{n+2} \cdots C_m) > (1-\epsilon)\nu_m(F_m) \quad \text{and}$$
$$\nu_m((F_n C_{n+1} C_{n+2} \cdots C_m) \cap gF_m) > (1-\epsilon)\nu_m(F_m).$$

Hence, $\nu_m(F_m \cap gF_m) > (1-2\epsilon)\nu_m(F_m)$. It follows that $\lim_{m\to\infty} \nu_m(F_m \triangle gF_m) = 0$, as desired.

(ii) Since $\mu$ is the Haar measure for $\mathcal{R}$, it follows that $\nu_n(A) = \#A/\#C_1 \cdots \#C_n$ for each subset $A \subset F_n$. Since $\mu$ is finite, there exists a limit

$$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} = \mu(X).$$

This fact and condition (i) yield that for each $g \in G$,

$$0 = \lim_{m\to\infty} \nu_n(F_n \triangle gF_n) = \lim_{m\to\infty} \frac{\#(F_n \triangle gF_n)}{\#C_1 \cdots \#C_n} = \mu(X) \lim_{m\to\infty} \frac{\#(F_n \triangle gF_n)}{\#F_n}.$$

Hence, $(F_n)_{n=1}^\infty$ is a left Følner sequence in $G$. Therefore, $G$ is amenable.

(iii) Suppose that $H$ is of infinite index in $G$. We first prove an auxiliary claim.

*Claim A.* For each finite subset $S \subset G$, there exists an element $g \in G$ such that $g \notin \bigcup_{a,b\in S} aHb^{-1}$.

By condition (ii), $G$ is amenable. Hence, there exists a left-invariant finitely additive measure $\xi$ on the $\sigma$-algebra of all subsets of $G$ such that $\xi(G) = 1$. We first observe that since $H$ is of infinite index, $\xi(H) = 0$. Indeed, for each $n > 0$, there are elements $g_1, \ldots, g_n \in G$ such that the cosets $g_1 H, \ldots, g_n H$ are mutually disjoint. Hence,

$$1 \geq \xi\left(\bigsqcup_{j=1}^n g_j H\right) = \sum_{j=1}^n \xi(g_j H) = \sum_{j=1}^n \xi(H) = n\xi(H).$$

This yields that $\xi(H) = 0$, as desired. As $g^{-1}Hg$ is also a subgroup of infinite index in $G$, it follows that $\xi(g^{-1}Hg) = 0$ for each $g \in G$. Since $\xi$ is left-invariant, $\xi(kHg) = 0$ for all $k, g \in G$. This implies that $\xi(\bigcup_{a,b\in S} aHb^{-1}) = 0$. Therefore, $G \neq \bigcup_{a,b\in S} aHb^{-1}$. Thus, Claim A is proved.

Since $\mu(X) < \infty$, there exists $n$ such that $\nu_n(F_n) > 0.5\nu_m(F_m)$ and $C_m \subset H$ for each $m \geq n$. Hence,

$$\nu_m((F_n H) \cap F_m) \geq \nu_m((F_n C_{n+1} C_{n+2} \cdots C_m) \cap F_m) = \nu_n(F_n) > 0.5\nu_m(F_m)$$

for each $m \geq n$. By Claim A, there is $g \in G$ such that $gF_n H \cap F_n H = \emptyset$. Since $\mu$ is the Haar measure, it follows that

$$\begin{aligned}\nu_m((gF_n H) \cap F_m) &\geq \nu_m((gF_n C_{n+1} C_{n+2} \cdots C_m) \cap F_m)\\ &= \nu_m((F_n C_{n+1} C_{n+2} \cdots C_m) \cap g^{-1} F_m).\end{aligned}$$

This inequality and condition (i) yield that $\nu_m((gF_n H) \cap F_m) > 0.5\nu_m(F_m)$ eventually in $m$. Therefore, $\nu_m(F_n H \cap gF_n H) > 0$ eventually in $m$, which is a contradiction. □

*Definition 2.12.* If $\mu(X \setminus X^G) = 0$, then the dynamical system $(X, \mu, T)$ (or simply $T$) is called *the non-singular $(C, F)$-action associated with* $(C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$.

From now on, we consider only the case where $\mu(X \setminus X^G) = 0$. As $X = X^G$ mod 0, we will assume that $T$ is defined on the entire space $X$. Then for each $n$ and every two elements $g, h \in F_n$, we have that $T_{hg^{-1}}[g]_n = [h]_n$ and the Radon–Nikodym derivative of the transformation $T_{hg^{-1}}$ is constant on the subset $[g]_n$. More precisely, this constant equals $\nu_n(h)/\nu_n(g)$.

We now prove the main result of this section.

THEOREM 2.13. *Each non-singular $(C, F)$-action is of rank one. Conversely, each rank-one non-singular $G$-action is isomorphic (via a measure preserving isomorphism) to a $(C, F)$-action.*

*Proof.* Let a sequence $(C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$ satisfy equations (2.1)–(2.3) and Proposition 2.10(ii). We claim that the $(C, F)$-action $T = (T_g)_{g\in G}$ associated with this sequence is of rank one along $(F_n)_{n=0}^\infty$. Let $X$ be the space of this action and let $\mu$ be the $(C, F)$-measure on $X$ determined by $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$. Then, $X = \bigcup_{n\geq 0} X_n$ and $\mathcal{R} = \bigcup_{n\geq 1} \mathcal{S}_n$, where $X_n$ and $\mathcal{S}_n$ were introduced in §2.2. Of course, for each $n \in \mathbb{N}$, the pair $([1_G]_n, F_n)$ is a Rokhlin tower for $T$. Moreover:

(a) $X_{[1_G]_n, F_n} = X_n$;
(b) $\xi_{[1_G]_n, F_n}$ is the partition of $X_n$ into cylinders $[f]_n$, $f \in F_n$; and
(c) if $x = (f_n, c_{n+1}, \ldots) \in X_n \cap X^G$, then $O_{[1_G]_n, F_n}(x) = \{T_g x \mid g \in F_n f_n^{-1}\} = \mathcal{S}_n(x)$.

We note that items (a) and (b) imply that Definition 2.1(i) holds. It follows from Proposition 2.10 that for a.e. $x \in X$ (or, more precisely, for each $x \in X^G$), the $T$-orbit of $x$ equals $\mathcal{R}(x)$. As $\mathcal{R}(x) = \bigcup_{n=1}^\infty \mathcal{S}_n(x)$, it follows that item (c) implies condition (ii) from Definition 2.1. Hence, $T$ is of rank one along $(F_n)_{n=1}^\infty$.

Conversely, suppose that $T$ is a non-singular $G$-action of rank one along an increasing sequence $(Q_n)_{n=0}^\infty$ of finite subsets in $G$ with $Q_0 = \{1_G\}$. Let $(B_n, Q_n)_{n=0}^\infty$ be the corresponding generating sequence of Rokhlin towers such that conditions (i) and (ii) of Definition 2.1 hold. We have to define a sequence $(C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$, satisfying equations (2.1)–(2.3) and Proposition 2.10(ii) such that the associated $(C, F)$-action is isomorphic to $T$. We first set $F_n := Q_n$ for each $n \geq 0$. By Definition 2.1(i), for each $n \geq 0$, there is a subset $R_{n+1} \subset Q_{n+1}$ such that $B_n = \bigsqcup_{f \in R_{n+1}} T_f B_{n+1}$. Without loss of generality, we may assume that $1_G \in R_{n+1}$. Indeed, if this is not the case, we replace $(B_{n+1}, Q_{n+1})$ with another Rokhlin tower $(T_s B_{n+1}, Q_{n+1}s^{-1})$ for an element $s \in R_{n+1}$. Then, $\xi_{B_{n+1},Q_{n+1}} = \xi_{T_s B_{n+1}, Q_{n+1}s^{-1}}$ and $O_{B_{n+1},Q_{n+1}}(x) = O_{T_s B_{n+1}, Q_{n+1}s^{-1}}(x)$ for each $x \in X$. Hence, such replacements will not affect conditions (i) and (ii) of Definition 2.1. We now set $C_{n+1} := R_{n+1}$. Thus, we defined the entire sequence $(C_n, F_{n-1})_{n \geq 1}$. It is straightforward to verify that equation (2.1) holds. Let $\nu_0$ be the Dirac measure supported at $1_G$. Next, for each $n > 0$ and $f \in F_n$, we let $\nu_n(f) := \mu(T_f B_n)$. Thus, we obtain a non-degenerated measure $\nu_n$ on $F_n$. Finally, we define a probability $\kappa_{n+1}$ on $C_{n+1}$ by setting

$$\kappa_{n+1}(c) := \frac{\nu_{n+1}(c)}{\nu_n(1_G)} \quad \text{for each } c \in C_{n+1} \text{ and } n \geq 0.$$

Thus, the entire sequence of measures $(\kappa_n, \nu_{n-1})_{n \geq 1}$ is defined. It follows from condition (iii) which is below Definition 2.1 that

$$\frac{\nu_{n+1}(fc)}{\nu_n(f)} = \frac{\nu_{n+1}(c)}{\nu_n(1_G)} = \kappa_{n+1}(c) \quad \text{for each } c \in C_{n+1} \text{ and } f \in F_n,$$

that is equation (2.3) holds. We note that for each $n > 0$, the restrictions of $\xi_{B_n,Q_n}$ to the subset $X_{B_0,Q_0} = B_0$ is the finite partition of $B_0$ into subsets $T_{c_1 \cdots c_n} B_n$, where $(c_1, \ldots, c_n)$ runs the subset $C_1 \times \cdots \times C_n$. As $\xi_{B_n,Q_n} \restriction B_0$ converges to the partition into singletons, we obtain that

$$\max_{c_1 \in C_1, \ldots, c_n \in C_n} \mu(T_{c_1 \cdots c_n} B_n) \to 0 \quad \text{as } n \to \infty.$$

Since $\mu(T_{c_1 \cdots c_n} B_n) = \mu(X_{B_0}) \prod_{j=1}^n \kappa_j(c_j)$, equation (2.2) follows.

Fix $n > 0$, $g \in G$ and $\epsilon > 0$. It follows from Definition 2.1(ii) that there exists $M > n$ such that for each $m > M$, there is a subset $A \subset X_{B_n,Q_n}$ such that $\mu(X_{B_n,Q_n} \setminus A) < \epsilon$ and $T_g x \in O_{B_m,Q_m}(x)$ for each $x \in A$. Hence, there exist $f_1, f_2 \in Q_m$ such that $T_g x = T_{f_1 f_2^{-1}} x$ and $T_{f_2^{-1}} x \in B_m$. As $T$ is free, $g f_2 = f_1 \in Q_m = F_m$. It follows that $T_{g f_2} B_m = T_{f_1} B_m \subset X_{B_m,Q_m}$. Since $T_{f_2} B_m \ni x$ and $x \in X_{B_n,Q_n}$, we obtain that $T_{f_2} B_m \subset X_{B_n,Q_n}$ because $\xi_{B_m,Q_m}$ is finer than $\xi_{B_n,Q_n}$. Thus, without loss of generality, we may assume that $A$ is measurable with respect to the partition $\xi_{B_m,Q_m}$. Since

$$X_{B_n,Q_n} = \bigsqcup_{f \in F_n} T_f B_n = \bigsqcup_{f \in F_n} T_f \bigg( \bigsqcup_{c \in C_{n+1} \cdots C_m} T_c B_m \bigg),$$

it follows that $T_{f_2} B_m \subset X_{B_n,Q_n}$ if and only if $f_2 \in F_n C_{n+1} \cdots C_m$. Hence,

$$\nu_m(\{f_2 \in F_m \mid g f_2 \in F_m, f_2 \in F_n C_{n+1} \cdots C_m\}) \geq \mu(A) > \mu(X_{B_n,Q_n}) - \epsilon.$$

This implies Proposition 2.10(ii).

Thus, $(C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$, satisfies equations (2.1)–(2.3) and Proposition 2.10(ii). Denote by $R$ the non-singular $(C, F)$-action associated with $(C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$. Let $(Y, \nu)$ be the space of this action. The correspondence

$$T_f B_n \longleftrightarrow R_f [1_G]_n \quad \text{where } f \text{ runs } F_n \text{ and } n \text{ runs } \mathbb{N},$$

gives rise to a Boolean measure preserving isomorphism of the underlying algebras of measurable subsets on $X$ and $Y$. The Boolean isomorphism is generated by a certain pointwise measure preserving isomorphism $\theta$ of $(X, \mu)$ onto $(Y, \nu)$. We claim that $\theta$ intertwines $T$ with $R$. Indeed, take $g \in G$. As was shown above, for each $n > 0$ and $\epsilon > 0$, there exists $M > n$ such that for each $m > M$, there is a subset $Q' \subset Q_m$ such that $\bigsqcup_{f \in Q'} T_f B_m \subset X_{B_n, Q_n}$, $\mu(X_{B_n,Q_n} \setminus \bigsqcup_{f \in Q'} T_f B_m) < \epsilon$ and $gQ' \subset Q_m$. Hence,

$$\theta(T_g T_f B_m) = R_g \theta(T_f B_m) \quad \text{for all } f \in Q'.$$

Passing to the limit as $\epsilon \to 0$ and using the fact that $\nu \circ \theta = \mu$, we obtain that $\theta(T_g x) = R_g \theta x$ for a.e. $x \in X_n$. Since $n$ is arbitrary, $\theta T_g = R_g \theta$, as claimed. □

*Remark 2.14.*

(a) Theorem 2.13 corrects [**Da3**, Theorem 1.6], where the particular case of $\sigma$-finite measure preserving rank-one actions was under consideration: the condition (ii) (see Definition 2.1) is missing in the definition of rank one in [**Da3**]. However, this condition cannot be omitted: counterexamples of non-rank-one action satisfying condition (i) (and hence not satisfying condition (ii)) is provided in Examples 4.4–4.6 below.

(b) It is worth noting that the condition on the Radon–Nikodym derivatives in the definition of Rokhlin tower in §2.1 is important and cannot be omitted either. Indeed, the associated flow of each non-singular $(C, F)$-system is AT in the sense of Connes and Woods [**CoWo**] (see also [**Ha**]). In [**DoHa**], Dooley and Hamachi constructed explicitly a Markov non-singular odometer ($\mathbb{Z}$-action) whose associated flow is non-AT. Hence, this Markov odometer is not isomorphic (if fact, it is not even orbit equivalent) to any rank-one non-singular $\mathbb{Z}$-action. (We do not provide definitions of orbit equivalence, associated flow and AT-flow because we will not use it anywhere below in this paper. Instead, we refer the interested reader to the survey [**DaSi**].) However, it is easy to see that this odometer satisfies a 'relaxed version' of Definition 2.1 in which we drop only the condition on the Radon–Nikodym derivatives.

2.4. *Telescopings and reductions.* Let a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ satisfy equations (2.1)–(2.3) and Proposition 2.10(ii). Denote by $T = (T_g)_{g \in G}$ the $(C, F)$-action of $G$ on $X$ associated with $\mathcal{T}$. Let $\mu$ stand for the non-singular $(C, F)$-measure on $X$ determined by $(\kappa_n)_{n=1}^{\infty}$ and $(\nu_n)_{n=0}^{\infty}$.

Given a strictly increasing infinite sequence of integers $\boldsymbol{l} = (l_n)_{n=0}^{\infty}$ such that $l_0 = 0$, we let

$$\widetilde{F}_n := F_{l_n}, \quad \widetilde{C}_{n+1} := C_{l_n+1} \cdots C_{l_{n+1}}, \quad \widetilde{\nu}_n := \nu_n, \quad \widetilde{\kappa}_{n+1} := \kappa_{l_n+1} * \cdots * \kappa_{l_{n+1}}$$

for each $n \geq 0$.

*Definition 2.15.* We call the sequence $\widetilde{\mathcal{T}} := (\widetilde{C}_n, \widetilde{F}_{n-1}, \widetilde{\kappa}_n, \widetilde{\nu}_{n-1})_{n=1}^{\infty}$ the $\boldsymbol{l}$*-telescoping* of $\mathcal{T}$.

It is easy to check that $\widetilde{\mathcal{T}}$ satisfies equations (2.1)–(2.3) and Proposition 2.10(ii). Hence, a non-singular $(C, F)$-action $\widetilde{T} = (\widetilde{T}_g)_{g \in G}$ of $G$ associated with $\widetilde{\mathcal{T}}$ is well defined. Let $\widetilde{X}$

denote the space of $\widetilde{T}$ and let $\widetilde{\mu}$ denote the non-singular $(C, F)$-measure on $\widetilde{X}$ determined by $(\widetilde{\kappa}_n)_{n=1}^\infty$ and $(\widetilde{\nu}_n)_{n=0}^\infty$. There is a *canonical* measure preserving isomorphism $\iota_{\boldsymbol{l}}$ of $(X, \mu)$ onto $(\widetilde{X}, \widetilde{\mu})$ that intertwines $T$ with $\widetilde{T}$. Indeed, if $x \in X$, then we select the smallest $n \geq 0$ such that $x = (f_{l_n}, c_{l_n+1}, c_{l_n+2}, \ldots) \in X_{l_n}$. Let

$$\iota_{\boldsymbol{l}}(x) := (f_{l_n}, c_{l_n+1} \cdots c_{l_{n+1}}, c_{l_{n+1}+1} \cdots c_{l_{n+2}}, \ldots) \in \widetilde{X}_n \subset \widetilde{X},$$

where $\widetilde{X}_n = \widetilde{F}_n \times \widetilde{C}_{n+1} \times \widetilde{C}_{n+2} \times \cdots$. It is a routine to verify that $\iota_{\boldsymbol{l}}$ is a homeomorphism of $X$ onto $\widetilde{X}$ such that $\iota_{\boldsymbol{l}} T_g = \widetilde{T}_g \iota_{\boldsymbol{l}}$ for each $g \in G$, as desired.

Let $\boldsymbol{l} = (l_n)_{n=0}^\infty$ and $\boldsymbol{m} = (m_n)_{n=0}^\infty$ be two strictly increasing sequences of integers such that $l_0 = m_0 = 0$. If $\widetilde{\mathcal{T}}$ is the $\boldsymbol{l}$-telescoping of $\mathcal{T}$ and $\mathcal{S}$ is the $\boldsymbol{m}$-telescoping of $\widetilde{\mathcal{T}}$, then $\mathcal{S}$ is the $\boldsymbol{l} \circ \boldsymbol{m}$-telescoping of $\mathcal{T}$, where $\boldsymbol{l} \circ \boldsymbol{m} := (l_{m_n})_{n=1}^\infty$ and $\iota_{\boldsymbol{m}} \circ \iota_{\boldsymbol{l}} = \iota_{\boldsymbol{l} \circ \boldsymbol{m}}$.

Given a sequence $\boldsymbol{A} = (A_n)_{n=1}^\infty$ of subsets $A_n \subset C_n$ such that $1_G \in A_n$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^\infty (1 - \kappa_n(A_n)) < \infty$, we let

$$\kappa_n'(a) := \frac{\kappa_n(a)}{\kappa_n(A_n)}, \quad a \in A_n.$$

Then, $\kappa_n'$ is a non-degenerated probability on $A_n$ for each $n \in \mathbb{N}$. We also define a measure $\nu_n'$ on $F_n$ by setting

$$\nu_n' = \frac{1}{\prod_{j=1}^n \kappa_j(A_j)} \cdot \nu_n$$

if $n > 0$ and $\nu_0' := \nu_0$. Let $\mathcal{T}' := (A_n, F_{n-1}, \kappa_n', \nu_{n-1}')_{n=1}^\infty$.

*Definition 2.16.* We call $\mathcal{T}'$ an $\boldsymbol{A}$*-reduction* of $\mathcal{T}$.

It is easy to check that $\mathcal{T}'$ satisfies equations (2.1)–(2.3). We note that

$$(\kappa_1' * \cdots * \kappa_m')(G \setminus g^{-1} F_m) \leq \frac{(\kappa_1 * \cdots * \kappa_m)(G \setminus g^{-1} F_m)}{\prod_{j=1}^m \kappa(A_j)}$$

for each $g \in G$. Passing to the limit and using Proposition 2.10(iii) for $\mathcal{T}$, we obtain that $(\kappa_1' * \cdots * \kappa_m')(G \setminus g^{-1} F_m) \to 0$ as $m \to \infty$. In other words,

$$\lim_{m \to \infty} (\kappa_1' * \cdots * \kappa_m')(g^{-1} F_m) = 1,$$

that is, Proposition 2.10(iii) holds for $\mathcal{T}'$. Hence, a non-singular $(C, F)$-action $T' = (T_g')_{g \in G}$ of $G$ associated with $\mathcal{T}'$ is well defined. Let $X'$ denote the space of $T'$ and let $\mu'$ denote the non-singular $(C, F)$-measure on $X'$ determined by $(\kappa_n')_{n=1}^\infty$ and $(\nu_n')_{n=0}^\infty$.

PROPOSITION 2.17. *There is a canonical measure scaling isomorphism $\iota_{\boldsymbol{A}}$ of $(X, \mu)$ onto $(X', \mu')$ that intertwines $T$ with $T'$ and $\mu \circ \iota_{\boldsymbol{A}}^{-1} = \prod_{m>0} \kappa_m(A_m) \cdot \mu'$.*

*Proof.* Indeed, fix $n \in \mathbb{N}$. Since $\mu \upharpoonright X_n = \nu_n \otimes \kappa_{n+1} \otimes \kappa_{n+2} \otimes \cdots$, it follows from the Borel–Cantelli lemma that for a.e. $x = (f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n$, there is $N = N_x > n$ such that $c_m \in A_m$ for each $m > N$. We then let

$$\iota_{\boldsymbol{A},n}(x) := (f_n c_{n+1} \cdots c_N, c_{N+1}, c_{N+2}, \ldots) \in X_N' \subset X'.$$

It is routine to verify that $\iota_{\boldsymbol{A},n} : X_n \ni x \mapsto \iota_{\boldsymbol{A},n}(x) \in X'$ is a well-defined non-singular mapping and

$$\frac{d\mu' \circ \iota_{\boldsymbol{A},n}}{d\mu}(x) = \frac{\nu'_N(f_n c_{n+1} \cdots c_N)}{\nu_N(f_n c_{n+1} \cdots c_N)} \prod_{m>N} \frac{1}{\kappa_m(A_m)} = \prod_{m>0} \frac{1}{\kappa_m(A_m)}.$$

Moreover, $\iota_{\boldsymbol{A},n+1} \restriction X_n = \iota_{\boldsymbol{A},n}$ for each $n \in \mathbb{N}$. Hence, a measurable mapping $\iota_{\boldsymbol{A}} : X \to X'$ is well defined by the restrictions $\iota_{\boldsymbol{A}} \restriction X_n = \iota_{\boldsymbol{A},n}$ for all $n \in \mathbb{N}$. It is straightforward to verify that $\iota_{\boldsymbol{A}}$ is an isomorphism of $(X, \mu)$ onto $(X', \mu')$ with $\mu \circ \iota_{\boldsymbol{A}}^{-1} = \prod_{m>0} \kappa_m(A_m) \cdot \mu'$ and $\iota_{\boldsymbol{A}} T_g = T'_g \iota_{\boldsymbol{A}}$ for each $g \in G$. □

2.5. *Locally compact models for rank-one non-singular systems.* Let $Z$ be a locally compact Polish $G$-space. We remind that a Borel mapping $\rho : G \times Z \to \mathbb{R}^*_+$ is called a $G$-cocycle if

$$\rho(g_2, g_1 z)\rho(g_1, z) = \rho(g_2 g_1, z) \quad \text{for all } g_1, g_2 \in G \text{ and } z \in Z.$$

The following definition is a dynamical analogue of Definition 2.6.

*Definition 2.18.* Fix a $G$-cocycle $\rho$. We say that the $G$-action on $Z$ is Radon *$\rho$-uniquely ergodic* if there exists a unique (up to scaling) Radon $G$-quasi-invariant measure $\gamma$ on $Z$ such that

$$\frac{d\gamma \circ g}{d\gamma}(z) = \rho(g, z) \quad \text{for all } g \in G \text{ and } z \in Z.$$

We now show that each rank-one non-singular action has a uniquely ergodic continuous realization on a locally compact Cantor space.

THEOREM 2.19. *Let a non-singular action $R$ of $G$ on a $\sigma$-finite standard non-atomic measure space $(Z, \eta)$ be of rank one along a sequence $(Q_n)_{n=1}^\infty$. Then there exist:*

(i) *a continuous, minimal, Radon uniquely ergodic $G$-action $T' = (T'_g)_{g\in G}$ defined on a locally compact Cantor space $X'$;*

(ii) *a $T'$-quasi-invariant Radon measure $\mu'$ on $X'$ such that the Radon–Nikodym derivative $d\mu' \circ T'_g/d\mu' : X' \to \mathbb{R}^*_+$ is continuous for each $g \in G$;*

(iii) *a measure preserving Borel isomorphism of $(Z, \eta)$ onto $(X', \mu')$ that intertwines $R$ with $T'$;*

(iv) *a sequence $\mathcal{T}' = (C'_n, F'_{n-1}, \kappa'_n, \nu'_{n-1})_{n=1}^\infty$ satisfying equations (2.1)–(2.4) such that $(X', \mu', T')$ is the $(C, F)$-action associated with $\mathcal{T}'$; and*

(v) *a sequence $(z_n)_{n=1}^\infty$ such that $z_n \in Q_n$ for each $n$ and $(F'_n)_{n=1}^\infty$ is a subsequence of $(z_n^{-1} Q_n)_{n=1}^\infty$.*

*Moreover, $T'$ is Radon $(d\mu' \circ T'_g/d\mu')_{g\in G}$-uniquely ergodic.*

*Proof.* By Theorem 2.13, there is a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$ satisfying equations (2.1)–(2.3) and Proposition 2.10(ii) such that the $(C, F)$-action $T$ of $G$ associated with $\mathcal{T}$ is isomorphic to $R$ via a measure preserving isomorphism. Denote by $(X, \mu)$ the space of $T$. Let $G = \{g_j \mid j \in \mathbb{N}\}$. It follows from Proposition 2.10(ii) that there are an increasing sequence $\boldsymbol{l} = (l_n)_{n=0}^\infty$ of integers and a sequence $(D_n)_{n=1}^\infty$ of subsets in $G$ such that $l_0 = 0$, $D_{n+1} := (C_{l_n+1} \cdots C_{l_{n+1}}) \cap \bigcap_{j=1}^n g_j^{-1} F_{l_{n+1}}$ and

$$\kappa_{l_n+1} * \cdots * \kappa_{l_{n+1}}(D_{n+1}) > 1 - \frac{1}{(n+1)^2}$$

for each $n \geq 0$. Denote by $\widetilde{\mathcal{T}} = (\widetilde{C}_n, \widetilde{F}_{n-1}, \widetilde{\kappa}_n, \widetilde{\nu}_{n-1})_{n=1}^\infty$ the $\boldsymbol{l}$-telescoping of $\mathcal{T}$. Let $(\widetilde{X}, \widetilde{\mu}, \widetilde{T})$ stand for the $(C, F)$-action of $G$ associated with $\widetilde{\mathcal{T}}$. Then, $D_n \subset \widetilde{C}_n$ and $\widetilde{\kappa}_n(D_n) > 1 - n^{-2}$ for each $n > 0$. Denote by $\iota_{\boldsymbol{l}}$ the canonical measure preserving isomorphism intertwining $T$ with $\widetilde{T}$. In general, $1_G \notin D_n$. Therefore, we need to modify the $(C, F)$-parameters $\widetilde{\mathcal{T}}$. First, we choose, for each $n > 0$, an element $c_n \in D_n$. Then, we let $z_0 := 1_G$ and $z_n := c_1 \cdots c_n$ for each $n > 0$. Finally, we define a new sequence $\widehat{\mathcal{T}} = (\widehat{C}_n, \widehat{F}_{n-1}, \widehat{\kappa}_n, \widehat{\nu}_{n-1})_{n=1}^\infty$ by setting

$$\widehat{C}_n := z_{n-1}\widetilde{C}_n z_n^{-1}, \quad \widehat{F}_{n-1} := \widetilde{F}_{n-1} z_{n-1}^{-1},$$

$\widehat{\kappa}_n$ is the image of $\kappa_n$ under the bijection $\widetilde{C}_n \ni c \mapsto z_{n-1} c z_n^{-1} \in \widehat{C}_n$ and $\widehat{\nu}_{n-1}$ is the image of $\nu_{n-1}$ under the bijection $\widetilde{F}_{n-1} \ni f \mapsto f z_{n-1}^{-1} \in \widehat{F}_{n-1}$. It is straightforward to verify that $\widehat{\mathcal{T}}$ satisfies equations (2.1)–(2.3) and Proposition 2.10(ii). Denote by $(\widehat{X}, \widehat{\mu}, \widehat{T})$ the $(C, F)$-action of $G$ associated with $\widehat{\mathcal{T}}$. Then there is a canonical continuous measure preserving isomorphism $\vartheta : (\widetilde{X}, \widetilde{\mu}) \to (\widehat{X}, \widehat{\mu})$ that intertwines $\widetilde{T}$ with $\widehat{T}$:

$$\widetilde{X} \supset \widetilde{X}_n \ni (f_n, c_{n+1}, \ldots) \mapsto (f_n z_n^{-1}, z_n c_{n+1} z_{n+1}^{-1}, z_{n+1} c_{n+2} z_{n+2}^{-1}, \ldots) \in \widehat{X}_n \subset \widehat{X}.$$

Let $\widehat{D}_n$ be the image of $D_n$ under the bijection $\widetilde{C}_n \ni c \mapsto z_{n-1} c z_n^{-1} \in \widehat{C}_n$. Then, $1 \in \widehat{D}_n$ and $\widehat{\kappa}_n(\widehat{D}_n) > 1 - n^{-2}$ for each $n \in \mathbb{N}$. Hence, $\sum_{n=1}^\infty (1 - \widehat{\kappa}_n(\widehat{D}_n)) < \infty$. We now set $\boldsymbol{D} := (D_n)_{n=1}^\infty$. Denote by $\mathcal{T}'$ the $\boldsymbol{D}$-reduction of $\widehat{\mathcal{T}}$. Then, $\mathcal{T}'$ satisfies not only equations (2.1)–(2.3) and Proposition 2.10(ii), but also equation (2.4). Let $(X', \mu', T')$ denote the $(C, F)$-action of $G$ associated with $\mathcal{T}'$ and let $\iota_{\boldsymbol{D}}$ stand for the canonical measure scaling isomorphism of $(\widehat{X}, \widehat{\mu})$ onto $(X', \mu')$ that intertwines $\widehat{T}$ with $T'$. Then, $\iota_{\boldsymbol{D}} \circ \vartheta \circ \iota_{\boldsymbol{l}}$ is a measure-scaling isomorphism of $(X, \mu, T)$ onto $(X', \mu', T')$. Replacing $\mu'$ with $a \cdot \mu'$ for an appropriate $a > 0$, we obtain that $\iota_{\boldsymbol{D}} \circ \vartheta \circ \iota_{\boldsymbol{l}}$ is measure preserving. It follows from Proposition 2.9 that $T'$ is a Radon uniquely ergodic minimal continuous action of $G$ on the locally compact Polish space $X'$. Thus, we proved conditions (i), (iii), (iv) and (v). The condition (ii) follows easily from equation (2.4) and the definition of $\mu'$.

The final claim of the theorem follows from condition (iv) and Proposition 2.7. □

2.6. *Non-singular $\mathbb{Z}$-actions of rank one along intervals and $(C, F)$-construction.* Let $G = \mathbb{Z}$. Suppose that a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$ satisfies equations (2.1)–(2.3) and Proposition 2.10(ii), and there is a sequence $(h_n)_{n=0}^\infty$ of positive integers such that $F_n = \{0, 1, \ldots, h_n - 1\}$. Denote by $(X, \mu, T)$ the $(C, F)$-dynamical system associated with $\mathcal{T}$.

We now show how to obtain $(X, \mu, T)$ via the classical inductive geometric cutting-and-stacking in the case of $\mathbb{Z}$-actions. On the initial step of the construction, we define a column $Y_0$ consisting of a single interval $[0, 1)$ equipped with Lebesgue measure. Assume that at the $n$th step, we have a column

$$Y_n = \{I(i, n) \mid i = 0, \ldots, h_n - 1\}$$

consisting of disjoint intervals $I(i, n) \subset \mathbb{R}$ such that $\bigsqcup_{i=0}^{h_n-1} I(i, n) = [0, \nu_n(F_n))$. Then we define a continuous *nth column mapping*

$$T^{(n)} : [0, \nu_n(F_n)) \setminus I(h_n - 1, n) \to [0, \nu_n(F_n)) \setminus I(0, n)$$

such that $T^{(n)} \restriction I(i, n)$ is the orientation preserving affine mapping of $I(i, n)$ onto $I(i + 1, n)$ for $i = 0, \ldots, h_n - 2$. It is convenient to think of $I(i, n)$ as a *level* of $Y_n$. The levels may be of different length, but they are parallel to each other and the $i$th level is above the $j$th level if $i > j$. The $n$th column mapping moves every level, except the highest one, one level up. By the move here, we mean the orientation preserving affine mapping. On the highest level, $T^{(n)}$ is not defined.

On the $(n + 1)$th step of the construction, we first cut each $I(i, n)$ into subintervals $I(i + c, n + 1)$, $c \in C_{n+1}$, such that $I(i + c, n + 1)$ is from the left of $I(i + c', n + 1)$ whenever $c < c'$ and

$$\frac{\text{the length of } I(i + c, n + 1)}{\text{the length of } I(i, n)} = \kappa_{n+1}(c) \quad \text{for each } c \in C_{n+1}.$$

Hence, we obtain that $\bigsqcup_{j \in F_n + C_{n+1}} I(j, n + 1) = [0, \nu_n(F_n))$. Next, we cut the interval $[\nu_n(F_n), \nu_{n+1}(F_{n+1}))$ into subintervals $I(j, n + 1)$, $j \in F_{n+1} \setminus (F_n + C_{n+1})$, such that the length of $I(j, n + 1)$ is $\nu_{n+1}(j)$ for each $j$. These new levels are called *spacers*. Thus, we obtain a new column $Y_{n+1} = \{I(j, n + 1) \mid j \in F_{n+1}\}$ with $\bigsqcup_{i \in F_{n+1}} I(i, n + 1) = [0, \nu_{n+1}(F_{n+1}))$. If an element $c \in C_{n+1}$ is not maximal in $C_{n+1}$, then we denote by $c^+$ the least element of $C_{n+1}$ that is greater than $c$. We define a *spacer mapping* $s_{n+1} : C_{n+1} \to \mathbb{Z}_+$ by setting

$$s_{n+1}(c) := \begin{cases} c^+ - c - h_n & \text{if } c \neq \max C_{n+1}, \\ h_{n+1} - c - h_n & \text{if } c = \max C_{n+1}. \end{cases}$$

The subcolumn $Y_{n,c} := \{I(i + c, n + 1) \mid i \in F_n\} \subset Y_{n+1}$ is called *the $c$-copy of $Y_n$*, $c \in C_{n+1}$. Thus, $Y_{n+1}$ consists of $\#C_{n+1}$ copies of $Y_n$, and spacers between them and above the highest copy of $Y_n$. More precisely, there are exactly $s_{n+1}(c)$ spacers above the $c$-copy of $Y_n$ in $Y_{n+1}$. The $(n + 1)$th column mapping

$$T^{(n+1)} : [0, \nu_{n+1}(F_{n+1})) \setminus I(h_{n+1} - 1, n + 1) \to [0, \nu_{n+1}(F_{n+1})) \setminus I(0, n + 1)$$

is defined in a similar way as $T^{(n)}$. Of course,

$$T^{(n+1)} \restriction ([0, \nu_n(F_n)) \setminus I(h_n - 1, n)) = T^{(n)}.$$

Passing to the limit as $n \to \infty$, we obtain a well-defined non-singular (piecewise affine) transformation $Q$ of the interval $[0, \lim_{n\to\infty} \nu_n(F_n)) \subset \mathbb{R}$ equipped with Lebesgue measure such that

$$Q \restriction ([0, \nu_n(F_n)) \setminus I(h_n - 1, n)) = T^{(n)} \quad \text{for each } n \in \mathbb{N}.$$

It is possible that $\lim_{n\to\infty} \nu_n(F_n) = \infty$ and then $T$ is defined on $[0, +\infty)$. Of course, this transformation (or, more precisely, the $\mathbb{Z}$-action generated by $Q$) is isomorphic to $(X, \mu, T)$. The according non-singular isomorphism is generated by the following correspondence:

$$X \supset [i]_n \longleftrightarrow I(i, n) \subset [0, \lim_{n\to\infty} \nu_n(F_n)), \quad i \in F_n, n \in \mathbb{N}.$$

Without loss of generality, we may assume $s_n(\max C_n) = 0$ for each $n > 0$ (see, for instance, [**Da4**]). This means that there are no spacers over the highest copy of $Y_n$ in $Y_{n+1}$.

2.7. *Non-singular rank-one $\mathbb{Z}$-actions as transformations built under function over non-singular odometer base.* Let $(X, \mu, T)$ be as in the previous subsection. We will assume that $\max F_{n+1} = \max F_n + \max C_{n+1}$ for each $n \geq 0$. In terms of the geometrical cutting-and-stacking (see §2.6), this means exactly that there are no spacers on the top of the $(n+1)$th column. We remind that $X_0 = C_1 \times C_2 \times \cdots$, $\mu \restriction X_0 = \kappa_1 \otimes \kappa_2 \otimes \cdots$ and

$$c^+ := \min\{d \in C_n \mid d > c\}$$

for each $n > 0$ and $c \in C_n$ such that $c \neq \max C_n$. Denote by $R$ the transformation induced by $T_1$ on $(X_0, \mu \restriction X_0)$. Since $T_1$ is conservative, $R$ is a well-defined non-singular transformation of $X_0$. Take $x = (c_1, c_2, \ldots) \in X_0$. Choose $n \geq 0$ such that $c_i = \max C_i$ for each $i = 1, \ldots, n$ and $c_{n+1} \neq \max C_{n+1}$. It is straightforward to verify that

$$Rx := (\underbrace{0, \ldots, 0}_{n \text{ times}}, c_{n+1}^+, c_{n+2}, c_{n+3} \ldots)$$

Thus, $R$ is a classical *non-singular odometer of product type* (see [**Aa**, **DaSi**] and references therein). Let $x^{\max} := (\max C_1, \max C_2, \ldots) \in X_0$. We now define a function $\vartheta : X_0 \setminus \{x^{\max}\} \to \mathbb{Z}_+$ by setting

$$\vartheta(x) := s_n(c_n)$$

if $x = (c_1, c_2, \ldots)$, $c_i = \max C_i$ for $i = 1, \ldots, n-1$ and $c_n \neq \max C_n$, where $s_n$ is the spacer mapping (see §2.6). Of course, $\vartheta$ is continuous. Then $(X, \mu, T_1)$ is isomorphic to the transformation $R^\theta$ built under $\vartheta$ over the base $R$. We do not provide a proof of this fact which is essentially folklore.

## 3. *Finite factors of non-singular $(C, F)$-actions*

Let a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$ satisfy equations (2.1)–(2.3) and Proposition 2.10(ii), and let $\lim_{n\to\infty} \nu_n(F_n) < \infty$. Let $\Gamma$ be a cofinite subgroup of $G$. We consider the left coset space $G/\Gamma$ as a homogeneous $G$-space on which $G$ acts by left translations. It is obvious that for each coset $g\Gamma \in G/\Gamma$, the subgroup $g\Gamma g^{-1} \subset G$ is the stabilizer of $g\Gamma$ in $G$.

*Definition 3.1.* Given a coset $g\Gamma \in G/\Gamma$, we say that $\mathcal{T}$ is *compatible with* $g\Gamma$ if

$$\sum_{n=1}^{\infty} \kappa_n(\{c \in C_n \mid c \notin g\Gamma g^{-1}\}) < \infty.$$

Denote by $(X, \mu, T)$ the $(C, F)$-action of $G$ associated with $\mathcal{T}$. Then, $\mu(X) = \lim_{n\to\infty} \nu_n(F_n) < \infty$. For a point $x = (c_1, c_2, \ldots) \in X_0 = C_1 \times C_2 \times \cdots$, we let

$$\pi_{(\mathcal{T},g\Gamma)}(x) := \lim_{n\to\infty} c_1 c_2 \cdots c_n g\Gamma \in G/\Gamma$$

whenever this limit exists. (The quotient space $G/\Gamma$ is endowed with the discrete topology.) It follows from the Borel–Cantelli lemma that if $\mathcal{T}$ is compatible with $g\Gamma \in G/\Gamma$, then $\pi_{(\mathcal{T},g\Gamma)}(x)$ is well defined for $\mu$-a.e. $x \in X_0$. It is straightforward to verify that for each $h \in G$,

$$\pi_{(\mathcal{T},g\Gamma)}(T_h x) = h\pi_{(\mathcal{T},g\Gamma)}(x)$$

whenever $\pi_{(\mathcal{T},g\Gamma)}(x)$ is well defined and $T_h x \in X_0$. It follows that the mapping

$$\pi_{(\mathcal{T},g\Gamma)} : X_0 \ni x \mapsto \pi_{(\mathcal{T},g\Gamma)}(x) \in G/\Gamma$$

extends uniquely (mod 0) to a measurable $G$-equivariant mapping from $X$ to $G/\Gamma$. We denote the extension by the same symbol $\pi_{(\mathcal{T},g\Gamma)}$. It is routine to verify that if $x = (f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n$ for some $n \in \mathbb{N}$, then

$$\pi_{(\mathcal{T},g\Gamma)}(x) := \lim_{m\to\infty} f_n c_{n+1} c_{n+2} \cdots c_m g\Gamma. \tag{3.1}$$

*Definition 3.2.* We call $\pi_{(\mathcal{T},g\Gamma)}$ *the* $(\mathcal{T}, g\Gamma)$*-factor mapping* for $T$.

We need some notation. Given $1 < n < m$, we denote the subset $C_n \cdots C_m$ of $G$ by $C_{n,m}$. Let $\kappa_{n,m}$ stand for the probability measure $\kappa_n * \cdots * \kappa_m$. It is supported on $C_n \cdots C_m$. We now state the main result of the section.

THEOREM 3.3. *The following are equivalent.*

(i) *There is a measurable factor map $\tau : X \to G/\Gamma$, that is, for each $g \in G$,*

$$\tau(T_g x) = g\tau(x) \quad \text{at } \mu - \text{a.e. } x \in X.$$

(ii) *There exists a sequence $(g_n)_{n>0}$ of elements of $G$ such that*

$$\lim_{N\to\infty} \sup_{m>n\geq N} \kappa_{n+1,m}(\{c \in C_{n+1,m} \mid c \notin g_n^{-1}\Gamma g_m\}) = 0.$$

(iii) *There exist a coset $g_0\Gamma \in G/\Gamma$ and a $g_0\Gamma$-compatible telescoping of $\mathcal{T}$.*

*It follows that $T$ has no factors isomorphic to $G/\Gamma$ if and only if there is no telescoping of $\mathcal{T}$ compatible with $g\Gamma g^{-1}$ for any $g \in G$.*

*Proof.* (i) $\Longrightarrow$ (ii) Let $Y_j := \tau^{-1}(j)$ for each $j \in G/\Gamma$. Then, $X = \bigsqcup_{j\in G/\Gamma} Y_j$. Consider $n$ large so that $\mu(Y_j \cap X_n) > 0$ for each $j \in G/\Gamma$. Let $g, h \in F_n$. Since $\tau$ is $G$-equivariant, it follows that $T_{hg^{-1}} Y_j = Y_{hg^{-1}j}$ and hence,

$$T_{hg^{-1}}([g]_n \cap Y_j) = [h]_n \cap Y_{hg^{-1}j}. \tag{3.2}$$

For each $j \in G/\Gamma$ and $g \in F_n$, let

$$d_{n,g}(j) := \mu([g]_n \cap Y_j)/\mu([g]_n).$$

Then, the set $\{d_{n,g}(j) \mid j \in G/\Gamma\} \subset (0, 1)$ does not depend on $g \in F_n$. Indeed, for each $h \in F_n$, the Radon–Nikodym derivative of the transformation $T_{gh^{-1}}$ is constant on $[h]_n$ and we deduce from equation (3.2) that

$$d_{n,h}(j) := \frac{\mu([h]_n \cap Y_j)}{\mu([h]_n)} = \frac{\mu([g]_n \cap Y_{gh^{-1}j})}{\mu([g]_n)} = d_{n,g}(gh^{-1}j). \tag{3.3}$$

Hence, $\{d_{n,h}(j) : j \in G/\Gamma\} = \{d_{n,g}(j) : j \in G/\Gamma\}$. Let

$$\delta_n := \max_{j \in G/\Gamma} d_{n,g}(j).$$

We claim that $\delta_m \to 1$ as $m \to \infty$. Indeed, for $m \geq n$, let $\mathcal{P}_m$ denote the finite $\sigma$-algebra generated by the $m$-cylinders (which are compact and open subsets of $X$) that are contained in $X_n$. Then, $\mathcal{P}_n \subset \mathcal{P}_{n+1} \subset \cdots$ and $\bigvee_{m>n} \mathcal{P}_m$ is the entire Borel $\sigma$-algebra on $X_n$. Hence, for each $j \in G/\Gamma$, there is $g_m \in F_m$ such that

$$\frac{\mu([g_m]_m \cap Y_j)}{\mu([g_m]_m)} \to 1 \quad \text{as } m \to \infty.$$

This implies that $\delta_m \to 1$ as $m \to \infty$, as claimed. In what follows, we consider $n$ large so that $\delta_n > 0.9$. Then for each $g \in F_n$, there is a unique $\Gamma$-coset $j_n(g) \in G/\Gamma$ such that $\delta_n = d_{n,g}(j_n(g))$. It is convenient to consider $G/\Gamma$ as a set of *colours*. Then, $j_n(g)$ is *the dominating colour* on $[g]_n$. It follows from equation (3.3) that $j_n(g) = gh^{-1} j_n(h)$ for all $g, h \in F_n$. Choose $g_n \in F_n$ such that $j_n(g_n) = \Gamma$. Then, $j_n(g) = gg_n^{-1}\Gamma$ for each $g \in F_n$. Given $\epsilon < \frac{1}{2}$, there is $N > 0$ such that $\delta_n > 1 - \epsilon^2$ for all $n > N$. Hence, for all $m > n > N$,

$$(1 - \epsilon^2)\mu([1_G]_n) < \mu([1_G]_n \cap Y_{j_n(1_G)}). \tag{3.4}$$

We recall that $[1_G]_n = \bigsqcup_{c \in C_{n+1,m}} [c]_m$. Let

$$D := \{c \in C_{n,m} \mid \mu([c]_m \cap Y_{j_n(1_G)}) > (1 - \epsilon)\mu([c]_m)\}.$$

It follows from equation (3.4) that

$$\begin{aligned}(1 - \epsilon^2)\mu([1_G]_n) &< \sum_{c \in D} \mu([c]_m) + (1 - \epsilon) \sum_{c \in C_{n+1,m} \setminus D} \mu([c]_m) \\ &= \sum_{c \in D} \mu([c]_m) + (1 - \epsilon)\bigg(\mu([1_G]_n) - \sum_{c \in D} \mu([c]_m)\bigg).\end{aligned}$$

This yields that $\sum_{c \in D} \mu([c]_m) > (1 - \epsilon)\mu([1_G]_n)$ or, equivalently,

$$\sum_{c \in D} \kappa_{n+1,m}(c) > (1 - \epsilon)\kappa_n(1_G). \tag{3.5}$$

By the definition of $D$, an element $c \in C_{n+1,m}$ belongs to $D$ if and only if $j_n(1_G)$ is the dominating colour on $[c]_m$, that is, $j_m(c) = j_n(1_G)$. Therefore, $cg_m^{-1}\Gamma = g_n^{-1}\Gamma$, that is, $c \in g_n^{-1}\Gamma g_m$. Hence, equation (3.5) yields that

$$\kappa_{n+1,m}(\{c \in C_{n+1,m} \mid c \notin g_n^{-1}\Gamma g_m\}) \le \epsilon$$

and property (ii) follows.

(ii) $\Longrightarrow$ (iii) As $\Gamma$ is cofinite and property (ii) holds, there exist an increasing sequence $0 = q_0 < q_1 < q_2 < \cdots$ of positive integers and an element $g_0 \in G$ such that $g_n^{-1} \in g_0\Gamma$ and

$$\kappa_{q_n+1,q_{n+1}}(\{c \in C_{q_n+1,q_{n+1}} \mid c \notin g_0\Gamma g_0^{-1}\}) < 2^{-n}$$

for each $n \in \mathbb{N}$. Let $\boldsymbol{q} := (q_n)_{n=0}^{\infty}$. Then $g_0\Gamma$ is compatible with the $\boldsymbol{q}$-telescoping of $\mathcal{T}$. This implies property (iii).

(iii) $\Longrightarrow$ (i) Denote by $\widetilde{\mathcal{T}}$ the $\boldsymbol{q}$-telescoping of $\mathcal{T}$. Then $\pi_{(\widetilde{\mathcal{T}},g_0\Gamma)} \circ \iota_{\boldsymbol{q}}$ is a factor mapping of $T$ onto $G/\Gamma$.

Thus, the first statement of the theorem is proved completely. The second statement follows from the first one and a simple observation that given two subgroups $\Gamma$ and $\Gamma'$ of $G$, the corresponding $G$-actions by left translations on $G/\Gamma$ and $G/\Gamma'$ are isomorphic if and only if $\Gamma$ and $\Gamma'$ are conjugate. □

The following important remark will be used essentially in the proof of the main result of §4.

*Remark 3.4.* In fact, we obtained more than what is stated in Theorem 3.3. We proved indeed that given a factor mapping $\tau : X \to G/\Gamma$, there exist a coset $g_0\Gamma$ and a $g_0\Gamma$-compatible $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ such that $\tau = \pi_{(\widetilde{\mathcal{T}},g_0\Gamma)} \circ \iota_{\boldsymbol{q}}$. To explain this fact, we use below the notation from the proof of Theorem 3.3. For $n \in \mathbb{N}$, let

$$X'_n := \bigsqcup_{f_n \in F_n} ([f_n]_n \cap Y_{j_n(f_n)}) \subset X_n.$$

We remind that $\mu(X) < \infty$. Since $\delta_n \to 1$ and $\mu(X_n) \to \mu(X)$ as $n \to \infty$, it follows that $\mu(X'_{q_n}) \to \mu(X)$ as $n \to \infty$. We can assume (passing to a subsequence of $(q_n)_{n=1}^{\infty}$ if needed) that $\sum_{n=1}^{\infty} \mu(X \setminus X'_{q_n}) < \infty$. Then, the Borel–Cantelli lemma yields that for a.e. $x \in X$, we have that $x \in X'_{q_n}$ eventually in $n$. Hence, for a.e. $x \in X$,

$$\tau(x) = \lim_{m\to\infty} j_{q_m}(f_{q_m}) = \lim_{m\to\infty} f_{q_m} g_{q_m}^{-1}\Gamma = \lim_{m\to\infty} f_{q_n} g_0\Gamma, \tag{3.6}$$

where $f_{q_m}$ is the first coordinate of $x$ in $X_{q_m}$, that is, $x = (f_{q_m}, c_{q_m+1}, \ldots) \in X_{q_m}$. Howerver, it follows from equations (3.1) and (3.2) that

$$\pi_{(\widetilde{\mathcal{T}},g_0\Gamma)}(\iota_{\boldsymbol{q}}(x)) = \lim_{m\to\infty} f_{q_m} g_0\Gamma \tag{3.7}$$

at a.e. $x \in X$. Therefore, equations (3.6) and (3.7) yield that $\tau = \pi_{(\widetilde{\mathcal{T}},g_0\Gamma)} \circ \iota_{\boldsymbol{q}}$ almost everywhere, as desired.

We note in this connection that if:

- $G$ is Abelian or $G$ is arbitrary but $\Gamma$ is normal in $G$; and
- the homogeneous space $G/\Gamma$ is a factor of an ergodic non-singular free action of $G$ on a standard measure space $(Y, \mathfrak{Y}, \nu)$,

then this factor (considered as an invariant sub-$\sigma$-algebra of $\mathfrak{Y}$) is defined uniquely by $\Gamma$. Indeed, if $\pi_1, \pi_2 : Y \to G/\Gamma$ are two $G$-equivariant measurable maps, then the mapping $Y \ni y \mapsto \pi_1(y)\pi_2(y)^{-1} \in G/\Gamma$ is invariant under $G$. Hence, it is constant. Therefore, there is $a \in G/\Gamma$ such that $\pi_1(y) = a\pi_2(y)$ for a.e. $y \in Y$. It follows that

$$\{\pi_1^{-1}(j) \in \mathfrak{Y} \mid j \in G/\Gamma\} = \{\pi_2^{-1}(j) \in \mathfrak{Y} \mid j \in G/\Gamma\}.$$

Therefore, the equality $\tau = \pi_{(\widetilde{\mathcal{T}}, g_0\Gamma)} \circ \iota_{\boldsymbol{q}}$ (at least, up to a rotation of $G/\Gamma$) stated in Remark 3.4 is a trivial fact. However, it is no longer true if $\Gamma$ is not normal.

*Example 3.5.* Let $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ and $\Gamma = \{0\} \times \mathbb{Z}_2$. Then $\Gamma$ is a non-normal cofinite subgroup of $G$ of index 3. We consider $\mathbb{Z}_3$ as a quotient $G/\Gamma$. Then $\mathbb{Z}_3$ is a $G$-space. Hence, the product space $\mathbb{Z}_3 \times \mathbb{Z}_3$ is also a $G$-space (we consider the diagonal $G$-action). Since the diagonal $D = \{(j, j) \mid j \in G/\Gamma\}$ is an invariant subspace of $\mathbb{Z}_3 \times \mathbb{Z}_3$, the complement $Y := (\mathbb{Z}_3 \times \mathbb{Z}_3) \setminus D$ of $D$ in $\mathbb{Z}_3 \times \mathbb{Z}_3$ is also $G$-invariant. It is easy to verify that the $G$-action on $Y$ is transitive and free. Endow $Y$ with the (unique) $G$-invariant probability measure $\nu$. Of course, the coordinate projections $\pi_1, \pi_2 : Y \to \mathbb{Z}_3$ are two-to-one $G$-equivariant maps. However, the corresponding $\sigma$-algebras of $\pi_1$-measurable and $\pi_2$-measurable subsets in $Y$ are different. Consider a rank-one $\mathbb{Z}$-action on a standard probability space $(Z, \mathfrak{Z}, \kappa)$. Then the product $(\mathbb{Z} \times G)$-action on $(Z \times Y, \kappa \otimes \nu)$ is of rank one. Denote it by $R$. The subgroup $\mathbb{Z} \times \Gamma$ of $\mathbb{Z} \times G$ is non-normal. It is of index 3. Hence, we can consider the corresponding finite quotient space $\mathbb{Z}_3$ as a $(\mathbb{Z} \times G)$-space. The mappings $1 \otimes \pi_1$ and $1 \otimes \pi_2$ from $Z \times Y$ onto $\mathbb{Z}_3$ are $(\mathbb{Z} \times G)$-equivariant. However, the corresponding factors of $R$, that is, the invariant sub-$\sigma$-algebras, are different.

A non-singular $G$-action is totally ergodic if and only if it has no non-trivial finite factors or, equivalently, each cofinite subgroup of $G$ acts ergodically. We thus deduce from Theorem 3.3 the following criterion of total ergodicity for the rank-one non-singular actions.

COROLLARY 3.6. *Let $T$ be a $(C, F)$-action of $G$ associated with a sequence $\mathcal{T}$ satisfying equations (2.1)–(2.3) and Proposition 2.10(ii). Then, $T$ is totally ergodic if no telescoping of $\mathcal{T}$ is compatible with any proper cofinite subgroup of $G$, that is, for each increasing sequence $n_1 < n_2 < \cdots$ of integers and each proper cofinite subgroup $\Gamma$ in $G$,*

$$\sum_{k=1}^{\infty} \kappa_{n_k+1} * \cdots * \kappa_{n_{k+1}}(\{c \in C_{n_k+1} \cdots C_{n_{k+1}} \mid c \notin \Gamma\}) = \infty.$$

## 4. *Non-singular odometer actions of residually finite groups*

4.1. *Non-singular odometers.* From now on, $G$ is residually finite. If $\Gamma$ is a cofinite subgroup in $G$, then the largest subgroup $\widetilde{\Gamma}$ of $\Gamma$ which is normal in $G$ is also cofinite in $G$. Of course, $\widetilde{\Gamma} = \bigcap_{g\in G} g\Gamma g^{-1}$. If $\Xi$ is cofinite subgroup in $\Gamma$, then $\widetilde{\Xi} \subset \widetilde{\Gamma}$. We now fix a decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups $\Gamma_n$ in $G$ such that

$$\bigcap_{n=1}^{\infty} \bigcap_{g\in G} g\Gamma_n g^{-1} = \{1_G\}. \tag{4.1}$$

It exists because $G$ is residually finite. We note that equation (4.1) means that the intersection of the maximal normal (in $G$) subgroups of $\Gamma_n$, $n \in \mathbb{N}$, is trivial. At the same time, the intersection of all $\Gamma_n$ can be non-trivial. Consider the natural inverse sequence of homogeneous $G$-spaces and $G$-equivariant mappings intertwining them:

$$G/\Gamma_1 \longleftarrow G/\Gamma_2 \longleftarrow \cdots . \tag{4.2}$$

Denote by $Y$ the projective limit of this sequence. A point of $Y$ is a sequence $(g_n\Gamma_n)_{n=1}^{\infty}$ such that $g_n\Gamma_n = g_{n+1}\Gamma_n$, that is, $g_n^{-1}g_{n+1} \in \Gamma_n$ for each $n > 0$. Endow $Y$ with the topology of projective limit. Then $Y$ is a compact Cantor $G$-space. Of course, the $G$-action on $Y$ is minimal and uniquely ergodic. Denote this action by $O = (O_g)_{g\in G}$. It follows from equation (4.1) that $O$ is *faithful*, that is, $O_g \neq I$ if $g \neq 1_G$. We note that a faithful action is not necessarily free.

*Definition 4.1.* The dynamical system $(Y, O)$ is called *the topological G-odometer associated with* $(\Gamma_n)_{n=1}^{\infty}$. If $\nu$ is a non-atomic Borel measure on $Y$ which is quasi-invariant and ergodic under $O$, then we call the dynamical system $(Y, \nu, O)$ a *non-singular G-odometer*. By the *Haar measure* for $(Y, O)$, we mean the unique $G$-invariant probability on $Y$.

In the finite measure preserving case, one can find the above definition in [**DaLe**] (see also [**LiSaUg**], where odometers are called 'subodometers'.)

We note that equation (4.1) is in no way restrictive. Indeed, let $\bigcap_{n=1}^{\infty} \bigcap_{g\in G} g\Gamma_n g^{-1} = N \neq \{1_G\}$. Define $(Y, O)$ as above. Then, $N$ is a proper normal subgroup of $G$ and $N = \{g \in G \mid O_g = I\}$. We now let $\widetilde{G} := G/N$ and $\widetilde{\Gamma}_n := \Gamma_n/N$. Then, $\widetilde{\Gamma}_n$ is a cofinite subgroup in $\widetilde{G}$ for each $n \in \mathbb{N}$, $\widetilde{\Gamma}_1 \supsetneq \widetilde{\Gamma}_2 \supsetneq \cdots$ and $\bigcap_{n=1}^{\infty} \bigcap_{g\in\widetilde{G}} g\widetilde{\Gamma}_n g^{-1} = \{1_{\widetilde{G}}\}$. Let $(\widetilde{Y}, \widetilde{O})$ denote the topological $\widetilde{G}$-odometer associated with the sequence $(\widetilde{\Gamma}_n)_{n=1}^{\infty}$. Then, of course, $Y = \widetilde{Y}$ and $O_g = \widetilde{O}_{gN}$ for each $g \in G$.

We now isolate a class of non-singular odometers of rank one. For each $n > 0$, we choose a finite subset $D_n \subset \Gamma_{n-1}$ such that $1_G \in D_n$ and each $\Gamma_n$-coset in $\Gamma_{n-1}$ intersects $D_n$ exactly once. (For consistency of the notation, we let $\Gamma_0 := G$.) We then call $D_n$ *a* $\Gamma_n$*-cross-section* in $\Gamma_{n-1}$. Then, the product $D_1 \cdots D_n$ is a $\Gamma_n$-cross-section in $G$. Hence, there is a unique bijection

$$\omega_n : G/\Gamma_n \to D_1 \cdots D_n$$

such that $\omega_n(g\Gamma_n)\Gamma_n = g\Gamma_n$ for each $g \in G$ and $\omega_n(\Gamma_n) = 1_G$. It follows, in particular, that

$$\text{if } \omega_n(g\Gamma_n) = h\omega_n(g'\Gamma_n) \text{ for some } g, g', h \in G, \text{ then } g\Gamma_n = hg'\Gamma_n. \tag{4.3}$$

It is straightforward to verify that the diagram

$$\begin{array}{ccccccc} G/\Gamma_1 & \longleftarrow & G/\Gamma_2 & \longleftarrow & G/\Gamma_3 & \longleftarrow & \cdots \\ {\scriptstyle\omega_1}\big\downarrow & & {\scriptstyle\omega_2}\big\downarrow & & {\scriptstyle\omega_3}\big\downarrow & & \\ D_1 & \xleftarrow{\phi_1} & D_1D_2 & \xleftarrow{\phi_2} & D_1D_2D_3 & \xleftarrow{\phi_3} & \cdots \\ {\scriptstyle\psi_1}\big\downarrow & & {\scriptstyle\psi_2}\big\downarrow & & {\scriptstyle\psi_3}\big\downarrow & & \\ D_1 & \xleftarrow{\phi'_1} & D_1 \times D_2 & \xleftarrow{\phi'_2} & D_1 \times D_2 \times D_3 & \xleftarrow{\phi'_3} & \cdots \end{array} \tag{4.4}$$

commutes. The horizontal arrows in the upper line denote the natural projections. The other mappings in the diagram are defined as follows:

$$\phi_n(d_1 \cdots d_{n+1}) := d_1 \cdots d_n,$$
$$\phi'_n(d_1, \ldots, d_{n+1}) := (d_1, \ldots, d_n) \quad \text{and}$$
$$\psi_n(d_1 \cdots d_n) := (d_1, \ldots, d_n)$$

for each $(d_1, \ldots, d_{n+1}) \in D_1 \times \cdots \times D_{n+1}$ and $n \geq 0$. It follows from equation (4.3) that there exists a natural homeomorphism of $Y$ onto the infinite product space $D := D_1 \times D_2 \times \cdots$. (The homeomorphism pushes down to a bijection between $G/\Gamma_n$ and $D_1 \times \cdots \times D_n$ for each $n$.)

PROPOSITION 4.2. *If, for each $n \in \mathbb{N}$, there is a $\Gamma_n$-cross-section $D_n$ in $\Gamma_{n-1}$ and a probability $\kappa_n$ on $G$ such that:*

(i) $\operatorname{supp} \kappa_n = D_n$ *for each $n$;*
(ii) $\prod_{n=1}^{\infty} \max_{d \in D_n} \kappa_n(d) = 0$*; and*
(iii) $\lim_{n\to\infty} (\kappa_1 * \cdots * \kappa_n)(gD_1 \cdots D_n) = 1$ *for each $g \in G$,*

*then there is a non-atomic probability Borel measure $\mu$ on $Y$ which is quasi-invariant under $O$ and such that the non-singular odometer $(Y, \mu, O)$ is of rank one along the sequence $(D_1 \cdots D_n)_{n=1}^{\infty}$.*

*Proof.* We set $F_0 := \{1_G\}$, $F_n := D_1 \cdots D_n$, $C_n := D_n$ and $\nu_n := \kappa_1 * \cdots * \kappa_n$ for each $n \in \mathbb{N}$. Then equations (2.1)–(2.3) hold for the sequence $(C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$. Moreover, Proposition 2.10(iii) is exactly property (iii) in the case under consideration. Hence, the $(C, F)$-action $T$ of $G$ associated with $(C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ is well defined. Let $(X, \mu)$ stand for the space of this action. It follows from the $(C, F)$-construction that

$$X = D_1 \times D_2 \times \cdots = D,$$

$\mu$ is non-atomic and $T$ is free (mod $\mu$). In view of equation (4.3), we can identify $X$ with $Y$. Hence, we consider $\mu$ as a probability on $Y$. Moreover, equation (4.2) yields that $T$ is conjugate to $O$. Thus, $(Y, \mu, O)$ is a non-singular odometer. It remains to apply Theorem 2.13. □

We now show that the classical non-singular $\mathbb{Z}$-odometers of product type are covered by Definition 4.1.

*Example 4.3.* Let $G = \mathbb{Z}$ and let $(a_n)_{n=1}^{\infty}$ be a sequence of integers such that $a_n > 1$ for each $n \in \mathbb{N}$. We set $\Gamma_n := a_1 \cdots a_n \mathbb{Z}$. Then, $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{0\}$. The set $D_n := a_1 \cdots a_{n-1} \cdot \{0, 1, \ldots, a_n - 1\}$ is a $\Gamma_n$-cross-section in $\Gamma_{n-1}$. Hence, in view of equation (4.4), the space $Y$ of the $\mathbb{Z}$-odometer $O = (O_n)_{n\in\mathbb{Z}}$ associated with $(\Gamma_n)_{n=1}^{\infty}$ is homeomorphic to the infinite product $D = D_1 \times D_2 \times \cdots$. We identify $D_n$ naturally with the set $\{0, 1, \ldots, a_n - 1\}$. Then,

$$D = \{0, 1, \ldots, a_1 - 1\} \times \{0, 1, \ldots, a_2 - 1\} \times \cdots .$$

To define $O$ explicitly on this space, we take $y = (y_n)_{n=1}^{\infty} \in D$. It is a routine to check that if there is $k > 0$ such that $y_j = a_j - 1$ for each $j < k$ and $y_k \neq a_k - 1$, then

$$O_1 y = (0, \ldots, 0, y_k + 1, y_{k+1}, y_{k+2}, \ldots).$$

If such a $k$ does not exist, that is, $y_j = a_j - 1$ for each $j > 0$, then $O_1 y = (0, 0, \ldots)$. Let $\kappa_n$ be a non-degenerated probability measure on $\{0, 1, \ldots, a_n - 1\}$ and let $\prod_{n>0} \max_{0\le d<a_n} \kappa_n(d) = 0$. This means that properties (i) and (ii) of Proposition 4.2 hold. Of course, Proposition 4.2(iii) holds also. Hence, by Proposition 4.2, the non-singular odometer

$$\left( \bigotimes_{n=1}^{\infty} \{0, \ldots, a_n - 1\}, \bigotimes_{n=1}^{\infty} \kappa_n, O \right)$$

is of rank one. Thus, in this case, our definition of non-singular odometer coincides with the classical definition of non-singular $\mathbb{Z}$-odometers of product type (see [**Aa**, **DaSi**]). Moreover, the $\mathbb{Z}$-odometers of product type are of rank one.

It is routine to verify that if $G = \mathbb{Z}^d$ with $d \in \mathbb{N}$, then each probability preserving $G$-odometer is of rank one. This follows from Proposition 4.2 if one chooses the $\Gamma_n$-cross-sections $D_n$ in $\Gamma_{n-1}$ in such a way that the sum $D_1 + \cdots + D_n$ is a parallelepiped $\{0, 1, \ldots, a_{1,n}\} \times \cdots \times \{0, 1, \ldots, a_{d,n}\}$ for some $a_{1,n}, \ldots, a_{d,n} \in \mathbb{N}$ with $\lim_{n\to\infty} a_{j,n} = \infty$ for each $j$. We leave details to the reader (see also [**JoMc**, Theorem 2.11]).

Note, however, that there exist probability preserving free $G$-odometers which are not of rank one.

*Example 4.4.* The free group with two generators $\boldsymbol{F}_2$ is residually finite. Hence, there is a sequence $N_1 \supsetneq N_2 \supsetneq \cdots$ of normal subgroups in $\boldsymbol{F}_2$ such that $\bigcap_{n=1}^{\infty} N_n = \{1_{\boldsymbol{F}_2}\}$. Then, the topological $\boldsymbol{F}_2$-odometer associated with $(N_n)_{n=1}^{\infty}$ is a free minimal $\boldsymbol{F}_2$-action by translation on a compact group $Y$. Let $\chi$ denote the Haar measure on $Y$. Then, $(K, \chi, O)$ is an ergodic probability preserving $\boldsymbol{F}_2$-odometer. If $(K, \chi, O)$ were of rank one, then $\boldsymbol{F}_2$ would be amenable by Corollary 2.11(ii), which is a contradiction. This argument works also for each non-amenable residually finite group in place of $\boldsymbol{F}_2$.

We also provide two examples of non-rank-one odometer actions for amenable groups $G$. In the first example, $G$ is locally finite, and in the second one, $G$ is non-locally finite periodic.

*Example 4.5.* Let $Z = \{0, 1\}^{\mathbb{N}}$. Endow $Z$ with the infinite product $\eta$ of the equidistributions on $\{0, 1\}$. Fix a sequence $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ of mappings

$$s_n : \{0, 1\}^n \to \mathrm{Homeo}(\{0, 1\}), \quad n \geq 0.$$

Consider the following transformation $T_{\boldsymbol{s}}$ of $Z$:

$$T_{\boldsymbol{s}}(z_1, z_2, \ldots) := (s_0 z_1, s_1(z_1)z_2, s_2(z_1, z_2)z_3, s_3(z_1, z_2, z_3)z_4, \ldots).$$

Of course, $T_{\boldsymbol{s}}$ preserves $\eta$. Let

$$G := \{T_{\boldsymbol{s}} \mid \boldsymbol{s} = (s_n)_{n=0}^{\infty} \text{ with } s_n \equiv I \text{ eventually}\}.$$

Then, $G$ is a locally finite (and hence amenable) countable group. We claim that the dynamical system $(Z, \eta, G)$ is a $G$-odometer. Indeed, for each $n \in \mathbb{N}$, we denote by $\pi_n : Z \to \{0, 1\}^n$ the projection to the first $n$ coordinates. Of course, there is a natural transitive action of $G$ on $\{0, 1\}^n$:

$$T_{\boldsymbol{s}} * (z_1, \ldots, z_n) := (s_0 z_1, s_1(z_1)z_2, \ldots, s_{n-1}(z_1, \ldots, z_{n-1})z_n).$$

Then, $\pi_n$ is a $G$-equivariant mapping. Thus, $\{0, 1\}^n$ is a finite factor of $(Z, \eta, G)$. Let

$$\Gamma_n := \{g \in G \mid g * (0, \ldots, 0) = (0, \ldots, 0)\}.$$

In other words, $\Gamma_n$ is the stabilizer of a point $(0, \ldots, 0) \in \{0, 1\}^n$. It is straightforward to verify that

$$\Gamma_n = \{T_{\boldsymbol{s}} \mid \boldsymbol{s} = (s_k)_{k=0}^{\infty} \text{ with } s_0 = s_1(0) = s_2(0, 0) = \cdots = s_{n-1}(0, \ldots, 0) = I\}.$$

Hence, $\Gamma_n$ is a cofinite subgroup in $G$ for each $n$. Moreover, $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$. Thus, we obtain, for each $n$, a $G$-equivariant bijection $\phi_n : \{0, 1\}^n \to G/\Gamma_n$ such that the following diagram commutes:

$$\begin{array}{ccccccc} G/\Gamma_1 & \longleftarrow & G/\Gamma_2 & \longleftarrow & G/\Gamma_3 & \longleftarrow & \cdots \\ {\scriptstyle\phi_1}\uparrow & & {\scriptstyle\phi_2}\uparrow & & {\scriptstyle\phi_3}\uparrow & & \\ \{0, 1\} & \xleftarrow{\tau_1} & \{0, 1\}^2 & \xleftarrow{\tau_2} & \{0, 1\}^3 & \xleftarrow{\tau_3} & \cdots \end{array} \tag{4.5}$$

where $\tau_n(z_1, \ldots, z_{n+1}) := (z_1, \ldots, z_n)$ for each $(z_1, \ldots, z_{n+1}) \in \{0, 1\}^{n+1}$ and all $n \in \mathbb{N}$. It is routine to check that

$$\widetilde{\Gamma}_n := \bigcap_{g \in G} g\Gamma_n g^{-1} = \{T_{\boldsymbol{s}} \mid \boldsymbol{s} = (s_k)_{k=0}^{\infty} \text{ with } s_0 = I, s_1 \equiv I, s_2 \equiv I, \ldots, s_{n-1} \equiv I\}$$

and $\bigcap_{n=1}^{\infty} \widetilde{\Gamma}_n = \{I\}$. Denote by $(Y, O)$ the topological $G$-odometer associated with the sequence $(\Gamma_n)_{n=1}^{\infty}$. Furnish it with the Haar measure $\nu$. Then, equation (4.5) yields a $G$-equivariant isomorphism $\phi : (Z, \eta) \to (Y, \nu)$, as desired.

We now show that $(Z, \eta, G)$ is not free. Take a point $z = (z_n)_{n=1}^{\infty} \in Z$. Then, the $G$-stabilizer $G_z$ of $z$ is the group

$$\bigcap_{n=1}^{\infty} \{T_{\boldsymbol{s}} \mid \boldsymbol{s} = (s_k)_{k=0}^{\infty} \text{ with } s_0 = s_1(z_1) = s_2(z_1, z_2) = \cdots = s_n(z_1, \ldots, z_n) = I\}.$$

Let $r_1 : \{0, 1\} \to \text{Homeo}(\{0, 1\})$ be the only mapping such that $r_1(z_1) = I$ but $r_1 \not\equiv I$. We define a transformation $R$ of $(Z, \eta)$ by setting

$$R(z_1, z_2, z_3, \ldots) := (z_1, r_1(z_1)z_2, z_3, z_4, \ldots).$$

Then, $G_z \ni R \neq I$. Hence, $O$ is not free. Therefore, $O$ is not of rank one.

*Example 4.6.* Let $(Z, \eta)$ and $\pi_n$ be as in Example 4.5. Denote by $\mathcal{R}$ the tail equivalence relation on $Z$. We let

$$\mathcal{A} := \{T_s \mid \text{for each } z \in Z, \text{ there is } N > 0 \text{ with } s_n(z_1, \ldots, z_n) = I \text{ if } n > N\}.$$

Then, $\mathcal{A}$ is a subgroup of $[\mathcal{R}]$. Of course, $\mathcal{A}$ generates $\mathcal{R}$. Let $\theta$ denote the non-identity bijection of $\{0, 1\}$. Define four transformations $a, b, c, d \in \mathcal{A}$ by the following formulae:

$$a(z_1, z_2, \ldots) := (\theta(z_1), z_2, \ldots),$$
$$b(1^n, 0, z_{n+2}, \ldots) := \begin{cases} (1^n, 0, \theta(z_{n+2}), z_{n+3}, \ldots) & \text{if } n \notin 3\mathbb{Z}_+, \\ (1^n, 0, z_{n+2}, z_{n+3}, \ldots) & \text{otherwise}, \end{cases}$$
$$c(1^n, 0, z_{n+2}, \ldots) := \begin{cases} (1^n, 0, \theta(z_{n+2}), z_{n+3}, \ldots) & \text{if } n \notin 1 + 3\mathbb{Z}_+, \\ (1^n, 0, z_{n+2}, z_{n+3}, \ldots) & \text{otherwise}, \end{cases}$$
$$d(1^n, 0, z_{n+2}, \ldots) := \begin{cases} (1^n, 0, \theta(z_{n+2}), z_{n+3}, \ldots) & \text{if } n \notin 2 + 3\mathbb{Z}_+, \\ (1^n, 0, z_{n+2}, z_{n+3}, \ldots) & \text{otherwise}. \end{cases}$$

We remind that the group $G$ generated by $a, b, c, d$ is called the Grigorchuk group. It was introduced in [**Gri**]. The group is residually finite, amenable, non-locally finite. Every proper quotient subgroup of $G$ is finite. Of course, $G \subset \mathcal{A}$. It is routine to verify $\mathcal{R}$ is the $G$-orbit equivalence relation. Hence, $G$ is an ergodic transformation group of $(Z, \eta)$. Since $\pi_n$ is a $G$-equivariant mapping of $Z$ onto $\{0, 1\}^n$ and the dynamical system $(Z, \eta, G)$ is ergodic, $G$ acts transitively on $\{0, 1\}^n$. Therefore, repeating our reasoning in Example 4.5 almost literally, we obtain that $(Z, \eta, G)$ is isomorphic to the probability preserving $G$-odometer associated with the following sequence $(\Gamma_n)_{n=1}^\infty$ of cofinite subgroups $\Gamma_n \subset G$:

$$\Gamma_n := \{T_s \in G \mid s = (s_k)_{k=0}^\infty \text{ with } s_0 = s_1(0) = \cdots = s_{n-1}(0, \ldots, 0) = I\},$$

and $\bigcap_{g \in G} \bigcap_{n=1}^\infty g\Gamma_n g^{-1} = \{I\}$. Furthermore, the stabilizer $G_z$ of this odometer at a point $z = (z_n)_{n=1}^\infty \in Z$ is the group

$$\{T_s \in G \mid s = (s_k)_{k=0}^\infty \text{ with } s_0 = s_1(z_1) = s_2(z_1, z_2) = \cdots = s_n(z) = I\}.$$

Hence, $G$ is not free. Therefore, $G$ is not of rank one.

However, we will show that each probability preserving $G$-odometer is a factor of a rank-one $\sigma$-finite measure preserving $G$-action.

THEOREM 4.7. *Let $(Y, O)$ be a topological $G$-odometer associated with a decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups in $G$ satisfying equation (4.1). Then there is a topological $(C, F)$-action $T$ of $G$ on a locally compact Cantor space $X$ and a continuous*

*$G$-equivariant mapping $\tau : X \to Y$. Moreover, $\tau$ maps the Haar measure (see Remark 2.5) on $X$ to a (non-$\sigma$-finite, in general) measure which is equivalent to the Haar measure on $Y$. (This means that the two measures have the same class of subsets of zero measure.)*

*Proof.* Construct inductively sequences $(C_n)_{n=1}^{\infty}$ and $(F_n)_{n=0}^{\infty}$ of finite subsets in $G$ such that equations (2.1) and (2.4) hold, $C_n \subset \Gamma_n$ and the projection

$$C_n \ni c \mapsto c\Gamma_{n+1} \in \Gamma_n / \Gamma_{n+1} \quad \text{is one-to-one and onto} \tag{4.6}$$

for each $n \in \mathbb{N}$. Let $T$ be the topological $(C, F)$-action of $G$ associated with $(C_n, F_{n-1})_{n=1}^{\infty}$. By Proposition 2.9, $T$ is defined on the entire locally compact space $X = \bigcup_{n=0}^{\infty} X_n$, where $X_n = F_n \times C_{n+1} \times C_{n+2} \times \cdots$. We define $\tau : X \to Y$ by setting

$$\tau(x) = (f_n\Gamma_1, f_n\Gamma_2, \ldots, f_n\Gamma_{n+1}, f_n c_{n+1}\Gamma_{n+2}, , f_n c_{n+1} c_{n+2}\Gamma_{n+3}, \ldots) \in Y$$

if $x = (f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n$ for some $n \in X_n$. Of course, $\tau$ is well defined, continuous and $G$-equivariant. Thus, the first claim of the proposition is proved.

Let $\mu$ denote the Haar measure on $X$ and let $\chi$ be the Haar measure on $Y$. Then, $\mu$ is the unique $T$-invariant $(C, F)$-measure such that $\mu(X_0) = 1$. It is determined by the sequence $(\kappa_n, \nu_{n-1})_{n=1}^{\infty}$, where $\kappa_n$ is the equidistribution on $C_n$, and $\nu_n(f) = \prod_{k=1}^{n} \kappa_k(1_G)$ for each $f \in F_n$ and $n \in \mathbb{N}$. We now show that $(\mu \restriction X_0) \circ \tau^{-1} = \chi$. Let $\tau_n$ stand for the mapping

$$C_1 \times \cdots \times C_n \ni (c_1, \ldots, c_n) \mapsto c_1 \cdots c_n \Gamma_{n+1} \in G/\Gamma_{n+1}.$$

We note that if

$$\tau_n(c_1, \ldots, c_n) = \tau_n(c_1', \ldots, c_n')$$

for some $c_1, c_1' \in C_1, \ldots, c_n, c_n' \in C_n$, then $c_j = c_j'$ for each $j = 1, \ldots, n$. Indeed,

$$c_1\Gamma_2 = c_1 \cdots c_n \Gamma_{n+1}\Gamma_2 = c_1' \cdots c_n' \Gamma_{n+1}\Gamma_2 = c_1'\Gamma_2.$$

Therefore, equation (4.6) yields that $c_1 = c_1'$ and hence $c_2 \cdots c_n \Gamma_{n+1} = c_2' \cdots c_n' \Gamma_{n+1}$. Arguing in a similar way, we obtain that $c_2 = c_2', \ldots, c_n = c_n'$, as claimed. It follows that $\tau_n$ is one-to-one. Moreover, $\tau_n$ is onto in view of equation (4.5). Then it is straightforward to verify that the diagram

$$\begin{array}{ccccccc}
G/\Gamma_2 & \longleftarrow & G/\Gamma_3 & \longleftarrow & G/\Gamma_4 & \longleftarrow & \cdots \\
{\scriptstyle \tau_1}\uparrow & & {\scriptstyle \tau_2}\uparrow & & {\scriptstyle \tau_3}\uparrow & & \\
C_1 & \longleftarrow & C_1 \times C_2 & \longleftarrow & C_1 \times C_2 \times C_3 & \longleftarrow & \cdots
\end{array}$$

commutes. Passing to the projective limit, we obtain that $\tau$ is a homeomorphism of $X_0$ onto $Y$. Since $\tau_n$ maps the equidistribution on $C_1 \times \cdots \times C_n$ to the equidistribution on $G/\Gamma_{n+1}$ for each $n$, it follows that $\tau$ maps $\mu \restriction X_0$ to $\chi$. Take a probability measure $\mu'$ on $X$ which is equivalent to $\mu$. Then,

$$\mu' \circ \tau^{-1} \gg (\mu' \restriction X_0) \circ \tau^{-1} \sim (\mu \restriction X_0) \circ \tau^{-1} = \chi. \tag{4.7}$$

Since $\mu'$ is quasi-invariant and ergodic under $T$, it follows that $\mu' \circ \tau^{-1}$ is quasi-invariant and ergodic under $O$. As the two probability Borel measures $\mu' \circ \tau^{-1}$ and $\chi$ on $Y$ are quasi-invariant and ergodic under $O$, they are either equivalent or mutually singular.

Therefore, equation (4.7) yields that $\mu' \circ \tau^{-1} \sim \chi$. Hence, $\mu \circ \tau^{-1}$ is equivalent to $\chi$, as desired. □

*Remark 4.8.* If we change the construction of $T$ in the proof of Theorem 4.7 in such a way that the mapping in equation (4.6) is one-to-one but $\#(C_n)/\#(\Gamma_n/\Gamma_{n+1}) \le 0.5$ for each $n \in \mathbb{N}$, then the first claim of Theorem 4.7 still holds: there is a $G$-equivariant continuous mapping $\tau : X \to Y$. However, the second claim fails: the $O$-quasi-invariant measure $\mu \circ \tau^{-1}$ on $Y$ will be singular with $\chi$.

We note that if $G$ is Abelian, then each ergodic non-singular $G$-action $T$ possesses the following property. Let $\Gamma_1 \subsetneq \Gamma_2 \subsetneq \cdots$ be a sequence of cofinite subgroups in $G$ with $\bigcap_{n\in\mathbb{N}} \Gamma_n = \{1_G\}$. If, for each $n \in \mathbb{N}$, $T$ has a finite factor $\mathfrak{F}_n$ isomorphic to the homogeneous $G$-space $G/\Gamma_n$, then $\mathfrak{F}_1 \subsetneq \mathfrak{F}_2 \subsetneq \cdots$ and $T$ has an odometer factor $\bigvee_{n>0} \mathfrak{F}_n$. This is no longer true if $G$ is non-Abelian (see Example 3.5). However, the following version of the aforementioned property holds for an arbitrary $G$.

THEOREM 4.9. *Let $T = (T_g)_{g\in G}$ be an ergodic non-singular $G$-action on a standard non-atomic probability space $(X, \mathfrak{B}, \mu)$. Let $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ be a sequence of cofinite subgroups in $G$ such that equation (4.1) holds. Denote by $(Y, O)$ the topological $G$-odometer associated with this sequence. Suppose that for each $n \in \mathbb{N}$, there exists a $T$-factor which is isomorphic to the homogeneous $G$-space $G/\Gamma_n$. Then there is an $O$-quasi-invariant measure $\nu$ on $Y$ such that the non-singular odometer $(Y, \nu, O)$ is a factor of $(X, \mu, T)$.*

*Proof.* We first prove an auxiliary claim.

CLAIM A. *Let $H$ be a cofinite subgroup in $G$. There exist no more than $\#(G/H)$ different factors of $T$ that are isomorphic to the homogeneous $G$-space $G/H$.*

*Proof.* Let $J := \#(G/H) + 1$. Suppose that there are $J$ pairwise different $T$-invariant $\sigma$-algebras $\mathfrak{F}_j \subset \mathfrak{B}$ such that $T \restriction \mathfrak{F}_j$ is isomorphic to $G/H$ for each $j \in J$. Denote by $\tau_j : X \to G/H$ the corresponding $G$-equivariant mapping. Then the mapping

$$\tau : X \ni x \mapsto (\tau_j(x))_{j\in J} \in (G/H)^J$$

is also $G$-equivariant. Denote by $E$ the support of the measure $\mu \circ \tau^{-1}$. Then:

- $E$ is a single $G$-orbit;
- the projection of $E$ onto each of the $J$ coordinates is onto.

Take a point $(g_j H)_{j\in J} \in E$. Since $J > \#(G/H)$, there are $i_0, j_0 \in J$ such that $i_0 \ne j_0$ but $g_{i_0} H = g_{j_0} H$. Hence, the projection of $E$ onto the 'plane' generated by the $i_0$ and $j_0$ coordinates is the diagonal $\{(gH, gH) \mid g \in G\}$ in $(G/H)^2$. Hence, $\mathfrak{F}_{i_0} = \mathfrak{F}_{j_0}$. This contradiction proves Claim A. □

To prove the theorem, we define a graded graph $\mathcal{G}$. The set $V$ of vertices of $\mathcal{G}$ is the union $\bigsqcup_{n\ge 0} V_n$, where $V_n$ is the set of all Borel $G$-equivariant maps from $X$ to $G/\Gamma_n$. For the consistency of notation, we let $\Gamma_0 := G$. Given $n \ge 0$, we denote by $\theta_n$ the projection

$$G/\Gamma_{n+1} \ni g\Gamma_{n+1} \mapsto g\Gamma_n \in G/\Gamma_n.$$

The set $E$ of edges of $\mathcal{G}$ is the union $\bigsqcup_{n\geq 0} E_n$, where an edge $e \in E_n$ joins a vertex $\pi \in V_n$ with a vertex $\tau \in V_{n+1}$ if $\pi = \theta_n \circ \tau$. It follows from Claim A that $V_n$ is finite for each $n$. Of course, every vertex from $V_n$ is adjacent (that is, connected by an edge) with a vertex in $V_{n-1}$ for each $n \in \mathbb{N}$. Hence, for each vertex of $\mathcal{G}$, there is a path connecting this vertex with the only vertex from $V_0$. Thus, $\mathcal{G}$ is connected. Of course, $\mathcal{G}$ is locally finite and infinite. Hence, by Kőnig's infinity lemma, $\mathcal{G}$ contains a ray. It follows that there exists a Borel $G$-equivariant mapping $\iota : X \to Y$. We set $\nu := \mu \circ \iota^{-1}$. Then, $(Y, \nu, O)$ is a factor of $(X, \mu, T)$, as desired. □

4.2. *Normal covers for non-singular odometers.* Let $(Y, O)$ be a topological $G$-odometer associated with a decreasing sequence $(\Gamma_n)_{n=1}^\infty$ of cofinite subgroups in $G$ such that equation (4.1) holds. If each $\Gamma_n$ is normal in $G$, then $(Y, O)$ is called *normal*. In this case, we have that $G/\Gamma_n$ is a finite group and hence $Y$ is a compact totally disconnected metric group. Moreover, there is a one-to-one group homomorphism $\phi : G \to Y$ such that $O_g y = \phi(g)y$ for all $g \in G$ and $y \in Y$. Of course, $\phi(g) = (g\Gamma_1, g\Gamma_2, \ldots) \in Y$ for each $g \in G$. This homomorphism embeds $G$ densely into $Y$. Every normal odometer is free.

Given a cofinite subgroup $\Gamma$ in $G$, the subgroup $\widetilde{\Gamma} := \bigcap_{g\in G} g\Gamma g^{-1}$ is the maximal normal (in $G$) subgroup of $\Gamma$. Of course, $\widetilde{\Gamma}$ is of finite index in $G$. The natural projection $G/\widetilde{\Gamma} \ni g\widetilde{\Gamma} \mapsto g\Gamma \in G/\Gamma$ is $G$-equivariant. Hence, for a decreasing sequence $\Gamma_1 \supset \Gamma_2 \supset \cdots$ of cofinite subgroups in $G$ satisfying equation (4.1), we obtain a decreasing sequence $\widetilde{\Gamma}_1 \supset \widetilde{\Gamma}_2 \supset \cdots$ of normal cofinite subgroups in $G$ with $\bigcap_{n=1}^\infty \widetilde{\Gamma}_n = \{1_G\}$. Let $(\widetilde{Y}, \widetilde{O})$ denote the normal topological $G$-odometer associated with $(\widetilde{\Gamma}_n)_{n=1}^\infty$. It is called *the topological normal cover* of $(Y, O)$. The natural projections

$$G/\widetilde{\Gamma}_n \ni g\widetilde{\Gamma}_n \mapsto g\Gamma_n \in G/\widetilde{\Gamma}_n, \quad n \in \mathbb{N},$$

generate a continuous projection $w : \widetilde{Y} \to Y$ that intertwines $\widetilde{O}$ with $O$. Let

$$H := \{(\widetilde{y}_n)_{n=1}^\infty \in \widetilde{Y} \mid \widetilde{y}_n \in \Gamma_n/\widetilde{\Gamma}_n \text{ for all } n \in \mathbb{N}\}.$$

Then, $H$ is a closed subgroup of $\widetilde{Y}$. We claim that $\omega$ is the quotient mapping

$$\widetilde{Y} \ni \widetilde{y} \mapsto \widetilde{y}H \in \widetilde{Y}/H.$$

Indeed, we first observe that $\omega(\widetilde{y}) = \omega(\widetilde{y}h)$ for all $\widetilde{y} \in \widetilde{Y}$ and $h \in H$. Second, if $\omega(\widetilde{y}) = \omega(\widetilde{z})$ for some $\widetilde{y}, \widetilde{z} \in \widetilde{Y}$, then $\widetilde{y}\widetilde{z}^{-1} \in H$. Finally, the subset $\omega(\widetilde{Y})$ is $G$-invariant and closed in $Y$. Hence, $\omega(\widetilde{Y}) = Y$.

It may seem that the coordinate projection $H \ni (\widetilde{y}_n)_{n=1}^\infty \mapsto \widetilde{y}_n \in \Gamma_n/\widetilde{\Gamma}_n$ is onto for each $n \in \mathbb{N}$. That is not true. A counterexample (in which $H$ is trivial but $\#(\Gamma_n/\widetilde{\Gamma}_n) = 2$ for each $n$) is constructed in Example 6.7 below.

The concepts of the topological normal cover and the normal cover in the finite measure preserving case can be found in [**CorPe**, **DaLe**]. We adapt it to the non-singular case in the following way.

*Definition 4.10.* Let $(Y, \nu, O)$ be a non-singular $G$-odometer, $(\widetilde{Y}, \widetilde{O})$ the topological normal cover of $(Y, O)$ and $\widetilde{\nu}$ an $\widetilde{O}$-quasi-invariant probability on $\widetilde{Y}$. We call the non-singular normal odometer $(\widetilde{Y}, \widetilde{\nu}, \widetilde{O})$ *the normal cover* of $(Y, \nu, O)$ if:

(i) $\widetilde{\nu} \circ \omega^{-1} = \nu$; and
(ii) $d\widetilde{\nu} \circ \widetilde{O}_g / d\widetilde{\nu} = (d\nu \circ O_g / d\nu) \circ \omega$ for each $g \in G$.

We note that property (ii) means that $\widetilde{O}$ is $\omega$-relatively finite measure preserving.

PROPOSITION 4.11. *Given a non-singular G-odometer $(Y, \nu, O)$, there is a G-quasi-invariant probability $\widetilde{\nu}$ on $\widetilde{Y}$ such that $(\widetilde{Y}, \widetilde{\nu}, \widetilde{O})$ is a normal cover of $(Y, \nu, O)$.*

*Proof.* Without loss of generality, we may assume that $\widetilde{Y} = Y \times H$ (as a set, not as a group) and there is a Borel map (1-cocycle) $s : G \times Y \to H$ such that

$$\widetilde{O}_g(y, h) = (O_g y, s(g, y)h) \quad \text{and} \quad \omega(y, h) = y \quad \text{for each } (y, h) \in Y \times H.$$

Denote by $\lambda_H$ the Haar measure on $H$. Then the direct product $\widetilde{\nu} := \nu \otimes \lambda_H$ satisfies properties (i) and (ii) from Definition 4.10. □

Let $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ be as above. Suppose that there is a sequence $(b_n)_{n=1}^{\infty}$ of $G$-elements such that

$$b_1 \Gamma_1 b_1^{-1} \supsetneq b_2 \Gamma_2 b_2^{-1} \supsetneq \cdots .$$

Denote by $(Y', O')$ the topological odometer associated with this sequence. Let $\nu$ and $\nu'$ stand for the Haar measures on $Y$ and $Y'$, respectively.

PROPOSITION 4.12. *The odometers $(Y', O', \nu')$ and $(Y, O, \nu)$ are isomorphic.*

*Proof.* It is easy to see that the normal covers of $(Y', O', \nu')$ and $(Y, O, \nu)$ are the same. Denote this common normal cover by $(\widetilde{Y}, \widetilde{\nu}, \widetilde{O})$. Then there are closed subgroups $H$ and $H'$ of $\widetilde{Y}$ such that $(Y, O, \nu)$ is the right $H$-quotient of $(\widetilde{Y}, \widetilde{\nu}, \widetilde{O})$ and $(Y', O', \nu')$ is the right $H'$-quotient of $(\widetilde{Y}, \widetilde{\nu}, \widetilde{O})$. It follows from Theorem 4.9 that $(Y', O', \nu')$ and $(Y, O, \nu)$ are weakly equivalent, that is, $(Y', O', \nu')$ is a factor of $(Y, O, \nu)$ and $(Y, O, \nu)$ is a factor of $(Y', O', \nu')$. Hence, there are compact subgroups $K$ and $K'$ of $\widetilde{Y}$ and elements $a, b \in \widetilde{Y}$ such that $K \supset H$, $K' \supset H'$, $K = aH'a^{-1}$ and $K' = bHb^{-1}$. Hence, $H \subset abHb^{-1}a^{-1}$. We claim that this implies that $H = abHb^{-1}a^{-1}$. Indeed, let $V := \{y \in \widetilde{Y} \mid H \subset yHy^{-1}\}$. Then $V$ is a closed subset of $\widetilde{Y}$. Of course, $V \ni (ab)^n$ for each $n \in \mathbb{N}$. Hence, $V$ includes the closure of the semigroup $\{(ab)^n \mid n \in \mathbb{N}\}$. It follows from [**HeRo**, Theorem 9.1] that the closure of $\{(ab)^n \mid n \in \mathbb{N}\}$ equals the closure of the group $\{(ab)^n \mid n \in \mathbb{Z}\}$. Hence, $(ab)^{-1} \in V$, that is, $H \supset abHb^{-1}a^{-1}$. Therefore, $H = abHb^{-1}a^{-1}$, as claimed. This yields that $K = H$ and $K' = H'$. Thus, we obtain that $H$ and $H'$ are conjugate. Hence, $(Y', O', \nu')$ and $(Y, O, \nu)$ are isomorphic. □

## 5. *Odometer factors of non-singular $(C, F)$-actions*

The following concept is an 'infinite' analogue of Definition 3.1.

*Definition 5.1.* Let a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ satisfy equations (2.1)–(2.3) and Proposition 2.10(ii), and let a sequence $(\Gamma_n)_{n=1}^{\infty}$ satisfy equation (4.1). Denote by

$(Y, O)$ the topological odometer associated with $(\Gamma_n)_{n=1}^\infty$. Given $y = (g_n\Gamma_n)_{n=1}^\infty \in Y$, we say that $\mathcal{T}$ is *compatible with y* if

$$\sum_{n=1}^{\infty} \kappa_n(\{c \in C_n \mid c \notin g_n\Gamma_n g_n^{-1}\}) < \infty.$$

Denote by $T$ the $(C, F)$-action of $G$ associated with $\mathcal{T}$. Let $X$ be the space of $T$ and let $\mu$ stand for the non-singular $(C, F)$-measure on $X$ determined by $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$. As

$$g_n\Gamma_n g_n^{-1} = g_{n+1}\Gamma_n g_{n+1}^{-1} \supset g_{n+1}\Gamma_{n+1} g_{n+1}^{-1} \quad \text{for each } n \in \mathbb{N},$$

it follows that if $\mathcal{T}$ is compatible with $y$, then $\mathcal{T}$ is compatible with the coset $g_n\Gamma_n \in G/\Gamma_n$ in the sense of Definition 3.1 for each $n \in \mathbb{N}$. Hence, the $(\mathcal{T}, g_n\Gamma_n)$-factor mapping $\pi_{(\mathcal{T}, g_n\Gamma_n)} : X \to G/\Gamma_n$ for $T$ is well defined (mod 0) for each $n \in \mathbb{N}$. Moreover, a measurable mapping

$$\pi_{(\mathcal{T},y)} := X \ni x \mapsto (\pi_{(\mathcal{T}, g_n\Gamma_n)}(x))_{n=1}^\infty \in Y$$

is well defined (mod 0) too. Of course, $\pi_{(\mathcal{T},y)} \circ T_g = O_g \circ \pi_{(\mathcal{T},y)}$ for each $g \in G$. Hence, the non-singular odometer $(Y, \mu \circ \pi_{(\mathcal{T},y)}^{-1}, O)$ is a factor of $(X, \mu, T)$.

*Definition 5.2.* We call $\pi_{(\mathcal{T},y)}$ *the* $(\mathcal{T}, y)$*-factor mapping* for $T$.

In the proposition below, we find necessary and sufficient conditions (in terms of the parameters $\mathcal{T}$) under which $\pi_{(\mathcal{T},y)}$ is one-to-one, that is, the dynamical systems $(X, \mu, T)$ and $(Y, \mu \circ \pi_{(\mathcal{T},y)}^{-1}, O)$ are isomorphic via $\pi_{(\mathcal{T},y)}$.

PROPOSITION 5.3. *Let $\mathcal{T}$ be compatible with y. The following are equivalent:*

(i) *$\pi_{(\mathcal{T},y)}$ is one-to-one (mod 0);*
(ii) *for each $n > 0$ and $\epsilon > 0$, there are $l > 0$ and a subset $D_l \subset G/\Gamma_l$ such that $\mu([1_G]_n \triangle \pi_{(\mathcal{T}, g_l\Gamma_l)}^{-1}(D_l)) < \epsilon$; and*
(iii) *for each $n > 0$ and $\epsilon > 0$, there are $l > 0$, a subset $D_l \subset G/\Gamma_l$ and $M > 0$ such that $\nu_m(C_{n+1} \cdots C_m \triangle \{f \in F_m \mid f g_l\Gamma_l \in D_l\}) < \epsilon$ for each $m > M$.*

*Proof.* (i)⇔(ii) Denote by $\mathfrak{B}$ and $\mathfrak{Y}$ the Borel $\sigma$-algebra on $X$ and $Y$, respectively. Of course, $\pi_{(\mathcal{T},y)}$ is one-to-one (mod 0) if and only if $\mathfrak{B} = \pi_{(\mathcal{T},y)}^{-1}(\mathfrak{Y})$ (mod 0). Since:

- $\mathfrak{B}$ is generated by the family of all cylinders in $X$; and
- $\mathfrak{B}$ is invariant under $T$,

it follows that $\mathfrak{B} = \pi_{(\mathcal{T},y)}^{-1}(\mathfrak{Y})$ if and only if $[1_G]_n \in \pi_{(\mathcal{T},y)}^{-1}(\mathfrak{Y})$ for each $n > 0$. Let $\mathfrak{Y}_l \subset \mathfrak{Y}$ denote the finite sub-$\sigma$-algebra of subsets that are measurable with respect to the canonical projection $G \to G/\Gamma_l$. Then, $\mathfrak{Y}_1 \subset \mathfrak{Y}_2 \subset \cdots$ and the union $\bigcup_{l=1}^\infty \mathfrak{Y}_l$ is dense in $\mathfrak{Y}$. It follows that $[1_G]_n \in \pi_{(\mathcal{T},y)}^{-1}(\mathfrak{Y})$ if and only if

$$\min_{A \in \mathfrak{Y}_l} \mu([1_G]_n \triangle \pi_{(\mathcal{T},y)}^{-1}(A)) \to 0 \quad \text{as } l \to \infty.$$

It remains to note that $\{\pi_{(\mathcal{T},y)}^{-1}(A) \mid A \in \mathfrak{Y}_l\} = \{\pi_{(\mathcal{T}, g_l\Gamma_l)}^{-1}(D) \mid D \subset G/\Gamma_l\}$.

(ii)⇔(iii) We note that for all $n > 0$, $l > 0$ and a subset $D \subset G/\Gamma_l$,

$$\mu([1_G]_n \triangle \pi^{-1}_{(\mathcal{T}, g_l\Gamma_l)}(D)) = \lim_{m\to\infty} \mu([1_G]_n \triangle \pi^{-1}_{(\mathcal{T}, g_l\Gamma_l)}(D) \cap [F_m]_m) \text{ and}$$

$$\lim_{m\to\infty} \bigotimes_{j\geq m} \kappa_j(\{x = (c_j)_{j\geq m} \in C_m \times C_{m+1} \times \cdots \mid c_j \in g_l\Gamma_l g_l^{-1} \quad \text{for all } j \geq m\}) = 1.$$

The latter follows from the fact that $\mathcal{T}$ is compatible with $y$. It implies that

$$\lim_{m\to\infty} \mu(\{x = (f_m, c_{n+1}, \ldots) \in [F_m]_m \mid \pi_{(\mathcal{T}, g_l\Gamma_l)}(x) = f_m g_l \Gamma_l\}) = \mu([F_m]_m).$$

Hence, for each $D \subset \Gamma_l$,

$$\begin{aligned}\mu([1_G]_n \triangle \pi^{-1}_{(\mathcal{T}, g_l\Gamma_l)}(D)) &= \lim_{m\to\infty} \mu\Big([1_G]_n \triangle \bigsqcup_{f\in F_m, fg_l\Gamma_l \in D} [f]_m\Big) \\ &= \lim_{m\to\infty} \nu_m(C_{n+1}\cdots C_m \triangle \{f \in F_m \mid fg_l\Gamma_l \in D\}).\end{aligned}$$

This equality implies the equivalence of properties (ii) and (iii). □

The following theorem is the main result of this section.

THEOREM 5.4. *Let a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ satisfy equations (2.1)–(2.3) and Proposition 2.10(ii). Let $T$ be the non-singular $(C, F)$-action of $G$ associated with $\mathcal{T}$ and let $(Y, O)$ be the topological $G$-odometer associated with a sequence $(\Gamma_n)_{n=1}^{\infty}$ satisfying equation (4.1). Then, for each $G$-equivariant measurable mapping $\tau : X \to Y$, there exist an increasing sequence $\boldsymbol{q}$ of non-negative integers and an element $y \in Y$ such that the $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}} = (\widetilde{C}_n, \widetilde{F}_{n-1}, \widetilde{\kappa}_n, \widetilde{\nu}_{n-1})_{n=1}^{\infty}$ of $\mathcal{T}$ is compatible with $y$ and*

$$\pi_{(\widetilde{\mathcal{T}}, y)} \circ \iota_{\boldsymbol{q}} = \tau. \tag{5.1}$$

*Moreover, $\tau$ is one-to-one (mod 0) if and only if for each $n > 0$ and $\epsilon > 0$, there are $l > 0$, a subset $D_l \subset G/\Gamma_l$ and $M > n$ such that for each $m > M$,*

$$\widetilde{\nu}_m(\widetilde{C}_{n+1}\cdots \widetilde{C}_m \triangle \{f \in \widetilde{F}_m \mid fg_l\Gamma_l \in D_l\}) < \epsilon.$$

We preface the proof of Theorem 5.4 with auxiliary simple but useful facts about factor mappings.

LEMMA 5.5. *Let $\Gamma, \Gamma_1$ be two cofinite subgroups in $G$ and $\Gamma_1 \subset \Gamma$. Then:*

(i) *if $\mathcal{T}$ is compatible with a coset $g\Gamma \in G/\Gamma$, then for each increasing sequence $\boldsymbol{a} = (a_n)_{n=0}^{\infty}$ of non-negative integers with $a_0 = 0$, the $\boldsymbol{a}$-telescoping $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ is also compatible with $g\Gamma$ and*

$$\pi_{(\mathcal{T}, g\Gamma)} = \pi_{(\widetilde{\mathcal{T}}, g\Gamma)} \circ \iota_{\boldsymbol{a}}; \tag{5.2}$$

(ii) *if $\mathcal{T}$ is compatible with two cosets $g\Gamma \in G/\Gamma$ and $g_1\Gamma_1 \in G/\Gamma_1$, then*

$$r \circ \pi_{(\mathcal{T}, g_1\Gamma_1)} = \pi_{(\mathcal{T}, g\Gamma)} \quad \text{if and only if } g_1 g^{-1} \in \Gamma,$$

*where $r : G/\Gamma_1 \to G/\Gamma$ denotes the natural projection.*

*Proof.* (i) For each $n > 0$, we let $\widetilde{C}_n := C_{a_{n-1}+1} \cdots C_{a_n}$ and $\widetilde{\kappa}_n := \kappa_{a_{n-1}} * \cdots * \kappa_{a_n}$. Then,

$$\widetilde{\kappa}_n(\{c \in \widetilde{C}_n \mid c \notin g\Gamma g^{-1}\}) \leq \sum_{j=a_{n-1}+1}^{a_n} \kappa_j(\{c \in C_j \mid c \notin g\Gamma g^{-1}\}).$$

Hence,

$$\sum_{n=1}^{\infty} \widetilde{\kappa}_n(\{c \in \widetilde{C}_n \mid c \notin g\Gamma g^{-1}\}) \leq \sum_{j=1}^{\infty} \kappa_j(\{c \in C_j \mid c \notin g\Gamma g^{-1}\}) < \infty.$$

Thus, $\widetilde{\mathcal{T}}$ is compatible with $g\Gamma$.

Equation (5.2) and claim (ii) are verified straightforwardly. □

*Proof of Theorem 5.4.* We note that $\tau(x) = (\tau_n(x))_{n=1}^{\infty}$ for each $x \in X$, where $\tau_n : X \to G/\Gamma_n$ is a $G$-equivariant mapping for every $n$.

From now on, we will argue inductively. At the first step we apply Remark 3.4 to $T$ and $\tau_1$: there exist a coset $g_1\Gamma_1 \in G/\Gamma_1$ and an increasing sequence $\boldsymbol{q}^1$ of non-negative integers such that the $\boldsymbol{q}^1$-telescoping $\mathcal{T}_1$ of $\mathcal{T}$ is $g_1\Gamma_1$-compatible and

$$\pi_{(\mathcal{T}_1, g_1\Gamma_1)} \circ \iota_{\boldsymbol{q}^1} = \tau_1. \tag{5.3}$$

At the second step, we apply Remark 3.4 to the $(C, F)$-action $\widetilde{T}$ of $G$ associated with $\mathcal{T}_1$ and the factor mapping $\tau_2 \circ \iota_{\boldsymbol{q}^1}{}^{-1}$ of $\widetilde{T}$: there exist a coset $g_2\Gamma_2 \in G/\Gamma_2$ and an increasing sequence $\boldsymbol{q}^2$ of non-negative integers such that the $\boldsymbol{q}^2$-telescoping $\mathcal{T}_2$ of $\mathcal{T}_1$ is $g_2\Gamma_2$-compatible and

$$\pi_{(\mathcal{T}_2, g_2\Gamma_2)} \circ \iota_{\boldsymbol{q}^2} = \tau_2 \circ \iota_{\boldsymbol{q}^1}{}^{-1}. \tag{5.4}$$

Consider the natural projection $\omega_{2,1} : G/\Gamma_2 \to G/\Gamma_1$. Since $\omega_{2,1} \circ \tau_2 = \tau_1$, it follows from equations (5.3) and (5.4) that

$$\omega_{2,1} \circ \pi_{(\mathcal{T}_2, g_2\Gamma_2)} \circ \iota_{\boldsymbol{q}^2} = \pi_{(\mathcal{T}_1, g_1\Gamma_1)}.$$

Then, Lemma 5.5(i),(ii) imply that $g_2 g_1^{-1} \in \Gamma_1$. Continuing inductively, we obtain a sequence $(g_n)_{n=1}^{\infty}$ of elements in $G$ and a sequence $(\boldsymbol{q}^n)_{n=1}^{\infty}$ of increasing sequences of non-negative integers such that for each $n > 0$:

($\alpha_1$) $g_n g_{n-1}^{-1} \in \Gamma_{n-1}$;

($\alpha_2$) the $\boldsymbol{q}^n$-telescoping $\mathcal{T}_n = (C_k^{(n)}, F_{k-1}^{(n)}, \kappa_k^{(n)}, \nu_{k-1}^{(n)})_{k=1}^{\infty}$ of $\mathcal{T}_{n-1}$ is compatible with the coset $g_n\Gamma_n$; and

($\alpha_3$) $\pi_{(\mathcal{T}_n, g_n\Gamma_n)} \circ \iota_{\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n} = \tau_n$.

We now choose an integer $a_n > 0$ large so that:

($\alpha_4$) $\sum_{k \geq a_n} \kappa_k^{(n)}(\{c \in C_k^{(n)} \mid c \notin g_n\Gamma_n g_n^{-1}\}) < 1/n^2$

for each $n > 0$. It follows from ($\alpha_1$) that the sequence $y := (g_n\Gamma_n)_{n=1}^{\infty}$ is a well-defined element of $Y$. Of course:

$\boldsymbol{q}^1 \circ \boldsymbol{q}^2$ is a subsequence of $\boldsymbol{q}^1$;
$\boldsymbol{q}^1 \circ \boldsymbol{q}^2 \circ \boldsymbol{q}^3$ is a subsequence of $\boldsymbol{q}^1 \circ \boldsymbol{q}^2$

and so on. Hence, using the diagonalization method, we can construct a sequence $\boldsymbol{q} = (q_n)_{n=1}^{\infty}$ of integers such that:

- $0 = q_0 < q_1 < \cdots < q_n$;
- $\boldsymbol{q}$ is a subsequence of $\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n$; and
- $q_n \geq a_n$

for every $n > 0$. Denote by $\widetilde{\mathcal{T}}$ the $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}} = (\widetilde{C}_n, \widetilde{F}_{n-1}, \widetilde{\kappa}_n, \widetilde{\nu}_{n-1})_{n=1}^{\infty}$ of $\mathcal{T}$. We are going to show that $\widetilde{\mathcal{T}}$ is compatible with $y$. By the construction of $\boldsymbol{q}$, for each $n > 0$, there are integers $d_{n,2} \geq d_{n,1} \geq a_n$ such that

$$\widetilde{C}_n = C_{d_{n,1}}^{(n)} \cdots C_{d_{n,2}}^{(n)} \quad \text{and} \quad \widetilde{\kappa}_n = \kappa_{d_{n,1}} * \cdots * \kappa_{d_{n,2}}.$$

Therefore, we deduce from $(\alpha_4)$ that

$$\widetilde{\kappa}_n(\{c \in \widetilde{C}_n \mid c \notin g_n \Gamma_n g_n^{-1}\}) \leq \sum_{k=d_{n,1}}^{d_{n,2}} \kappa_k^{(n)}(\{c \in C_k^{(n)} \mid c \notin g_n \Gamma_n g_n^{-1}\}) < \frac{1}{n^2}.$$

Hence, $\sum_{n=1}^{\infty} \widetilde{\kappa}_n(\{c \in \widetilde{C}_n \mid c \notin g_n \Gamma_n g_n^{-1}\}) < \infty$, that is, $\widetilde{\mathcal{T}}$ is compatible with $y$, as desired.

We now prove equation (5.1). Of course, $\widetilde{\mathcal{T}}$ is a telescoping of the $\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n$-telescoping of $\mathcal{T}$ for each $n$. In view of $(\alpha_2)$, the $\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n$-telescoping of $\mathcal{T}$ equals $\mathcal{T}_n$. Thus, $\widetilde{\mathcal{T}}$ is a telescoping of $\mathcal{T}_n$. Denote by $\theta_n$ the canonical isomorphism corresponding to this telescoping. Then $\iota_{\boldsymbol{q}} = \theta_n \circ \iota_{\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n}$. It follows from this and $(\alpha_3)$ that

$$\pi_{(\widetilde{\mathcal{T}}, g_n \Gamma_n)} \circ \iota_{\boldsymbol{q}} = \pi_{(\widetilde{\mathcal{T}}, g_n \Gamma_n)} \circ \theta_n \circ \iota_{\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n} = \pi_{(\mathcal{T}_n, g_n \Gamma_n)} \circ \iota_{\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n} = \tau_n$$

for each $n \in \mathbb{N}$. Hence, $\pi_{(\widetilde{\mathcal{T}}, y)} = \tau$, as desired.

The second (the last) claim of the theorem follows from the first one and Proposition 5.3. □

As a corollary, we obtain a criterion for the existence (or non-existence) of odometer factors for rank-one non-singular actions.

COROLLARY 5.6. *Let $T$ be the non-singular $(C, F)$-action of $G$ associated with a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ satisfying equations (2.1)–(2.3) and Proposition 2.10(ii). Then $T$ has no non-singular odometer factors if and only if for each decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups in $G$ satisfying equation (4.1), no telescoping of $\mathcal{T}$ is compatible with the sequence $(\Gamma_n)_{n=1}^{\infty}$, that is, for each sequence $0 = q_1 < q_2 < \cdots$,*

$$\sum_{n=0}^{\infty} \kappa_{q_n+1} * \cdots * \kappa_{q_{n+1}}(\{c \in C_{q_n+1} \cdots C_{q_{n+1}} \mid c \notin \Gamma_n\}) = \infty.$$

*Proof.* It is sufficient to use Theorem 5.4 and the following remark.

Let $Y$ be a $G$-odometer associated with a decreasing sequence $(\Gamma_n)_{n=1}^{\infty}$ of cofinite subgroups in $G$ such that equation (4.1) holds. Let $y \in Y$. Then, $y = (g_n \Gamma_n)_{n=1}^{\infty}$ with $g_n g_{n+1}^{-1} \in \Gamma_n$ for each $n \in \mathbb{N}$. Of course, $g_1 \Gamma_1 g_1^{-1} \supset g_2 \Gamma_2 g_2^{-1} \supset \cdots$ and the sequence $(g_n \Gamma_n g_n^{-1})_{n=1}^{\infty}$ satisfies equation (4.1). Denote by $Y_y$ the space of the $G$-odometer associated with $(g_n\Gamma_n g_n^{-1})_{n=1}^\infty$. Then there is a canonical $G$-equivariant homeomorphism $\varphi_y : Y \to Y_y$. It is well defined by the formula

$$\varphi_y((z_n\Gamma_n)_{n=1}^\infty) := (z_n\Gamma_n g_n^{-1})_{n=1}^\infty = (z_n g_n^{-1}(g_n\Gamma_n g_n^{-1}))_{n=1}^\infty.$$

It follows that there is a $G$-equivariant map from $X$ to $Y$ if and only if there is a $G$-equivariant map from $X$ to $Y_y$. □

In a similar way, we obtain a criterion when a non-singular $(C, F)$-action is not isomorphic to any non-singular odometer.

COROLLARY 5.7. *Let $T$ be the non-singular $(C, F)$-action of $G$ associated with a sequence $\mathcal{T}$ satisfying equations (2.1)–(2.3) and Proposition 2.10(ii). Then $T$ is not isomorphic to any non-singular odometer if and only if for each decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups in $G$ satisfying equation (4.1) and each increasing sequence $\boldsymbol{q}$ of non-negative integers such that the $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ is compatible with the point $(\Gamma_n)_{n=1}^\infty \in \operatorname{proj\,lim}_{n\to\infty} G/\Gamma_n$, there exist $n > 0$ and $\epsilon_0 > 0$ such that for each $l > 0$, $D_l \subset G/\Gamma_l$ and $M > n$, there is $m > M$ with*

$$\widetilde{\nu}_m(\widetilde{C}_{n+1}\cdots\widetilde{C}_m \triangle \{f \in \widetilde{F}_m \mid f g_l\Gamma_l \in D_l\}) > \epsilon_0.$$

We state one more corollary from Theorems 5.4 and 2.19 on the existence of minimal Radon uniquely ergodic topological models for rank-one non-singular extensions of non-singular odometers.

COROLLARY 5.8. *Let $(X, \mu, T)$ be a rank-one non-singular action of $G$. Let $T$ have a non-singular odometer factor $(Y, \nu, O)$ and let $\pi : X \to Y$ stand for the corresponding $G$-equivariant factor mapping with $\nu = \mu \circ \pi^{-1}$. Then there exist a locally compact Cantor space $\widetilde{X}$, a minimal Radon uniquely ergodic free continuous action $\widetilde{T}$ of $G$ on $\widetilde{X}$, a continuous $G$-equivariant mapping $\widetilde{\pi} : \widetilde{X} \to Y$ and a Borel isomorphism $R : X \to \widetilde{X}$ such that:*

- *$\widetilde{\mu} := \mu \circ R^{-1}$ is a Radon measure on $\widetilde{X}$;*
- *$RT_g = \widetilde{T}_g R$ for each $g \in G$;*
- *the function $d\widetilde{\mu} \circ \widetilde{T}_g/d\widetilde{\mu} : \widetilde{X} \to \mathbb{R}^*$ is continuous for each $g \in G$;*
- *$\widetilde{T}$ is Radon $(d\widetilde{\mu} \circ \widetilde{T}_g/d\widetilde{\mu})_{g\in G}$-uniquely ergodic; and*
- *$\widetilde{\pi} R = \pi$.*

We can also characterize the class of quasi-invariant measures for odometers that appear as factors of rank-one actions. Let $(Y, O)$ be the topological $G$-odometer associated with a decreasing sequence $(\Gamma_n)_{n=1}^\infty$ of cofinite subgroups $\Gamma_n$ of $G$ satisfying equation (4.1). Let a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$ satisfy equation (2.1)–(2.4) and $C_n \subset \Gamma_n$ for each $n > 0$. Denote by $\mu_{\mathcal{F}}$ the $(C, F)$-measure determined by the sequence $(\kappa_n, \nu_{n-1})_{n=1}^\infty$. Let $X$ stand for the space of $\mu_{\mathcal{F}}$. We define a mapping $\pi_{\mathcal{F}} : X \to Y$ by setting

$$\pi_{\mathcal{F}}(x) = (f_n\Gamma_1, \ldots, f_n\Gamma_{n+1}, f_n c_{n+1}\Gamma_{n+2}, f_n c_{n+1} c_{n+2}\Gamma_{n+3}, \ldots)$$

if $x = (f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n \subset X$ for some $n \geq 0$. Then $\pi_{\mathcal{F}}$ is well defined and continuous. Let

$$\mathcal{M}_Y := \{\mu_{\mathcal{F}} \circ \pi_{\mathcal{F}}^{-1} \mid \mathcal{F} \text{ satisfies equations (2.1)–(2.4) and } C_n \subset \Gamma_n \text{ for all } n > 0\}.$$

We deduce the following claim from Corollary 5.8.

COROLLARY 5.9. *Each measure $\nu \in \mathcal{M}_Y$ is quasi-invariant under $O$. A Borel measure $\nu$ on $Y$ is equivalent to a measure belonging to $\mathcal{M}_Y$ if and only if there is a rank-one non-singular $G$-action $T$ such that $(Y, O, \nu)$ is a measurable factor of $T$.*

## 6. *Examples*

### 6.1. *Non-odometer rank-one $\mathbb{Z}$-action with odometer factor.*

In [Fo–We], an example of classical rank-one finite measure preserving $\mathbb{Z}$-action $T$ is constructed such that:

- $T$ has the 2-adic odometer as a factor; but
- $T$ is not isomorphic to any odometer.

We remind that given a prime $p$, the *p-adic odometer* is associated with the sequence

$$p\mathbb{Z} \supset p^2\mathbb{Z} \supset p^3\mathbb{Z} \supset \cdots$$

of cofinite subgroups in $\mathbb{Z}$. The argument in [Fo–We] is based on their description of the odometer factors of rank-one transformations (that result is generalized in our Theorem 5.4). We now consider their example from another point of view, bypassing the use of any version of Theorem 5.4. Our approach is more direct and leads to stronger results.

*Example 6.1.* Let $G = \mathbb{Z}$. We construct a measure preserving (classical) rank-one $\mathbb{Z}$-action $T$ on a probability space $(X, \mu)$ such that:

- $(X, \mu, T)$ has a proper 2-adic odometer factor $(Y, \nu, O)$;
- $(Y, \nu, O)$ is the *Kronecker factor* of $(X, \mu, T)$, that is, $O$ is the maximal factor of $T$ with a pure discrete spectrum;
- the projection $(X, \mu) \to (Y, \nu)$ is uncountable-to-one (mod 0), that is, the corresponding conditional measures on fibres are non-atomic.

We set $h_0 := 0$ and $h_{n+1} := 4h_n + 2^{n+1}$ for each $n \in \mathbb{N}$. It follows that $h_n = 2^n(2^{n+1} - 1)$ for each $n \geq 0$. We let

$$F_n := \{0, \ldots, h_n - 1\}, \quad C_{n+1} := \{0, h_n, 2h_n + 2^{n+1}, 3h_n + 2^{n+1}\},$$
$$\nu_n(f) = \frac{1}{4^n} \quad \text{for each } f \in F_n \quad \text{and} \quad \kappa_n(c) = \frac{1}{4} \quad \text{for each } c \in C_{n+1}$$

for every $n \geq 0$. Then the sequence $\mathcal{T} := (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ satisfies equations (2.1)–(2.3) and Proposition 2.10(ii). Denote by $(X, \mu, T)$ the $(C, F)$-action of $\mathbb{Z}$ associated with $\mathcal{T}$. Then $T$ is of classical rank one along $(F_n)_{n=0}^{\infty}$. We note that $T$ is the transformation that was studied in [Fo–We]. Of course, $T$ preserves $\mu$ and $\mu(X) < \infty$. We have that

$$\mu(X) = \mu(X_0) + \sum_{n=1}^{\infty} 2^n \mu([0]_n) = 1 + \sum_{n=1}^{\infty} \frac{2^n}{4^n} = 2.$$

Denote by $(Y, O)$ the 2-adic $\mathbb{Z}$-odometer. Then the transformation $O_1$ acts on the compact metric group $Y := \operatorname{proj\,lim}_{n\to\infty} \mathbb{Z}/2^n\mathbb{Z}$ by translation with the element $(1 + 2\mathbb{Z}, 1 + 2^2\mathbb{Z}, 1 + 2^3\mathbb{Z}, \dots) \in Y$. Let $\nu$ stand for the Haar measure on $Y$. Since each element of $C_n$ is divisible by $2^{n-1}$ for every $n > 0$, it follows that $O$ is a factor of $T$. The corresponding $(\mathcal{T}, 0)$-factor mapping $\pi : X \to Y$ is well defined by the formula:

$$\pi(x) = (f_n + 2^n\mathbb{Z}, f_n + c_{n+1} + 2^{n+1}\mathbb{Z}, f_n + c_{n+1} + c_{n+2} + 2^{n+2}\mathbb{Z}, \dots) \in Y,$$

if $x = (f_n, c_{n+1}, c_{n+1}, \dots) \in X_n = F_n \times C_{n+1} \times C_{n+2} \times \cdots \subset X$ for some $n \geq 0$ (see Definition 3.2). Since the measure $\mu \circ \pi^{-1}$ is invariant under $O$, it follows that $\mu \circ \pi^{-1}$ is proportional to $\nu$. More precisely, $\mu \circ \pi^{-1} = \mu(X) \cdot \nu = 2\nu$.

We now show that if $\lambda$ is an eigenvalue of $T$, then there is $n > 0$ such that $\lambda^{2^n} = 1$. Since $\#C_m = 4$ for each $m$, it follows from [**DaVi**, Corollary 3.8] that

$$\lim_{m\to\infty} \max_{c\in C_m} |1 - \lambda^c| = 0.$$

As $4^m = 2h_{m-1} + 2^m \in C_m$, we obtain that $\lim_{m\to\infty} \lambda^{4^m} = 1$. This is only possible if $\lambda$ is a dyadic root of 1, as desired. (Indeed, observe that $\lambda^{4^{m+1}} = (\lambda^{4^m})^4$ and iterate.) However, denote by $\mathfrak{Y}$ the $\sigma$-algebra of all measurable subsets in $Y$. Let $\mathfrak{F} := \{\pi^{-1}(B) \mid B \in \mathfrak{Y}\}$. Then each eigenfunction of $T$ whose eigenvalue is a 2-adic root of 1 is $\mathfrak{F}$-measurable. It follows that the $\mathfrak{F}$ is the Kronecker factor of $T$.

We now show that the Kronecker factor is proper, that is, that the spectrum of $T$ has a continuous component. Moreover, we prove that the extension $T \to O$ is uncountable-to-one. For each $n > 0$, we let $C_n^{(1)} := \{0, h_{n-1}\}$ and $C_n^{(2)} := \{0, 2h_{n-1} + 2^n\} = \{0, 4^n\}$. Then, $C_n = C_n^{(1)} + C_n^{(2)}$. For $j = 1, 2$, let $X_0^{(j)} := C_1^{(j)} \times C_2^{(j)} \times \cdots$. Then, $X_0^{(j)}$ is a compact subset of $X_0$. Given $x = (c_1, c_2, \dots) \in X_0^{(1)}$ and $z = (d_1, d_2, \dots) \in X_0^{(2)}$, the sum

$$x + z := (c_1 + d_1, c_2 + d_2, \dots) \in X_0$$

is well defined. (Note that $X_0^{(1)}$, $X_0^{(2)}$ and $X_0$ are compact subsets of the Polish Abelian group $\mathbb{Z}^{\mathbb{N}}$.) Moreover, the mapping $(x, z) \mapsto x + z$ is a homeomorphism of the Cartesian product $X_0^{(1)} \times X_0^{(2)}$ onto $X_0$ and

$$\pi(x + z) = \pi(x) + \pi(z) \quad \text{for all } x, z \in X_0. \tag{6.1}$$

Endow $X_0^{(1)}$ and $X_0^{(2)}$ with the infinite products $\mu^{(1)}$ and $\mu^{(2)}$ of the equidistributions on $C_n^{(1)}$ and $C_n^{(2)}$, respectively, $n \in \mathbb{N}$. We claim that the restriction of $\pi$ to $X_0^{(j)}$ is one-to-one for $j = 1, 2$. It is straightforward to verify that for each $x = (c_m)_{m=1}^\infty \in X_0$,

$$\pi(x) = \left(\left(\sum_{m=1}^{n} c_m^1 + \sum_{1\leq m<n/2} c_m^2\right) + 2^n\mathbb{Z}\right)_{n=1}^{\infty} \in Y, \tag{6.2}$$

where $c_m^j \in C_m^{(j)}$ and $c_m = c_m^1 + c_m^2$ for each $m$. Take two points $x = (c_1^1, c_2^1, \dots) \in X_0^{(1)}$ and $y = (d_1^1, d_2^1, \dots) \in X_0^{(1)}$ such that $\pi(x) = \pi(y)$. It follows from equation (6.2) that $c_1^1 = d_1^1 (\mathrm{mod}\ 2), c_1^1 + c_2^1 = d_1^1 + d_2^1 (\mathrm{mod}\ 2^2), \dots$. The first equality implies that $d_1^1 = c_1^1$.

Therefore, the second equality is equivalent to $c_2^1 = d_2^1 (\mathrm{mod}\ 2^2)$, which, in turn, yields that $d_2^1 = c_2^1$. By the induction, $d_n^1 = c_n^1$ for each $n > 0$, that is, $x = y$, as desired. It now follows that the mapping

$$C_1^{(1)} \times \cdots \times C_n^{(1)} \ni (c_1^1, \ldots, c_n^1) \mapsto \left( \sum_{m=1}^{n} c_m^1 \right) + 2^n\mathbb{Z} \in \mathbb{Z}/2^n\mathbb{Z}$$

is bijective for each $n > 0$. Hence, $\pi$ maps $X_0^{(1)}$ bijectively (and homeomorphically) onto $Y$. Moreover, $\pi$ maps $\mu^{(1)}$ to $\nu$. In a similar way, one can check that $\pi$ maps $X_0^{(2)}$ bijectively (and homeomorphically) onto the closed subset $\pi(X_0^{(2)}) \subset Y$. Given $y \in Y$ and $z \in X_0^{(2)}$, we define an element $Q(y, z) \in X_0^{(1)}$ by the formula

$$\pi(Q(y, z)) = y - \pi(z).$$

Of course, $Q(y, z)$ is well defined. Moreover, the mapping

$$Q : Y \times X_0^{(2)} \ni (y, z) \mapsto Q(y, z) \in X_0^{(1)}$$

is continuous. If we fix $y \in Y$, then the mapping $X_0^{(2)} \ni z \mapsto Q(y, z) \in X_0^{(1)}$ is one-to-one. Of course, for each $y \in Y$,

$$X_0 \cap \pi^{-1}(y) = \{Q(y, z) + z \mid z \in X_0^{(2)}\}.$$

Hence, $\pi^{-1}(y)$ is uncountable. Moreover, the corresponding conditional measure on the fibre $X_0 \cap \pi^{-1}(y)$ is the image of $\mu^{(2)}$ under the mapping

$$X_0^{(2)} \ni z \mapsto Q(y, z) + z \in X_0.$$

Hence, the conditional measure on $X_0 \cap \pi^{-1}(y)$ is non-atomic for each $y \in Y$. In particular, $\pi$ is uncountable-to-one.

*Remark 6.2.*

(i) The existence of the 2-adic odometer factor $O$ of $T$ in Example 6.1 was proved in [**Fo–We**]. It was shown there that $O$ is maximal in the family of odometer factors of $T$. We refine this result by showing in Example 6.1 that $O$ is the Kronecker factor of $T$. The claims that $T$ is an uncountable-to-one extension of $O$ and that $T$ has a continuous part in the spectrum are new.

(ii) Perhaps the simplest example of non-odometer rank-one $\mathbb{Z}$-action with the maximal odometer factor $O = (O_n)_{n\in\mathbb{Z}}$ is the following one. Let $R$ be an irrational rotation on the circle. Then the transformation $S := O_1 \times R$ is an ergodic transformation with pure discrete spectrum. Hence, it is of rank one [**dJ1**]. Of course, $O_1$ is the maximal odometer factor of $S$. Of course, $S$ is not isomorphic to any odometer (as the discrete spectrum of $S$ has elements of infinite order). Obviously, $S$ is rigid. However, in contrast with $T$ from Example 6.1, the spectrum of $S$ is purely discrete.

6.2. *Non-singular counterparts of Example 6.1.* We first recall briefly the concepts of the associated flow of an ergodic equivalence relation, Krieger's type and AT-flow. For details, we refer to the survey [**DaSi**] and references therein. Let $\mathcal{R}$ be a countable

Borel equivalence relation on a standard $\sigma$-finite measure space $(Z, \gamma)$. Assume that $\mathcal{R}$ is $\gamma$-non-singular and ergodic. This means that if a Borel subset $A \subset Z$ is $\gamma$-null, then the $\mathcal{R}$-saturation of $A$ is also $\gamma$-null, and each $\mathcal{R}$-invariant (that is, $\mathcal{R}$-saturated) Borel subset of $Z$ is either $\gamma$-null or $\gamma$-conull. Denote by $\rho_\gamma : \mathcal{R} \to \mathbb{R}_+^*$ the Radon–Nikodym cocycle of $\mathcal{R}$. Endow the product space $Z \times \mathbb{R}$ with the product measure $\gamma \otimes \text{Leb}$. We define an equivalence relation $\mathcal{R}(\log \rho_\gamma)$ on $Z \times \mathbb{R}$ by setting

$$(z, t) \sim_{\mathcal{R}(\log \rho_\gamma)} (z', t') \quad \text{if } (z, z') \in \mathcal{R} \quad \text{and} \quad t' = t - \log \rho_\gamma(z, z').$$

Then, $\mathcal{R}(\log \rho_\gamma)$ is countable, $(\mu \otimes \text{Leb})$-non-singular but not necessarily ergodic. Denote by $\mathfrak{I}$ the $\sigma$-algebra of $\mathcal{R}(\log \rho_\gamma)$-invariant subsets. Let $V = (V_s)_{s\in\mathbb{R}}$ denote the action of $\mathbb{R}$ on $Z \times \mathbb{R}$ by translations along the second coordinate, that is, $V_s(z, t) = (z, t + s)$. Of course, $\mathfrak{I}$ is invariant under $V$. The dynamical system $(Z \times \mathbb{R}, \mathfrak{I}, \mu \otimes \text{Leb}, V)$ is called *the flow associated with* $\mathcal{R}$. The associated flow is non-singular and ergodic. We denote it by $W^{\mathcal{R}}$. We will need the following two well-known properties of the associated flows.

(*) If $A \subset Z$ is of positive measure $\gamma$, then the associated flow of $\mathcal{R} \restriction A$ is isomorphic to the associated flow of $\mathcal{R}$.

(**) If $(Z, \gamma) = (Z_1, \gamma_1) \otimes (Z_2, \gamma_2)$, $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ and $W^{\mathcal{R}_1}$ is free and transitive, then $W^{\mathcal{R}}$ is isomorphic to $W^{\mathcal{R}_2}$.

An ergodic non-singular flow $V$ is called an *AT-flow* if there is a sequence $(A_n, \alpha_n)_{n=1}^\infty$ of finite subsets $A_n$ and non-degenerated probability measures $\alpha_n$ on $A_n$ such that the infinite product measure $\bigotimes_{n=1}^\infty \alpha_n$ is non-atomic and $V$ is isomorphic to the associated flow of the tail equivalence relation on the probability space $\bigotimes_{n=1}^\infty (A_n, \alpha_n)$. If, moreover, $\#A_n = 2$ for each $n$, then we call the corresponding AT-flow *finitary*. (The class of finitary AT-flows coincides with the class of flows of weights for the $\text{ITPFI}_2$ factors (in the sense of the theory of von Neumann algebras).) For instance, every ergodic flow with pure point spectrum is a finitary AT-flow [**BeVa**]. If $S$ is an ergodic non-singular $G$-action on a standard $\sigma$-finite measure space, then the associated flow of the $S$-orbit equivalence relation is called *the associated flow of S*. If the associated flow of $S$ is transitive and aperiodic, then $S$ is called *of Krieger type II*. If the associated flow of $S$ is transitive and periodic with the least positive period $-\log \lambda$ for some $\lambda \in (0, 1)$, then $S$ is called *of Krieger type* $III_\lambda$. If the associated flow of $S$ is the trivial flow on a singleton, then $S$ is called *of Krieger type* $III_1$. If $S$ is neither of type *II* nor of type $III_\lambda$ for any $\lambda \in (0, 1]$, then $S$ is called *of Krieger type* $III_0$.

Let $(X, T)$, $(Y, \nu, O)$ and $\pi : X \to Y$ be the same as in Example 6.1.

PROPOSITION 6.3. *For each finitary AT-flow V, there exists a* $(C, F)$*-measure* $\mu$ *on X such that:*

(i) $\mu$ *is quasi-invariant under T and hence the non-singular system* $(X, \mu, T)$ *is of rank one;*

(ii) *the associated flow of the system* $(X, \mu, T)$ *is isomorphic to V;*

(iii) $\mu \circ \pi^{-1} \sim \nu$, *that is, the 2-adic probability preserving odometer* $(Y, \nu, O)$ *is a factor of* $(X, \mu, T)$;

(iv) $(Y, \nu, O)$ *is the maximal odometer factor of* $(X, \mu, T)$*; and*

(v) *the factor mapping $\pi$ is uncountable-to-one (mod 0). Hence, $(X, \mu, T)$ is not an odometer.*

*In particular, for each $\lambda \in [0, 1]$, there is a $(C, F)$-measure $\mu$ on $X$ such that $(X, \mu, T)$ is of Krieger type $III_\lambda$ satisfying properties (i), (iii)–(v).*

*Proof.* We will use below the notation from §6.1.

Let $\kappa_n^1$ stand for the equidistribution on $C_n^{(1)}$ and let $\kappa^1 := \bigotimes_{n=1}^\infty \kappa_n^1$. As $V$ is finitary AT, there is a sequence $(\kappa_n^2)_{n=1}^\infty$ of non-degenerated probability measures $\kappa_n^2$ on $C_n^{(2)}$ such that the infinite product measure $\kappa^2 := \bigotimes_{n=1}^\infty \kappa_n^2$ is non-atomic and the associated flow of the tail equivalence relation on $(X_0^{(2)}, \kappa^2)$ is isomorphic to $V$. Denote by $\kappa_n$ the convolution $\kappa_n^1 * \kappa_n^2$. Then, $\kappa_n$ is a non-degenerated probability measure on $C_n$ for each $n \in \mathbb{N}$. We now select inductively a sequence $(\nu_n)_{n=1}^\infty$ of measures on $G$ such that $\nu_n$ is supported on $F_n$ for each $n \in \mathbb{N}$ and equation (2.3) holds. Consider now the sequence $\mathcal{T} := (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$. Of course, equation (2.2) holds. It is straightforward to verify that Proposition 2.10(iii) holds for $\mathcal{T}$. Let $\mu$ denote the $(C, F)$-measure determined by $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$. It follows from Proposition 2.10 that $\mu$ is quasi-invariant under $T$. Thus, property (i) holds.

According to $(*)$, the associated flow of $T$ is isomorphic to the associated flow of the tail equivalence relation on $(X_0, \mu \restriction X_0)$. Then $(**)$ yields that the later flow is isomorphic to the associated flow of the tail equivalence relation on $(X_0^{(2)}, \kappa^2)$. Hence, the associated flow of $T$ is isomorphic to $V$, that is, property (ii) is proven.

We deduce from equation (6.1) that

$$(\mu \restriction X_0) \circ \pi^{-1} = (\kappa^1 \circ \pi^{-1}) * (\kappa^2 \circ \pi^{-1}).$$

It was shown in §6.1 that $\kappa^1 \circ \pi^{-1} = \nu$. As $\nu$ is the Haar measure on the compact Abelian group $Y$, we obtain that $\nu * (\kappa^2 \circ \pi^{-1}) = \nu$. Thus, $(\mu \restriction X_0) \circ \pi^{-1} = \nu$. Hence, $\mu \circ \pi^{-1} \gg \nu$. As the two measures, $\mu \circ \pi^{-1}$ and $\nu$ on $Y$, are quasi-invariant and ergodic under $T$, it follows that $\mu \circ \pi^{-1} \sim \nu$. Thus, property (iii) is proven.

If $O$ is not the maximal odometer factor of $T$, then there is a prime number $p > 2$ such that the homogeneous $\mathbb{Z}$-space $\mathbb{Z}/p\mathbb{Z}$ is a factor of $T$. Since $\#C_n = 4$ for each $n > 0$, it can be deduced from Theorem 3.3 that each element of $C_n$ is divisible by $p$ eventually in $n$. However, $4^n \in C_n$ for each $n$, which is a contradiction. Thus, property (iv) is proven.

As for property (v), it is proved almost literally in the same way as in Example 6.1.

The second claim of the proposition follows directly from the first one. □

6.3. *Rank-one $\mathbb{Z}^2$-action without odometer factors but whose generators have $\mathbb{Z}$-odometer factors.* In this section, $G = \mathbb{Z}^2$. Only finite measure preserving actions of $G$ are considered in this section. In [**JoMc**, §6], a rank-one $\mathbb{Z}^2$-action $T$ is constructed such that:

(a) each of the generators $T_{(1,0)}$ and $T_{(0,1)}$ of $T$ has an odometer factor (as a $\mathbb{Z}$-action); but

(b) $T$ has no $\mathbb{Z}^2$-odometer factors.

The corresponding construction is rather involved (see [**JoMc**, Theorem 6.1]). We provide a different, elementary example of $T$ possessing properties (a) and (b). To prove that, we do not use any machinery developed in the previous sections.

*Example 6.4.* Let $R$ be an irrational rotation on the circle $(\mathbb{T}, \lambda_{\mathbb{T}})$ and let $S$ be an ergodic rotation on a compact totally disconnected Abelian infinite group $Y$ endowed with the Haar measure $\lambda_Y$. Then the dynamical system $O = (S^n)_{n\in\mathbb{Z}}$ is a classical odometer on $(Y, \lambda_Y)$. We now define a $\mathbb{Z}^2$-action $T = (T_g)_{g\in\mathbb{Z}^2}$ on the product space $(\mathbb{T} \times Y, \lambda_{\mathbb{T}} \otimes \lambda_Y)$ by setting $T_{(n,m)} := R^n \times S^m$. Of course, $T$ is of rank-one along a Følner sequence of rectangles $(F_n)_{n=1}^{\infty}$ in $\mathbb{Z}^2$. Let $\theta := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then the action $\widetilde{T} := (T_{\theta(g)})_{g\in\mathbb{Z}^2}$ is of rank-one along the sequence $(\theta(F_n))_{n=1}^{\infty}$. Of course, $(\theta(F_n))_{n=1}^{\infty}$ is also Følner. The generators $\widetilde{T}_{(1,0)} = R \times S$ and $\widetilde{T}_{(0,1)} = I \times S$ of $\widetilde{T}$ have $S$ as a factor. Thus, property (a) holds for $\widetilde{T}$.

It remains to show that $\widetilde{T}$ satisfies property (b), that is, that $\widetilde{T}$ has no $\mathbb{Z}^2$-odometer factors. Suppose that $\widetilde{T}$ has an odometer factor $\widetilde{O}$. Let $K$ be the space of $\widetilde{O}$ and let $\pi : \mathbb{T} \times Y \to K$ denote the corresponding $\mathbb{Z}^2$-equivariant factor mapping. Then $K$ is a totally disconnected compact Abelian group. Since $\widetilde{T}$ has pure discrete spectrum, there is a compact subgroup $H \subset \mathbb{T} \times Y$ such that $K = (\mathbb{T} \times Y)/H$ and $\pi(g) = gH$ for each $g \in \mathbb{T} \times Y$. Since $\mathbb{T}$ is connected and $K$ is totally disconnected, the closed subgroup $\pi(\mathbb{T} \times \{0\}) \subset K$ is trivial. This means that $H$ contains the subgroup $\mathbb{T} \times \{0\}$. Hence, $K$ is a quotient group of $Y = (\mathbb{T} \times Y)/(\mathbb{T} \times \{0\})$ indeed and $\pi$ is the corresponding projection map. It follows that $\widetilde{O}$ is not faithful:

$$\widetilde{O}_{(n,-n)} = \pi \circ \widetilde{T}_{(n,-n)} = \pi \circ T_{(n,0)} = \pi \circ (S^n \times I) = I$$

for each $n \in \mathbb{Z}$. Since each odometer action is faithful, $\widetilde{O}$ is not an odometer. Thus, property (b) is proven.

According to the terminology of [**JoMc**] (which is different from ours), $\mathbb{Z}^2$-odometers can be non-free. Hence, property (b) can be interpreted as 'every rank-one factor of $T$ is non-free'.

*Remark 6.5.* We note that in the example [**JoMc**, §6], the generators $T_{(0,1)}$ and $T_{(1,0)}$ are non-ergodic. In Example 6.4, $\widetilde{T}_{(1,0)}$ is ergodic but $\widetilde{T}_{(0,1)}$ is not. However, if we change $\theta$ with the matrix $\theta' := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, then we obtain a new example of rank-one $\mathbb{Z}^2$-action $\widetilde{T}$ possessing properties (a), (b) and

(c) each of the generators $\widetilde{T}_{(1,0)}$ and $\widetilde{T}_{(0,1)}$ of $\widetilde{T}$ is ergodic.

6.4. *Heisenberg group actions of rank one.* In this section, we consider the three-dimensional discrete Heisenberg group $H_3(\mathbb{Z})$. We recall that

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{Z} \right\}.$$

This group is non-Abelian, nilpotent and residually finite. For brevity, we will write $(x, y, z)$ for the matrix $\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)$. It is straightforward to verify that

$$(x, y, z) \cdot (x_1, y_1, z_1) = (x + x_1, y + y_1, z + z_1 + xy_1),$$
$$(x, y, z) \cdot (x_1, y_1, z_1) \cdot (x, y, z)^{-1} = (x_1, y_1, z_1 + xy_1 - yx_1).$$

The centre of $H_3(\mathbb{Z})$ is $\{(0,0,z) \mid z \in \mathbb{Z}\}$. Given $a, b, c \geq 0$, we let

$$\Pi(a,b,c) := \{(x,y,z) \in H_3(\mathbb{Z}) \mid 0 \leq x < a, 0 \leq y < b, 0 \leq z < c\}.$$

It is straightforward to verify that if $a_n \to +\infty$, $b_n \to +\infty$, $c_n \to +\infty$ and $b_n/c_n \to 0$ as $n \to \infty$, then $(\Pi(a_n, b_n, c_n))_{n=1}^{\infty}$ is a left Følner sequence in $H_3(\mathbb{Z})$. If, in addition, $a_n/c_n \to 0$, then $(\Pi(a_n, b_n, c_n))_{n=1}^{\infty}$ is a 2-sided Følner sequence in $H_3(\mathbb{Z})$.

*Example 6.6.* We construct a probability preserving $(C, F)$-action $T$ of $H_3(\mathbb{Z})$ that has an odometer factor, but $T$ itself is not isomorphic to any odometer. For that, we first define recurrently a sequence $(h_n)_{n=0}^{\infty}$ of positive integers by setting

$$h_0 := 1 \quad \text{and} \quad h_{n+1} := 16h_n + 9 \cdot 4^{n+1}.$$

It is easy to check that $h_n = 4^{2n+1} - 3 \cdot 4^n$ for each $n \geq 0$. We set

$$C_{n+1}^{(1)} := \{(a,b,0) \mid a, b \in \{0, 2^n\}\} \quad \text{and}$$
$$C_{n+1}^{(2)} := \{(0,0,jh_n) \mid j = 0, \ldots, 7\} \cdot \{(0,0,0), (0,0,8h_n + 2 \cdot 4^{n+1})\}.$$

We now define a sequence $(C_n, F_{n-1})_{n=1}^{\infty}$ by setting

$$F_n := \Pi(2^n, 2^n, h_n) \quad \text{and} \quad C_{n+1} := C_{n+1}^{(1)} C_{n+1}^{(2)} \quad \text{for each } n \geq 0.$$

It is straightforward to check that equation (2.1) is satisfied. We define measures $\kappa_n$ on $C_n$ and $\nu_n$ on $F_n$ by setting $\nu_0(1) = 1$ and

$$\kappa_n(c) := \frac{1}{\#C_n} = \frac{1}{64}, \quad \nu_n(f) := \frac{1}{64^n} \quad \text{for each } c \in C_n,\ f \in F_n,\ n > 0.$$

Let $\mathcal{T} := (C_n, F_{n-1}, \kappa_n, \nu_{n-1})$. Then (2.2) and (2.3) hold for $\mathcal{T}$. Since $\#F_n = 4^n(4^{2n+1} - 3 \cdot 4^n)$ and $\#C_{n+1} = 64$, we obtain that

$$\prod_{n>0} \frac{\#F_{n+1}}{\#F_n \#C_{n+1}} = \prod_{n>0} \frac{4^{n+2} - 3}{4(4^{n+1} - 3)} = \prod_{n>0} \left(1 + \frac{9}{4(4^{n+1} - 3)}\right) < \infty. \tag{6.3}$$

Of course, $(F_n)_{n=1}^{\infty}$ is a 2-sided Følner sequence in $H_3(\mathbb{Z})$. Hence, Proposition 2.10(ii) holds. Then the $(C, F)$-action $T = (T_g)_{g \in H_3(\mathbb{Z})}$ associated with $\mathcal{T}$ is well defined on a measure space $(X, \mu)$, where $\mu$ is the $(C, F)$-measure determined by $(\kappa_n, \nu_{n-1})_{n=1}^{\infty}$. Moreover, $T$ preserves $\mu$, that is, $\mu$ is the Haar measure for the $(C, F)$-equivalence relation on $X$, and $\mu(X) < \infty$ in view of equation (6.3) (see Remark 2.5).

Next, we define a measure preserving odometer action of $H_3(\mathbb{Z})$. Given $n > 0$, we let

$$\Gamma_n := \{(i \cdot 2^n, j \cdot 2^n, k \cdot 2^n) \in H_3(\mathbb{Z}) \mid i, j, k \in \mathbb{Z}\}.$$

Then, $\Gamma_n$ is a normal cofinite subgroup of $H_3(\mathbb{Z})$, $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. Denote by $O$ the $H_3(\mathbb{Z})$-odometer associated with the sequence $(\Gamma_n)_{n=1}^{\infty}$. We call it *the 2-adic odometer* action of $H_3(\mathbb{Z})$. This odometer is normal. It is defined on the compact metric group

$$Y := \operatorname*{proj\,lim}_{n \to \infty} H_3(\mathbb{Z})/\Gamma_n.$$

Denote by $\nu$ the Haar measure on $Y$. Since $C_{n+1} \subset \Gamma_n$ for each $n \in \mathbb{N}$, it follows that $\mathcal{T}$ is compatible with $(\Gamma_n)_{n=1}^{\infty} \in Y$. Hence, the $(\mathcal{T}, (\Gamma_n)_{n=1}^{\infty})$-factor mapping $\pi_{(\mathcal{T},(\Gamma_n)_{n=1}^{\infty})} : X \to Y$ intertwines $T$ with $O$. For brevity, instead of $\pi_{(\mathcal{T},(\Gamma_n)_{n=1}^{\infty})}$, below we will write $\pi$. The measure $\mu \circ \pi^{-1}$ on $Y$ is finite and invariant under $O$. Since $O$ is uniquely ergodic, it follows that $\mu \circ \pi^{-1} = \mu(X) \cdot \nu$. Thus, the 2-adic $H_3(\mathbb{Z})$-odometer is a finite measure preserving factor of $(X, \mu, T)$.

We claim that $(Y, \nu, O)$ is the *maximal odometer factor* of $T$, that is, every odometer factor of $T$ is a factor of $O$. Let $\Lambda_1 \supset \Lambda_2 \supset \cdots$ be a sequence of cofinite subgroups in $H_3(\mathbb{Z})$ with $\bigcap_{n=1}^{\infty} \bigcap_{g \in G} g\Lambda_n g^{-1} = \{1\}$ and let $Q = (Q_g)_{g \in H_3(\mathbb{Z})}$ stand for the associated $H_3(\mathbb{Z})$-odometer which is defined on the space $Z := \text{proj lim}_{n\to\infty} H_3(\mathbb{Z})/\Lambda_n$ equipped with Haar measure. If there is an $H_3(\mathbb{Z})$-equivariant mapping $\tau : X \to Z$ then, by Theorem 5.4, there exist a sequence $\boldsymbol{q} = (q_n)_{n=0}^{\infty}$ and an element $z = (g_n\Lambda_n)_{n=1}^{\infty} \in Z$ such that $0 = q_0 < q_1 < q_2 < \cdots$, the $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ is compatible with $z$ and $\tau = \pi_{(\widetilde{\mathcal{T}},z)} \circ \iota_{\boldsymbol{q}}$. Replacing $\Lambda_n$ with $g_n^{-1}\Lambda_n g_n$ for each $n \in \mathbb{N}$, we pass to an isomorphic (to $Q$) odometer as a factor of $X$ (see the proof of Corollary 5.6). We denote it by the same symbol $Z$. Therefore, without loss of generality, we may assume that $\widetilde{\mathcal{T}}$ is compatible with $z = (\Lambda_n)_{n=1}^{\infty}$. It follows that there is $N > 0$ such that for each $n > N$,

$$\#\{c \in C_{q_n+1} \cdots C_{q_{n+1}} \mid c \notin \Lambda_n\} < 64^{-1}.$$

As $\#C_{q_n+1} = 64$ and $\#(C_{q_n+1} \cdots C_{q_{n+1}}) = \#(C_{q_n+1})\#(C_{q_n+2} \cdots C_{q_{n+1}})$, a version of Fubini's theorem yields that there exists at least one element $d \in C_{q_n+2} \cdots C_{q_{n+1}}$ such that $C_{q_n+1}d \subset \Lambda_n$. Hence,

$$\{(2^{q_n}, 0, 0), (0, 2^{q_n}, 0)\} \subset \{\widetilde{c}c^{-1} \mid \widetilde{c}, c \in C_{q_n+1}\} \subset \Lambda_n.$$

Therefore, the subgroup

$$\Sigma_n := \{(i \cdot 2^{q_n}, j \cdot 2^{q_n}, k \cdot 4^{q_n}) \in H_3(\mathbb{Z}) \mid i, j, k \in \mathbb{Z}\}$$

is contained in $\Lambda_n$. This implies that $\Gamma_{2q_n} \subset \Sigma_n \subset \Lambda_n$ for each $n > 0$. The natural projection

$$H_3(\mathbb{Z})/\Gamma_{2q_n} \to H_3(\mathbb{Z})/\Lambda_n$$

is $H_3(\mathbb{Z})$-equivariant for each $n$. Passing to the projective limit as $n \to \infty$, we obtain an $H_3(\mathbb{Z})$-equivariant projection $\eta : Y \to Z$. It is straightforward to verify that $\tau = \eta \circ \pi$, as desired.

It remains to show that $\pi$ is not one-to-one (mod 0). For that, it suffices to show that the restriction of $\pi$ to $X_0$ is not one-to-one. Let $\kappa_n^1$ and $\kappa_n^2$ be the equidistributions on $C_n^{(1)}$ and $C_n^{(2)}$, respectively, $n \in \mathbb{N}$. We let

$$(X_0^{(1)}, \kappa^1) := \left(C_1^{(1)} \times C_2^{(1)} \times \cdots, \bigotimes_{n=1}^{\infty} \kappa_n^1\right) \quad \text{and}$$

$$(X_0^{(2)}, \kappa^2) := \left(C_1^{(2)} \times C_2^{(2)} \times \cdots, \bigotimes_{n=1}^{\infty} \kappa_n^2\right).$$

Then, $X_0$, $X_0^{(1)}$ and $X_0^{(2)}$ are compact subsets of the Polish group $H_3(\mathbb{Z})^{\mathbb{N}}$. The mapping

$$\alpha : X_0^{(1)} \times X_0^{(2)} \ni (x^1, x^2) \mapsto x^1 x^2 \in X_0$$

is a homeomorphism that maps the product measure $\kappa^1 \otimes \kappa^2$ to $\mu \restriction X_0$. Moreover, for each $(x^1, x^2) \in X_0^{(1)} \times X_0^{(2)}$,

$$\pi(x^1 x^2) = \pi(x^1)\pi(x^2). \tag{6.4}$$

Let $\mathcal{Z}$ stand for the centre of $H_3(\mathbb{Z})$. Denote by $Y_0$ the centre of $Y$. It is routine to verify that

$$Y_0 = \operatorname*{proj\,lim}_{n\to\infty} \mathcal{Z}\Gamma_n/\Gamma_n = \operatorname*{proj\,lim}_{n\to\infty} \mathcal{Z}/(\mathcal{Z} \cap \Gamma_n).$$

Denote by $Y_2$ the quotient group $Y/Y_0$ and by $\omega$ the quotient homomorphism $Y \to Y_2$. Then we obtain a short exact sequence of compact totally disconnected groups

$$1 \longrightarrow Y_0 \longrightarrow Y \overset{\omega}{\to}\longrightarrow Y_2 \longrightarrow 1.$$

Denote by $\lambda_0$ and $\lambda_2$ the Haar measures on $Y_0$ and $Y_2$, respectively. It is straightforward to verify that

$$Y_2 = \operatorname*{proj\,lim}_{n\to\infty} H_3(\mathbb{Z})/(\mathcal{Z}\Gamma_n) = \operatorname*{proj\,lim}_{n\to\infty} \mathbb{Z}^2/2^n\mathbb{Z}^2 \quad \text{and}$$
$$\omega \circ \pi(x^1) = ((a_1, b_1) + 2\mathbb{Z}^2, (a_2, b_2) + 2^2\mathbb{Z}^2, \dots)$$

for each $x^1 = ((a_1, b_1, 0), (a_2, b_2, 0), \dots) \in X_0^{(1)}$. Hence, $\omega \circ \pi$ is a measure preserving homeomorphism of $(X_0^{(1)}, \kappa^1)$ onto $(Y_2, \lambda_2)$. We now define a continuous mapping $s : Y_2 \to Y$ by setting

$$s(\omega(\pi(x^1))) := \pi(x^1). \tag{6.5}$$

Then, $s$ is a cross-section of $\omega$. Hence, the mapping

$$\beta : Y \ni y \mapsto (ys(\omega(y))^{-1}, \omega(y)) \in Y_0 \times Y_2$$

is a well-defined measure preserving homeomorphism of $(Y, \nu)$ onto the product measure space $(Y_0 \times Y_2, \lambda_0 \otimes \lambda_2)$. It follows from equations (6.4) and (6.5) that

$$\beta \circ \pi \circ \alpha(x^1, x^2) = \omega(\pi(x^1)\pi(x^2)) = (\pi(x^2), \omega \circ \pi(x^1))$$

for each $(x^1, x^2) \in X_0^{(1)} \times X_0^{(2)}$. Moreover,

$$(\kappa^1 \otimes \kappa^2) \circ (\beta \circ \pi \circ \alpha)^{-1} = \lambda_0 \otimes \lambda_2.$$

Therefore, $\pi$ is not one-to-one ($\mu$-mod 0) if and only if the mapping $\pi \restriction X_0^{(2)} \to Y_0$ is not one-to-one ($\mu^{(2)}$-mod 0).

Our purpose now is to show that $\pi \restriction X_0^{(2)} \to Y_0$ is not one-to-one. Let

$$C_{n+1}^{(3)} := \{0, h_n, 2h_n, 3h_n\},$$
$$C_{n+1}^{(4)} := \{0, 4h_n\} + \{0, 8h_n + 2 \cdot 4^{n+1}\},$$

$\kappa_n^3$ and $\kappa_n^4$ are the equidistributions on $C_n^{(3)}$ and $C_n^{(4)}$ respectively for each $n \geq 0$, and let

$$(X_0^{(3)}, \kappa^3) := \left(C_1^{(3)} \times C_2^{(3)} \times \cdots, \bigotimes_{n=1}^{\infty} \kappa_n^3\right) \quad \text{and}$$
$$(X_0^{(4)}, \kappa^4) := \left(C_1^{(4)} \times C_2^{(4)} \times \cdots, \bigotimes_{n=1}^{\infty} \kappa_n^4\right).$$

Then, $X_0^{(3)}$ and $X_0^{(4)}$ are compact subsets of $H_3(\mathbb{Z})^{\mathbb{N}}$. The mapping

$$X_0^{(3)} \times X_0^{(4)} \ni (x^3, x^4) \mapsto x^3 x^4 \in X_0^{(2)}$$

is a well-defined homeomorphism that maps the product measure $\kappa^3 \otimes \kappa^4$ to $\kappa^2$. Moreover, for each $(x^3, x^4) \in X_0^{(3)} \times X_0^{(4)}$,

$$\pi(x^3 x^4) = \pi(x^3)\pi(x^4).$$

In view of that, it suffices to show that the mapping $\pi \upharpoonright X^{(3)} \to Y_0$ is a bijection and $\kappa^3 \circ (\pi \upharpoonright X^{(3)})^{-1} = \lambda_0$. We leave a routine verification of these facts to the reader.

With the example below, we illustrate that the common concept of normality for odometers is not invariant under isomorphism. The normality depends on the choice of the underlying sequence of $(\Gamma_n)_{n=1}^{\infty}$ of cofinite subgroups in $G$.

*Example 6.7.* Let $\Gamma_n := \{(i2^{n-1}, j2^n, k2^n) \in H_3(\mathbb{Z}) \mid i, j, k \in \mathbb{Z}\}$. Of course, $\Gamma_n$ is a non-normal cofinite subgroup of $H_3(\mathbb{Z})$ and $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ with $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. It is easy to see that the largest normal subgroup $\widetilde{\Gamma}_n$ of $\Gamma_n$ is

$$\widetilde{\Gamma}_n := \{(i2^n, j2^n, k2^n) \in H_3(\mathbb{Z}) \mid i, j, k \in \mathbb{Z}\}.$$

Thus, $\widetilde{\Gamma}_n$ is of index 2 in $\Gamma_n$ for each $n > 0$. Denote by $(Y, O)$ and $(\widetilde{Y}, \widetilde{O})$ the topological odometers associated with $(\Gamma_n)_{n=1}^{\infty}$ and $(\widetilde{\Gamma}_n)_{n=1}^{\infty}$, respectively. Then, $(\widetilde{Y}, \widetilde{O})$ is a normal odometer while $(Y, O)$ is not. Moreover, $(\widetilde{Y}, \widetilde{O})$ is the topological normal cover of $(Y, O)$. As was shown in §4.2, $(Y, O)$ is a factor of $(\widetilde{Y}, \widetilde{O})$ under the natural projection $\omega : \widetilde{Y} \ni \widetilde{y} \mapsto \widetilde{y}H \in \widetilde{Y}/H = Y$, where

$$H := \{(\widetilde{y}_n)_{n=1}^{\infty} \in \widetilde{Y} \mid \widetilde{y}_n \in \Gamma_n/\widetilde{\Gamma}_n \text{ for all } n \in \mathbb{N}\}$$

is a closed subgroup of $\widetilde{Y}$. Since $\Gamma_{n+1} \subset \widetilde{\Gamma}_n \subset \Gamma_n$ for each $n$, it follows that $\omega$ is one-to-one. Hence, $\omega$ is an isomorphism of $\widetilde{O}$ with $O$, and $H = \{1\}$.

## 7. *Comments on the paper* [JoMc]

The article [JoMc] by Johnson and McClendon is also devoted to a generalization of [Fo–We]. However, [JoMc] deals only with probability preserving actions of amenable groups. Moreover, the odometers that appear in their paper are always normal. In this section, we discuss the results from [JoMc] and compare them with ours.

(1) 'Følner rank one' is rank one according to Definition 2.1. By [JoMc], a measure preserving $G$-action $T$ on a probability space $(X, \mu)$ is called *of Følner rank one* if

Definition 2.1(i) holds and $(F_n)_{n=1}^{\infty}$ is a Følner sequence in $G$. We first claim that then, $T$ is free. Indeed, denote by $H_x \subset G$ the stabilizer of $T$ at $x \in X$, that is,

$$H_x := \{g \in G \mid T_g x = x\}.$$

Let $(B_n, F_n)_{n=1}^{\infty}$ be the sequence of Rokhlin towers satisfying Definition 2.1(i). Fix a finite subset $K \subset G \setminus \{1_G\}$. Let $F_n^K := \{f \in F_n \mid Kf \subset F_n\}$. Then,

$$\lim_{n\to\infty} \#F_n^K/\#F_n = 1 \quad \text{and} \quad \lim_{n\to\infty} \mu\bigg(\bigsqcup_{f\in F_n} T_f B_n\bigg) = 1.$$

We let $X_n := \bigsqcup_{f\in F_n^K} T_f B_n$. Then, $\mu(X_n) \to 1$ as $n \to \infty$. Hence, $\mu$-a.e. $x$ belongs to $X_n$ for some $n = n(x) > 0$. It follows that the points $T_k x$, $k \in K$, belong to different levels of the tower $(B_n, F_n)$. Hence, $T_k x \neq x$ for each $k \in K$, that is, $K \cap H_x = \emptyset$. Since $K$ is arbitrary, we conclude that $H_x \cap G = \{1_g\}$ at a.e. $x$. Thus, $T$ is free. Since $(F_n)_{n=1}^{\infty}$ is Følner, Definition 2.1(ii) is satisfied. Thus, $T$ is of rank one according to Definition 2.1. The converse follows from Corollary 2.11(ii) and Theorem 2.13. Thus, we obtain that a probability preserving $G$-action $T$ is of Følner rank-one if and only if $T$ is of rank one according to Definition 2.1. Therefore, the main results of [**JoMc**]: their Theorems 3.1, 4.7 and 5.1 (which are stated for the Følner rank-one $\mathbb{Z}^d$-actions) follow from our Theorems 3.3 and 5.4.

(2) An ergodic $G$-action is totally ergodic if and only if it has no non-trivial finite factors. However, it is claimed in [**JoMc**, Theorem 3.3] that there exist non-totally ergodic Følner rank-one $\mathbb{Z}^2$-actions without non-trivial finite factors. This seeming contradiction is caused by the non-standard definition of total ergodicity in [**JoMc**]. As we understood from the proof of [**JoMc**, Theorem 3.3], by the total ergodicity of a $\mathbb{Z}^d$-action $T$, they mean the *individual ergodicity* of $T$, that is, that the transformation $T_g$ is ergodic for each non-zero $g \in \mathbb{Z}^d$. The proof of [**JoMc**, Theorem 3.3] given there is based on their analysis of finite factors for rank-one systems. However, an easy alternative proof follows from the joining theory (see [**dJRu**, **Ru**]) and has no direct relation to the rank one. Indeed, let $S$ be a transformation with MSJ. Let $(X, \mathfrak{B}, \mu)$ be the space of $S$. Denote by $T$ the following $\mathbb{Z}^2$-action: $T_{(n,m)} = S^n \times S^m$, $n, m \in \mathbb{Z}$, on $(X \times X, \mathfrak{B} \otimes \mathfrak{B}, \mu \otimes \mu)$. Of course, each factor of $T$ is a factor of the transformation $T_{(1,1)} = S \times S$. However, $S \times S$ has only three non-trivial proper factors: $\mathfrak{B} \otimes \{\emptyset, X\}$, $\{\emptyset, X\} \otimes \mathfrak{B}$ and $\mathfrak{B}^{\odot 2}$ [**dJRu**]. The first two $\sigma$-algebras are also factors of $T$, while the latter one is not. Thus, $T$ has only two proper factors which are weakly mixing. Hence, $T$ is totally ergodic. Since $T_{(1,0)}$ is not ergodic, $T$ is not individually ergodic. It remains to note that the 3-cut Chacon transformation, used in the proof of [**JoMc**, Theorem 3.3], has MSJ [**dJRaSw**].

(3) Theorem 4.1 of [**JoMc**] is a particular case of our Theorem 4.9: $G$ is amenable and the actions are probability preserving. Despite that, our proof of Theorem 4.9 is shorter than the proof of [**JoMc**, Theorem 4.1] because we use the Kőnig infinity lemma.

(4) Section 6 from [**JoMc**] is devoted entirely to construction of a $\mathbb{Z}^2$-action $T$ that has no $\mathbb{Z}^2$-odometer factors but whose generators $T_{(0,1)}$ and $T_{(1,0)}$ have $\mathbb{Z}$-odometer factors. We provide a simpler construction in Example 6.4.

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