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ON THE FUNDAMENTAL LEMMA FOR STANDARD ENDOSCOPY: REDUCTION TO UNIT ELEMENTS

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ABSTRACT. The fundamental lemma for standard endoscopy follows from the matching of unit elements in Hecke algebras. A simple form of the stable trace formula, based on the matching of unit elements, shows the fundamental lemma to be equivalent to a collection of character identities. These character identities are established by comparing them to a compact-character expansion of orbital integrals.

1. **Introduction.** L. Clozel has deduced the *fundamental lemma* for stable base change from the corresponding result for the unit elements of Hecke algebras [Cl2]. J.-P. Labesse has offered a second proof of this result [La]. This paper adapts Clozel's argument to standard endoscopy, thereby reducing this fundamental lemma for reductive groups to the unit elements of Hecke algebras. Unlike stable base change, the matching of units is not currently known.

One of the main purposes of this paper is to clarify the set of local conditions that imply the fundamental lemma. These local conditions are formalized as *local data* in Section 4.1. Local arguments reduce the fundamental lemma to groups G with connected anisotropic centers. For such G, local data are a collection of finite character identities between a reductive group G and an endoscopic group H. If local data exist, and if the fundamental lemma is known for the Levi factors of G, then the fundamental lemma holds for G.

Although the conditions we formulate are local, the only methods currently known to establish the existence of local data are global. In the final section we show how a simple stable trace formula may be used to prove the existence of local data. This argument assumes the matching of unit elements of Hecke algebras at almost all places of a global situation that we construct.

2. **Notation and terminology.** Let F, $\bar{F}$, and ϖ denote a p-adic field of characteristic zero, a fixed algebraic closure $\bar{F}$, and a uniformizing element in the ring of integers O_F of F. Background on the next few definitions can be found in the survey article by Borel [B]. If G is a reductive group, then $\hat{G}$ is the connected component of the complex dual group

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${}^L G$. The group $\hat{G}_{\text{der}}$ is its derived group and is not to be confused with the connected complex dual of G_{der}. We write $Z(G)$ for the center of G. For any torus S, let $X^*(S)$ and $X_*(S)$ be the character and cocharacter groups of S. In fact, $X^*(S)$ is defined for any diagonalizable group S. Let T_d be the split subtorus of a maximally split Cartan subgroup T_G of G. The complex dual to T_d is conventionally denoted Y. Dual to the inclusion $T_d \subset T_G$ is the canonical projection $\hat{T} \longrightarrow Y$.

Let W' is the subgroup of the Weyl group of T_G that is invariant under the action of the Galois group. The group W' is denoted ${}_kW$ in [B]. The Hecke algebras in this paper are understood to be the spherical Hecke algebras, composed of functions bi-invariant by a fixed hyperspecial maximal compact subgroup. Initially, we will use the Hecke algebra of functions of compact support, but we will also have occasion to use Hecke algebras of $Z(G)^0(F)$-invariant functions that are compactly supported modulo the center. The Satake transform $f^\wedge$ of a compactly supported Hecke function f is a W'-invariant regular function on Y. Under the bijection between Y/W' and $\text{Int}({}^L G^0)$-orbits on $({}^L G^0 \rtimes \text{Fr})_{\text{ss}}$, it is also viewed as a function on a component of ${}^L G$. (See [M] and [B].) Let $c(f, \lambda)$, for $\lambda \in X^*(Y)$, be the coefficient: $f^\wedge(t) = \sum_\lambda c(f, \lambda)\lambda(t)$. Associated with $\lambda \in X^*(Y)$, there is a compactly supported Hecke function ϕ_λ, determined by the requirement that $c(\phi_\lambda, \cdot)$ is the characteristic function of the W'-orbit of λ. The functions ϕ_λ form a linear basis of the Hecke algebra of compactly supported functions. When the group G is defined over O_F and $G(O_F)$ is hyperspecial, the Hecke algebra is to be defined relative to the hyperspecial subgroup $G(O_F)$.

Endoscopic data are attached to a triple (G, θ, ω) consisting of a reductive group G defined over F, an automorphism θ of G over F, and a quasicharacter ω of $G(F)$. For background material on twisted endoscopy, we refer the reader to the work of Kottwitz and Shelstad [KS1]. *Standard endoscopy* refers to data obtained when θ and ω are trivial. This paper treats a slightly larger class, obtained by requiring only θ to be trivial; call this enlarged class *standard endoscopy with quasicharacters*. The fundamental lemma in this paper refers exclusively to the fundamental lemma for standard endoscopy with quasicharacters. In fact, we also deal exclusively with the fundamental lemma for strongly G-regular semisimple elements. As explained in greater detail below, we assume that the endoscopic data is unramified.

We let Fr denote the Frobenius element in the Galois group of the maximal unramified extension of F. When G splits over an unramified extension, the Frobenius element acts on the connected dual $\hat{G}$, and various functorially related dual objects.

A quasicharacter is *unramified* if its Langlands parameter in $H^1\left(W_F, Z(\hat{G})\right)$, the first continuous cohomology group of the Weil group W_F of F with coefficients in $Z(\hat{G})$, is unramified (see [B]). An unramified cocycle in this group is determined by its value at the Frobenius element, and thus unramified quasicharacters correspond bijectively to the set $Z(\hat{G}) \rtimes \text{Fr}$ modulo conjugation by $Z(\hat{G})$.

Two definitions of transfer factors have been given, one an extension of the other. The Langlands-Shelstad factor, defined for standard endoscopy [LS1], extends readily to the slightly larger class of standard endoscopy with quasicharacters, but not to the fully twisted situation. The general twisted transfer factor of Kottwitz and Shelstad [KS1] agrees with the Langlands-Shelstad definition when it is restricted to standard endoscopy with quasicharacters. Thus, we may use results from either paper, depending on which better suits our purposes.

The paper [H2] treats standard endoscopy, not twisted endoscopy. Nevertheless, just as the definition of Langlands and Shelstad extends readily to encompass quasicharacters, so also the results of [H2] extend. We will make use of the canonical normalization of transfer factors, descent for Levi factors, results on the unramified central character, and other minor results from that paper.

The definition of unramified endoscopic group should be amended as follows to include standard endoscopy with quasicharacters. Let $(H, \mathcal{H}, s, \xi)$ be endoscopic data for G in the sense of [KS1]. We say that $(H, \mathcal{H}, s, \xi)$ are *unramified* endoscopic data for (G, ω) if

(1) G is defined over O_F, $G(O_F)$ is hyperspecial, and G is unramified,
(2) H is defined over O_F, $H(O_F)$ is hyperspecial, and H is unramified,
(3) $\mathcal{H}$ is the L-group of H,
(4) The embedding ξ descends to an unramified field extension E/F, and
(5) ω is an unramified quasicharacter of $G(F)$.

The changes to the definition from the paper [H2] are minor. The reference to [LS1] has been changed to [KS1], the quasicharacter ω has been added, along with Condition 5. An unnecessary hypothesis, the finiteness of the extension E/F, has been eliminated from Condition 4. Finally, a redundant condition [H2, 6-Condition 4] has been eliminated.

The map of Hecke algebras (of compactly supported functions) associated with an embedding ξ of L-groups will be denoted $f \longmapsto b_\xi(f)$, or simply $f \longmapsto b(f)$ when the context is clear. Let $\Phi(\gamma_G, f)$ denote the orbital integral of f on the conjugacy class of γ_G for the quasicharacter ω. If γ_G is strongly regular with centralizer T, this orbital integral is defined as

$$\Phi(\gamma_G, f) = \int_{T(F)\backslash G(F)} f(g^{-1}\gamma_G g)\omega(g)\, dg,$$

where dg is an invariant measure on $T(F)\backslash G(F)$. Similarly, let $\Phi^{\mathrm{st}}\big(\gamma_H, b(f)\big)$ be the stable orbital integral of $b(f)$. It is defined as

$$\Phi^{\mathrm{st}}\big(\gamma_H, b(f)\big) = \int_{(T_H\backslash H)(F)} b(f)(h^{-1}\gamma_H h)\, dh,$$

where T_H is the centralizer of γ_H, and dh is an invariant measure on $(T_H\backslash H)(F)$ obtained from an invariant form on $T_H\backslash H$. The distinction between $(T_H\backslash H)(F)$ and $T_H(F)\backslash H(F)$ is essential. We take the measures on G and H to be compatibly normalized. We let $\Lambda = 0$ denote the identity of the fundamental lemma. Namely, set

$$\Lambda(\gamma_H, f) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f) - \Phi^{\mathrm{st}}\big(\gamma_H, b(f)\big),$$

where Δ is the transfer factor of Kottwitz and Shelstad with the canonical normalization given in [H2, 7]. Let $H(F)_{G\text{-reg}}$ denote the set of strongly G-regular elements in $H(F)$

(see [LS1]). The fundamental lemma conjecturally asserts that $\Lambda(\gamma_H, f) = 0$, for all $\gamma_H \in H(F)_{G\text{-reg}}$ and all compactly supported Hecke functions f.

Next, we elaborate on what it means for the fundamental lemma to hold for $Z(G)^0(F)$-invariant Hecke functions. For each basis function ϕ_λ of the Hecke algebra, we define a $Z(G)^0(F)$-extension ϕ'_λ by the the condition $\phi'_\lambda(\gamma) = \phi_\lambda(z\gamma)$ if there exists $z \in Z(G)^0(F)$ such that $z\gamma \in \mathrm{supp}(\phi_\lambda)$, and $\phi'_\lambda(\gamma) = 0$ otherwise. Lemma 3.2 will prove that each unramified character of $G(F)$ is constant on the support of ϕ_λ. Thus, the ambiguity of the expression $z\gamma$ is by an element of $Z(G)^0(F)$ lying in the kernel of all unramified characters. Such an element belongs to $G(O_F) \cap Z(G)^0(F)$, so that ϕ'_λ is well-defined. The functions ϕ'_λ span the space of $Z(G)^0(F)$-invariant Hecke functions that are compactly supported modulo the center. Equivalently, ϕ'_λ is given by

$$\sum_{z \in Z(G)^0(F)/Z(G)^0(O_F)} R_z \phi_\lambda,$$

where R_z is defined by $R_z f(g) = f(zg)$. Set $b'_\xi(\phi'_\lambda) = \sum b_\xi(R_z \phi_\lambda)$, where the locally finite sum extends over $z \in Z(G)^0(F)/Z(G)^0(O_F)$. Use b' to extend the notion of the fundamental lemma. Write $\Lambda'(\gamma_H, f)$, when f is a $Z(G)^0(F)$-invariant Hecke function, for the expression obtained from $\Lambda(\gamma_H, \cdot)$ by replacing b with b'.

3. **Routine facts and routine reductions.**

LEMMA 3.1. *Every quasicharacter on G is constant on each stable conjugacy class.*

PROOF. By a z-extension argument [Ko1, 3.1.2], we may assume that the derived group of G is simply connected. If γ and γ^g are stably conjugate, then $\gamma^{-1}\gamma^g$ lies in both $G_{\mathrm{der}}(\bar{F})$ and $G(F)$, and hence also in $G_{\mathrm{sc}}(F)$. Any quasicharacter vanishes on $G_{\mathrm{sc}}(F)$, and from this the result follows. ∎

LEMMA 3.2. *Every unramified character is constant on the support of each Hecke function ϕ_λ.*

PROOF. Fix an unramified character θ. The function $\gamma \longmapsto \tilde{\phi}_\lambda(\gamma)$, defined to be $\phi_\lambda(\gamma)$ when $\theta(\gamma) = \theta(\varpi^\lambda)$ and zero otherwise, belongs to the Hecke algebra and has the same orbital integrals on the maximally split torus as ϕ_λ. If two functions in the Hecke algebra have equal orbital integrals on this torus, then the functions are equal. Hence $\phi_\lambda = \tilde{\phi}_\lambda$. Consequently, the unramified character θ is constant on the support of ϕ_λ. ∎

The rest of the section carries out some routine reductions that simplify the exposition in later sections. For instance, nothing is lost by assuming that H is an elliptic endoscopic group, because the fundamental lemma for a nonelliptic endoscopic group is equivalent to a fundamental lemma for an elliptic endoscopic group obtained by descent (see [H2], [LS2]).

LEMMA 3.3. *If the fundamental lemma holds for one choice of embedding ξ, then it holds for all choices of embeddings.*

PROOF. Compare two embeddings of L-groups ξ and ξ'. Because of the standing assumption that the endoscopic data is unramified, both ξ and ξ' are unramified. Write $\Delta^{\xi}(\gamma_H, \gamma_G)$ for the transfer factor associated with the embedding ξ. The ratio of the transfer factors depends only on the term Δ_{III_2}, defined in [LS1]. The ratio has the form

$$\Delta^{\xi'}(\gamma_H, \gamma_G)/\Delta^{\xi}(\gamma_H, \gamma_G) = \langle a_{\xi'}/a_{\xi}, \gamma\rangle,$$

where γ, inside a Cartan subgroup $T = T_G$, is an image of γ_G in the quasisplit form of G, and a_ξ and $a_{\xi'}$ are cocycles, defined in [LS1], in the Weil cohomology group $H^1(W_F, \hat{T})$. According to Langlands' theory of abelian groups, this cohomology group pairs canonically with $T(F)$.

The embeddings $\xi, \xi' \colon {}^L H \longrightarrow {}^L G$ have the form $\xi(1 \rtimes \mathrm{Fr}) = x_1 m_0 \rtimes \mathrm{Fr}$ and $\xi'(1 \rtimes \mathrm{Fr}) = x_1' m_0 \rtimes \mathrm{Fr}$, for some elements x_1, x_1', and $m_0 \in \hat{G}$. Both $x_1 m_0 \rtimes \mathrm{Fr}$ and $x_1' m_0 \rtimes \mathrm{Fr}$ fix a splitting in $\hat{H}$, and this is possible only if x_1'/x_1 belongs to $Z(\hat{H})$. It follows by consulting the definition of a_ξ and $a_{\xi'}$ in [LS1], discussed further in [H2], that $(x_1'/x_1)a_\xi(w) = a_{\xi'}(w)$, for all elements w in the Weil group W_F lying over the Frobenius element Fr. If θ is the unramified character on $H(F)$ whose parameter in $Z(\hat{H}) \rtimes \mathrm{Fr}$ is $(x_1'/x_1) \rtimes \mathrm{Fr}$, then we conclude that the ratio of transfer factors $\langle a_{\xi'}/a_\xi, \gamma\rangle$ simplifies to $\theta(\gamma)$.

Now assume that the fundamental lemma holds for the embedding ξ. Then

$$\sum_{\gamma_G} \Delta^{\xi'}(\gamma_H, \gamma_G)\Phi(\gamma_G, f) = \theta(\gamma_H)\sum_{\gamma_G} \Delta^{\xi}(\gamma_H, \gamma_G)\Phi(\gamma_G, f) = \theta(\gamma_H)\Phi^{\mathrm{st}}\big(\gamma_H, b_\xi(f)\big).$$

Lemma 3.3 will then follow from the identity

$$(*) \qquad \theta(\gamma_H)\Phi^{\mathrm{st}}\big(\gamma_H, b_\xi(f)\big) = \Phi^{\mathrm{st}}\big(\gamma_H, b_{\xi'}(f)\big),$$

for all γ_H in $H(F)_{G\text{-reg}}$. There are three separate cases to consider in the proof of this identity.

CASE 1. Suppose that the endoscopic group H is an elliptic torus of G. We may identify $\hat{H}$ with the complex dual of T_G in $\hat{G}$. An ordered pair (t, z) will represent the element tz of $\hat{G}$ according to the decomposition $\hat{G} = \hat{G}_{\mathrm{der}} Z(\hat{G})^0$. With a slight shift in notation, the embedding $\xi \colon {}^L H \longrightarrow {}^L G$ takes the form $(t, z) \rtimes \mathrm{Fr}_H \longmapsto (tx_0 m_0, zx) \rtimes \mathrm{Fr}_G$, where m_0 lies in the normalizer of $\hat{H}$ and is independent of the embedding. The subscripts H and G have been added to the Frobenius element to distinguish the L-actions coming from H and G. Set $\hat{T}_{\mathrm{der}} = \hat{G}_{\mathrm{der}} \cap \hat{T}_G$. Since $\hat{T}_{\mathrm{der}}$ with its H-induced Frobenius action is dual to an unramified elliptic torus, $tx_0 m_0$ is Fr_G-conjugate to an element $\rho \in \hat{T}_{\mathrm{der}}$ that is independent of t and x_0. In other words, the map $\hat{T}_{\mathrm{der}} \longrightarrow \hat{T}_{\mathrm{der}}$ given by $t \longmapsto t^{-1} m_0 \mathrm{Fr}_G(t) m_0^{-1} = t^{-1}\mathrm{Fr}_H(t)$ is surjective. We obtain the conjugate element $(\rho, zx) \rtimes \mathrm{Fr}_G \in \hat{T} \rtimes \mathrm{Fr}_G$.

Recall that Y is the complex dual to the split torus T_d. If $\lambda \in X^*(Y)$, then it pulls back to a character, also denoted λ, in $X^*(\hat{T})$ satisfying $\mathrm{Fr}_G(\lambda) = \lambda$. By definition, the

Satake transform of $b_\xi(\phi_\lambda)$, viewed as a function on $\hat{H} \rtimes \mathrm{Fr}_H$, evaluated at $(t,z) \rtimes \mathrm{Fr}_H$ is $\sum_{w \in W'} w \cdot \lambda(\rho, zx)$, where, as usual, W' is the subgroup of the Weyl group fixed by the Frobenius element Fr_G. The group W' acts trivially on $Z(\hat{G})^0$, so that this expression simplifies to $c_\lambda \lambda(1, zx)$, where $c_\lambda = \sum_{w \in W'} w \cdot \lambda(\rho, 1)$. The ambiguity by $\hat{T}_{\mathrm{der}} \cap Z(\hat{G})^0$ in the decomposition $\hat{T} = \hat{T}_{\mathrm{der}} Z(\hat{G})^0$ may be used to show that $c_\lambda = 0$, unless the restriction of λ to $\hat{T}_{\mathrm{der}} \cap Z(\hat{G})^0$ is trivial. Hence, for nonvanishing terms, the character λ descends to a character on $Z(\hat{G})^0 / \big(Z(\hat{G})^0 \cap \hat{T}_{\mathrm{der}}\big)$, which is the complex dual to $Z(G)^0$. The actions of Fr_G and Fr_H coincide on $Z(\hat{G})^0$. Thus, the invariance $\lambda\big(\mathrm{Fr}_G(z)\big) = \lambda(z)$ gives $\lambda\big(\mathrm{Fr}_H(z)\big) = \lambda(z)$, for $z \in Z(\hat{G})^0$. The complex torus Y_H, defined as the dual to the maximally split subtorus of H, is a quotient of the complex dual to $Z(G)^0$, because H is elliptic in G. This shows that λ descends to a character $\bar{\lambda}$ in $X^*(Y_H)$. The Satake transform of $b_\xi(\phi_\lambda)$ becomes $\bar{z} \longmapsto c_\lambda \bar{\lambda}(\bar{z}) \bar{\lambda}(\bar{x})$, where $\bar{z}$ and $\bar{x}$ are the images of $(1,z)$ and $(1,x)$ in Y_H. The Satake transform on a torus H is essentially trivial; we find that $b_\xi(\phi_\lambda)$ is the characteristic function of the double coset $H(O_F)\varpi^{\bar{\lambda}} H(O_F)$ times the constant $c_\lambda \bar{\lambda}(\bar{x})$. For $\gamma = \varpi^{\bar{\lambda}}$, the identity $(*)$ becomes

$$c_\lambda \theta(\varpi^{\bar{\lambda}}) \bar{\lambda}(\bar{x}) = c_\lambda \bar{\lambda}(\bar{x}'),$$

where $\bar{x}'$ corresponds to ξ'. Recall that θ is the unramified character defined by comparing the embeddings ξ and ξ'; they differ by an element $(1, x'/x) \in Z(\hat{G})^0$. Thus, $\theta(\varpi^{\bar{\lambda}}) = \bar{\lambda}(\bar{x}'/\bar{x})$. The identity $(*)$ is now evident.

CASE 2. Suppose that the element γ_H lies in the maximally split Cartan subgroup T_H of H and that $H \neq T_H$. In this case T_H is not elliptic, and a routine descent argument (for instance, [H2, 9]) reduces this case to the previous case.

CASE 3. Suppose that the element γ_H does not lie in the maximally split Cartan subgroup. Fix a function f of the Hecke algebra, and write $b_\xi(f)$ as a finite linear combination $\sum_\lambda c_{\xi,\lambda} \phi^H_\lambda$, where ϕ^H_λ is the basis function on the group H corresponding to $\lambda \in X^*(Y_H)$, analogous to the function ϕ_λ on G. Do the same for $b_{\xi'}(f)$. By Lemma 3.2 an unramified character θ is constant on the support of ϕ^H_λ.

Recall that θ is also constant on each stable conjugacy class (Lemma 3.1). Thus, $(*)$ may be rewritten, when γ_H belongs to the maximally split torus (Case 2), as the collection of identities $\theta(\varpi^\lambda) c_{\xi,\lambda} = c_{\xi',\lambda}$, for all λ. Select λ_0 such that $\theta(\gamma_H) = \theta(\varpi^{\lambda_0})$. Then $(*)$ holds generally, because

$$\begin{aligned}\theta(\gamma_H)\Phi^{\mathrm{st}}\big(\gamma_H, b_\xi(f)\big) &= \sum_\lambda \theta(\gamma_H) c_{\xi,\lambda} \Phi^{\mathrm{st}}(\gamma_H, \phi^H_\lambda) = \sum_\lambda \theta(\varpi^{\lambda_0}) c_{\xi,\lambda} \Phi^{\mathrm{st}}(\gamma_H, \phi^H_\lambda) \\ &= \sum c_{\xi',\lambda} \Phi^{\mathrm{st}}(\gamma_H, \phi^H_\lambda) = \Phi^{\mathrm{st}}\big(\gamma_H, b_{\xi'}(f)\big).\end{aligned}$$

■

Let θ be an unramified character of $G(F)$ with parameter $t \rtimes \mathrm{Fr}$ in $Z(\hat{G}) \rtimes \mathrm{Fr}$. The canonical inclusion of $Z(\hat{G}) \longrightarrow Z(\hat{H})$ leads to a character θ_H on $H(F)$ with the same parameter, now in $Z(\hat{H}) \rtimes \mathrm{Fr}$.

LEMMA 3.4. *With θ_H constructed as above, θ_H is constant on the support of $b(\phi_\lambda)$. Moreover, $\theta\big(\mathrm{supp}(\phi_\lambda)\big) = \theta_H\Big(\mathrm{supp}\big(b(\phi_\lambda)\big)\Big)$.*

PROOF. Certainly $b(\phi_\lambda)$ is determined by its values on the elements $\varpi^{\lambda'}$, for $\lambda' \in X^*(Y_H)$, and θ_H is constant on every double coset of $H(O_F)$. Thus, we consider $\theta_H(\varpi^{\lambda'})$, for $\varpi^{\lambda'} \in \mathrm{supp}\big(b(\phi_\lambda)\big)$. By a descent argument we reduce to the case that H is an elliptic torus of G. The argument of Lemma 3.3 (Case 1) shows that $b(\phi_\lambda)$ is supported on the double coset of $\varpi^{\bar\lambda}$. The character $\bar\lambda \in X^*(Y_H)$ gives a character on $\widehat{Z(G)^0}$ and hence a character λ' on $Z(\hat G)$, by dualizing the inclusion $T_d^H \subset Z(G)^0$, where T_d^H is the maximally split torus in the elliptic Cartan subgroup H. One then easily finds that $\theta_H(\varpi^{\bar\lambda}) = \langle t, \lambda'\rangle = \theta(\varpi^{\lambda})$. ∎

LEMMA 3.5. *The fundamental lemma is true for a reductive group if it is true for the $Z(G)^0(F)$-invariant Hecke functions on the group.*

PROOF. By Lemma 3.3, we may pick whatever embedding is the most convenient. As the proof of Lemma 3.6 will explain in greater detail, there exists an embedding for which the transfer factor is invariant by the connected center: $\Delta(z\gamma_H, z\gamma_G) = \Delta(\gamma_H, \gamma_G)$, for $z \in Z(G)^0(F)$. We refer the reader to the definition of $b'(\phi'_\lambda)$ in Section 2. We have

$$(*) \qquad \Lambda'(\gamma_H, \phi'_\lambda) = \sum_z \Lambda(\gamma_H, R_z\phi_\lambda) = \sum_z \Lambda(z\gamma_H, \phi_\lambda).$$

The second equality makes use of the compatibility of the fundamental lemma with translations by $Z(G)^0(F)$, as proved in [H2, 11]. (With a different choice of embedding, a character of $Z(G)^0(F)$ would appear in the right-hand term.) The hypothesis of the lemma means that $\Lambda'(\gamma_H, \phi'_\lambda) = 0$, for all λ. By translating γ_H by a central element of G, we may assume that γ_G lies in the same coset of ${}^0G$ as the support of ϕ_λ (see Lemma 3.2), where ${}^0G$ is the intersection of the kernels of all unramified characters on G. Then the stable conjugacy class of $z\gamma_G$ does not meet the support of ϕ_λ unless z belongs to $Z(G)^0(O_F)$.

By Lemma 3.4, the stable conjugacy class of $z\gamma_H$ does not meet the support of $b(\phi_\lambda)$ unless z belongs to $Z(G)^0(O_F)$. Consequently, $(*)$ becomes $\Lambda(\gamma_H, \phi_\lambda) = 0$, and the fundamental lemma holds. ∎

LEMMA 3.6. *If the fundamental lemma holds whenever G is an adjoint group, then it holds in general.*

PROOF. By Lemma 3.3, we may pick whatever embedding is the most convenient. For our purposes, it is best to pick an embedding for which the image in ${}^LG$ of the Frobenius element Fr lies in $\hat G_{\mathrm{der}} \rtimes \mathrm{Fr}$. Such embeddings exist (see for example [H2, 6.1]). By adjusting the choice of the element s, which is a given of the unramified endoscopic data, by a central element in $\hat G$, we may assume that s lies in the derived group of $\hat G$. With these choices the endoscopic data $(H, \mathcal{H}, s, \xi)$ easily lead to endoscopic data $(\bar H, {}^L\bar H, s, \bar\xi)$ of the semisimple group $\bar G$ dual to $\hat G_{\mathrm{der}}$.

As an initial step toward the proof, let us deduce the fundamental lemma associated with the data $(H, \mathcal{H}, s, \xi)$ from the fundamental lemma associated with the data

$(\bar{H}, {}^L\bar{H}, s, \bar{\xi})$. There is nothing difficult here, but a number of small facts must be checked. The sequence $1 \longrightarrow \hat{G}_{\text{der}} \longrightarrow \hat{G} \longrightarrow \hat{G}/\hat{G}_{\text{der}} \longrightarrow 1$ is dual to the sequence $1 \longrightarrow Z(G)^0 \longrightarrow G \longrightarrow G/Z(G)^0 \longrightarrow 1$. In fact, in the sequence $1 \longrightarrow A \longrightarrow G \longrightarrow \bar{G} \longrightarrow 1$ dual to $1 \longrightarrow \hat{G}_{\text{der}} \longrightarrow \hat{G} \longrightarrow \hat{G}/\hat{G}_{\text{der}} \longrightarrow 1$, we know that A is central and connected, and $\bar{G}$ is semisimple, and these properties characterize the sequence $1 \longrightarrow Z(G)^0 \longrightarrow G \longrightarrow G/Z(G)^0 \longrightarrow 1$. Thus $\bar{G}$ is isomorphic to the quotient $G/Z(G)^0$. Similarly, $\bar{H}$ is isomorphic to $H/Z(G)^0$, under the canonical embedding of $Z(G)^0$ into H. Thus, we may project a pair (γ_H, γ_G) in $H \times G$ to a pair $(\bar{\gamma}_H, \bar{\gamma}_G)$ in $\bar{H} \times \bar{G}$. Because of the choice of ξ made above, the cocycle a_ξ restricts to a character that is trivial on $Z(G)^0$. (See [H2, 11]). It is then a mere formality to check that $\Delta^{\xi}_{H,G}(\gamma_H, \gamma_G) = \Delta^{\bar{\xi}}_{\bar{H},\bar{G}}(\bar{\gamma}_H, \bar{\gamma}_G)$. Subscripts to Δ have been added to distinguish transfer factors on different groups.

The image of a stable conjugacy class in G is a stable conjugacy class in $\bar{G}$, and likewise for H. This simple fact follows from the observation that the image of $G(F)$ in $\bar{G}(F)$ is the kernel of a collection of quasicharacters, and that quasicharacters are constant on stable conjugacy classes (Lemma 3.1). By pulling Hecke functions on $\bar{G}$ and $\bar{H}$ back to G and H, the functions are only compactly supported modulo the center. But Lemma 3.5 states that this does not matter. In this way the fundamental lemma is displaced to $\bar{G}$ and $\bar{H}$.

Now we assume that G is semisimple and reduce to the case where G is adjoint. The endoscopic data $(H, {}^LH, s, \xi)$ easily lift to data for the simply connected cover $\hat{G}_{\text{sc}}$ It is essential to allow the the quasicharacters ω to be nontrivial at this point, otherwise the lift would not always exist. The character θ on the center of G, defined by $\theta(z) = \Delta(z\gamma_H, z\gamma_G)/\Delta(\gamma_H, \gamma_G)$, is trivial. (The existence of such a character θ is proved in [LS2]). It is trivial because it is the restriction, of an unramified character on the maximally split torus, to the compact (finite) group of F-rational points in $Z(G)$ (see [H2, 11]). There is a canonical injection from the Hecke algebra on G to that on G_{adj}, since the center of G is contained in $G(O_F)$. There is a corresponding injection on the Hecke algebra of H. The remaining details are similar to those already given for the reduction to the semisimple case. ∎

Not only may we assume that G is adjoint, we may also assume that it is simple over the algebraic closure of F. Orbital integrals, endoscopy, transfer factors, Hecke algebras, and the map b of Hecke algebras are all compatible with products and are all compatible with the restriction of scalars.

It is convenient to reduce to standard endoscopy, for which the unramified quasicharacter ω is trivial. The next lemma describes the group to be used for this. The reductive groups G constructed in the next lemma will be called the *basic cases*.

LEMMA 3.7. *For any simple unramified adjoint group G_{adj}, there is an unramified reductive group G (whose adjoint group is G_{adj}) with the following properties:*

(1) The center of G is connected and anisotropic;

(2) The image of $G(F)$ in the adjoint group is equal to the kernel of ω.

PROOF. If we construct an unramified group G_1 whose center is connected, whose derived group is G_{sc}, and whose image in $G_{adj}(F)$ is the kernel of ω, then we may define G to be the quotient of G_1 by the split component of the center.

An embedding $G_{sc} \subset G_1$ is determined by a homomorphism α from $X^*\big(Z(G_1)\big)$ *onto* $X^*\big(Z(G_{sc})\big)$. The center of G_1 is connected if $X^*\big(Z(G_1)\big)$ is torsion free. There is an exact sequence

$$1 \longrightarrow \mathrm{Im}\big(G_1(F) \longrightarrow G_{adj}(F)\big) \setminus G_{adj}(F) \longrightarrow H^1\big(F, Z(G_1)\big) \longrightarrow H^1(F, D),$$

where D is the torus G_1/G_{sc}; see [Ko3, 1.5]. The group $X_*(D)$ may be identified with the lattice dual to the kernel of α. By Tate-Nakayama duality, the quotient of $G_{adj}(F)$ by $G_1(F)$ is identified with the subgroup of $H^{-1}\Big(F, X_*\big(Z(G_1)\big)\Big)$ of elements whose image in $H^{-1}\big(F, X_*(D)\big)$ is trivial. We list the homomorphisms $\alpha\colon X^*\big(Z(G_1)\big) \longrightarrow X^*\big(Z(G_{sc})\big)$ and the action of the Frobenius on $X^*\big(Z(G_1)\big)$, and leave the verification of the properties of the lemma to the reader.

When ω is trivial, we take G_1 to be any unramified z-extension of G_{adj}. Every unramified quasicharacter is trivial on adjoint groups of types ${}^2A_{2k}$, ${}^3D_4$, ${}^2E_6$, E_8, F_4, and G_2.

CASE 1. (split, but not of type $D_{2\ell}$) ω has order ℓ dividing n, $X^*\big(Z(G_{sc})\big) = \mathbb{Z}/n\mathbb{Z}$, and Fr acts trivially on $\mathbb{Z}/n\mathbb{Z}$.

$$1 \longleftarrow \mathbb{Z}/n\mathbb{Z} \stackrel{\alpha}{\longleftarrow} \mathbb{Z}^{\ell+1}/(n, 1, \ldots, 1)\mathbb{Z}.$$

The homomorphism α is projection onto the first factor, and $\mathrm{Fr}(y, x_1, \ldots, x_\ell) = (y, x_2, \ldots, x_\ell, x_1)$ on $X^*\big(Z(G_1)\big)$.

CASE 2. $({}^2A_{2k-1}, {}^2D_{2\ell+1})$ ω has order 2, $X^*\big(Z(G_{sc})\big) = \mathbb{Z}/2k\mathbb{Z}$, and Fr acts by $\mathrm{Fr}(x) = -x$ on $X^*\big(Z(G_{sc})\big)$.

$$1 \longleftarrow \mathbb{Z}/2k\mathbb{Z} \stackrel{\alpha}{\longleftarrow} \mathbb{Z}.$$

Let Fr act by $\mathrm{Fr}(x) = -x$ on $\mathbb{Z}$.

CASE 3. $(D_{2\ell}, {}^2D_{2\ell})$ ω has order 2, $X^*\big(Z(G_{sc})\big) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and Fr acts by order 1 or 2. If Fr acts trivially, then we proceed as in Case 2 with $k = 1$. Now assume that $\mathrm{Fr}(x, y) = (y, x)$ on $X^*\big(Z(G_{sc})\big)$. We induce the data from Case 1. Let $X^*\big(Z(G_1)\big) = \mathbb{Z}^6/\big((2,1,1,0,0,0)\mathbb{Z} + (0,0,0,2,1,1)\mathbb{Z}\big)$, $\alpha(y, x_1, x_2, y', x_1', x_2') = (y, y')$, $\mathrm{Fr}(y, x_1, x_2, y', x_1', x_2') = (y', x_1', x_2', y, x_2, x_1)$, and so forth.

Cases 1, 2, and 3 cover the only possibilities that arise when ω is nontrivial. ∎

LEMMA 3.8. *If the fundamental lemma holds for the basic cases G, then it holds in general.*

PROOF. The fundamental lemma holds for elementary reasons for functions on G_{adj} whose support does not meet the kernel of the quasicharacter ω. To see this, consider

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a strongly regular semisimple element γ that is not in this kernel. By the ω-invariance of the transfer factor [KS1], making the change of variables $g \longmapsto \gamma g$ in the integrand $f(g^{-1}\gamma g)\omega(g) = f\big((\gamma g)^{-1}\gamma(\gamma g)\big)\omega(g)$ of the orbital integral, we find that the orbital integral is $\omega(\gamma)$ times itself and is hence zero. The transfer $b(f)$ of a function f that does not meet the kernel of ω is zero. To see this using Lemma 3.4, we have the trivial character ω_H on $H(F)$, and, if the support of $b(\phi_\lambda)$ is nonempty, then $1 = \omega_H\Big(\mathrm{supp}\big(b(\phi_\lambda)\big)\Big) = \omega\big(\mathrm{supp}(\phi_\lambda)\big)$. Hence, the fundamental lemma is true of such functions. Since the kernel of ω is bi-invariant by the hyperspecial maximal compact subgroup, we may assume now that the support of f lies in the kernel of ω.

Waldspurger has explained the rest of this reduction [W, 3.1.2]. He gives the argument for GL(n), but the argument is general. The orbital integral on the adjoint group is equal to a κ-orbital integral of a Hecke function on G. This lift to G is compatible with endoscopy. ■

4. **Local data.** We are now ready to undertake the nontrivial part of the local argument needed for the fundamental lemma. Local data, discussed in this section, are the local character identities, pertaining to the fundamental lemma, that can be obtained from the trace formula. Throughout this section, we assume that G is a basic case. Let $R(G)$ denote the set of irreducible admissible representations of $G(F)$ with an Iwahori fixed vector. Let $R(H)$ denote the set of irreducible admissible representations of $H(F)$ with an Iwahori fixed vector.

4.1. *Local data* for (G, H) consist of the data (a), (b), and (c) subject to Conditions 1 and 2 below.

(a) An indexing set I (possibly infinite)

(b) A collection of complex constants $a_i^G(\pi)$ for $i \in I$ and $\pi \in R(G)$

(c) A collection of complex constants $a_i^H(\pi')$ for $i \in I$ and $\pi' \in R(H)$

(1) For i fixed, the constants $a_i^G(\pi)$ and $a_i^H(\pi')$ are zero for all but finitely many π and π'.

(2) For every function f in the Hecke algebra of G, the following are equivalent:

  (A) for all $i \in I$, we have $\sum_\pi a_i^G(\pi)\,\mathrm{trace}\,\pi(f) = \sum_{\pi'} a_i^H(\pi')\,\mathrm{trace}\,\pi'\big(b(f)\big)$, and

  (B) for all $\gamma_H \in H(F)_{G\text{-reg}}$, we have $\Lambda(\gamma_H, f) = 0$.

The essential part of the definition is Condition 2. Roughly, local data indicate how to translate the fundamental lemma into a collection of character identities. Nothing would change if $R(G)$ and $R(H)$ were taken to be spherical representations, since f and $b(f)$ belong to Hecke algebras.

The justification of local data comes from the following theorem, which will be proved in the rest of this section. Langlands and Shelstad have shown how to obtain an endoscopic group of a Levi factor by descent from an endoscopic group of G (see [LS2]).

THEOREM 4.2. *Let G be a basic case. Suppose that there exist local data for (G, H). Suppose that the fundamental lemma holds for all proper Levi factors of G for the endoscopic groups obtained by descent from H. Then the fundamental lemma holds for the endoscopic group H of G.*

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Let local data be given. It consists of an indexing set I, and functions a_i^G and a_i^H, for $i \in I$. In light of the equivalence expressed by Condition 2, the fundamental lemma holds if the identities $\sum_\pi a_i^G(\pi)\,\mathrm{trace}\,\pi(f) = \sum_{\pi'} a_i^H(\pi')\,\mathrm{trace}\,\pi'\big(b(f)\big)$ hold for all $i \in I$. To show this, we fix our attention on a single identity, for some $i \in I$, and drop i from the notation. By elementary properties of spherical functions, for each $\pi' \in R(H)$, there exist $\pi \in R(G)$ and a parameter $s \in Y$ such that

$$\mathrm{trace}\,\pi'\big(b(f)\big) = \mathrm{trace}\,\pi(f) = f^\wedge(s).$$

This allows us to rewrite the desired identity as $0 = A(f)$, where $A(f)$ is a finite sum of the form $A(f) = \sum a(s) f^\wedge(s)$, for appropriate functions $a(s)$ on Y. If we show that A is zero (Lemma 4.4), then the theorem is proved.

The compact trace, denoted $\mathrm{trace}_c\,\pi(f)$, is defined in [Cl1]; in brief, it is equal to $\mathrm{trace}\,\pi(1_c f)$, where 1_c is the characteristic function of the compact elements in $G(F)$. Similarly, we form $\mathrm{trace}_c\,\pi'$, for $\pi' \in R(H)$.

PROPOSITION 4.3. *Under the hypotheses of Theorem 4.2, the linear functional $f \longmapsto A(f)$ on the Hecke algebra is a finite linear combination of the linear functionals $\Lambda(\gamma_H, \cdot)$, for $\gamma_H \in H(F)_{G\text{-reg}}$. There is also an expression for $A(f)$ as a finite linear combination of linear functionals of the form*

$$f \longmapsto \mathrm{trace}_c\,\pi(f) \textit{ and } f \longmapsto \mathrm{trace}_c\,\pi'\big(b(f)\big), \quad \textit{for } \pi \in R(G) \textit{ and } \pi' \in R(H).$$

PROOF. By the hypothesis on Levi factors in Theorem 4.2, we may assume that $\Lambda(\gamma_H, f) = 0$, if γ_H is not elliptic. Thus, the expansion to be produced in Proposition 4.3 will involve functionals $\Lambda(\gamma_H, \cdot)$, for γ_H elliptic. By the Howe conjecture, proved by Clozel, applied to both G and H, we find for any elliptic element $\gamma_H \in H(F)_{G\text{-reg}}$ that $\Lambda(\gamma_H, f)$ has an expansion in terms of compact traces of the sort given in the proposition. (Details of this are given in [H1, 1].)

Turn to the first statement of Proposition 4.3. Again, by the Howe conjecture, the space of distributions $f \longmapsto \Lambda(\gamma_H, f)$ on the Hecke algebra of G is finite dimensional. Thus, the Condition 2.B in the definition of local data may be replaced with the condition

$$\text{(B}'\text{)} \qquad \Lambda(\gamma_j, f) = 0, \quad \text{for } j = 1, \ldots, k,$$

for an appropriate finite collection $\{\gamma_j\}$ of strongly G-regular semisimple elements in H. Condition 2 now implies that, if $\Lambda(\gamma_j, f) = 0$, for $j = 1, \ldots, k$, then $A(f) = 0$. This means that the functional A is a linear combination of the functionals $\Lambda(\gamma_j, \cdot)$. ■

LEMMA 4.4. *The functional A vanishes identically on the Hecke algebra.*

PROOF THE PARAMETERS s, FINITE IN NUMBER, FOR WHICH $a(s) \neq 0$ ARE TEMPERED. This temperedness argument is given by Clozel [Cl2, 5.5]. In the present context the argument is even easier because we avoid the complications of base change. The argument relies on Proposition 4.3 expressing the functional A as a finite linear combination of the distributions $\Lambda(\gamma_j, \cdot)$ and the temperedness of orbital integrals.

In the compact-trace expansion of $A(f)$ given in Proposition 4.3, we may assume that each of the representations π and π' comes from a nonunitary point in the spectrum. To see this, we consider various cases. Begin with H. Since the orbital integrals on H are stable, we may assume that π' is obtained by pulling back a representation on the adjoint group. A tempered representation on the adjoint group with an Iwahori fixed vector is a full induced unitary principal series representation [Ke]. By examining the principal series character formula, we see that all principal series representations have the same compact trace; in particular, π' has the same compact trace as a nonunitary point in the spectrum.

Next, consider the representation π in the special case that $G_{\text{adj}} = \text{PGL}(n)$ and ω has order n. The fundamental lemma has been established in this case by Kazhdan [Ka]. Thus, $\Lambda(\gamma_j, f) = 0$, for all γ_j, so that, by the definition of local data, $A(f)$ is also zero.

Finally, consider any representation π in any case not yet treated. Keys has analyzed the reducibility of unitary principal series representations. By analyzing the parameters of [Ke] case by case, we see that it is always possible to deform the inducing parameter away from a unitary point in the spectrum, except in the one case already treated above ($G_{\text{adj}} = \text{PGL}(n)$, ω of order n). Keys actually treats only the semisimple simply connected case, but the other cases are an easy consequence of this, the most reducible case. In fact, in other cases, reducibility is understood by looking at which constituents have vectors fixed by the various hyperspecial subgroups.

To work one example in more detail, we consider the group $\text{Sp}(2n)$ and review some of the results of Keys. We may think of the unitary parameter s as lying in the diagonal subgroup $\{(s_1, s_2, \ldots, s_n, 1, s_n^{-1}, \ldots, s_1^{-1})\}$ of the complex dual group $\text{SO}(2n+1, \mathbb{C})$. Even if there is reducibility, there will be at most two constituents. A unitary parameter s that gives reducibility satisfies, for instance, $s_n = -1$. Unitarity implies that $|s_i| = 1$, for all i. But these representations remain reducible when the unitarity constraint $|s_i| = 1$ is dropped. A calculation with intertwining operators similar to that given in [H1, 2] shows that the compact trace of each constituent remains unchanged as s varies under the constraint $s_n = -1$. Therefore, there is a nonunitary parameter s satisfying the constraint except in the rank one situation, where $s_n = -1$ determines s. But when the rank is one, we fall within the case previously considered ($\text{PGL}(n)$, ω of order n, $n = 2$).

In every other case, we observe that the intertwining operators $\mathcal{A}(w, \lambda)$ forming the commuting algebra are formed by R-group elements $w \in W$ that are realized in proper Levi subgroups. In particular, the constituents are not elliptic by the results of Arthur [A2]. Thus, there is a Levi subgroup to which we may apply the arguments of [H1, 2].

The final step follows the argument provided by a referee to Clozel's base change paper [Cl2, p. 257]. We will rewrite the identity

$$A(\phi_\lambda) = \sum c_i \operatorname{trace}_c \pi(\phi_\lambda) + \sum c_i' \operatorname{trace}_c \pi'\big(b(\phi_\lambda)\big),$$

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produced by Proposition 4.3, in a more suggestive form. For $\lambda \in X^*(Y)$, we have

$$A(\phi_\lambda) = \sum_s a(s)\phi_\lambda^\wedge(s) = \sum_s \sum_{W'} a(s)\lambda(w\cdot s) = \sum_i a_i\lambda(s_i),$$

for some finite collection of tempered parameters s_i and complex constants a_i.

Next we consider a term $\mathrm{trace}_c\, \pi(\phi_\lambda)$. By a theorem of Clozel and Waldspurger, the compact trace is a linear combination of forms $(\hat{\chi}_N\phi_\lambda^{(P)})^\wedge(z)$. The superscript (P) indicates the function obtained from integration over the unipotent radical of a standard parabolic subgroup $P = MN$. The function $\hat{\chi}_N$ factors as a product of three maps

$$M(F) \longrightarrow \mathbf{a}_M \longrightarrow \mathbf{a}_{M_0} \longrightarrow \mathbb{R},$$

where $\mathbf{a}_M$ is the real Lie algebra of the split center of M, and $M_0 \subset M$ is a Levi subgroup of a minimal parabolic subgroup $P_0 \subset P$. The first map is the Harish-Chandra map $H_M: M \longrightarrow \mathbf{a}_M$. The second map is a natural identification of $\mathbf{a}_M$ with a subspace of $\mathbf{a}_{M_0}$ (defined by Arthur [A1]). The third map is the characteristic function $\hat{\tau}_P^G$ of the *obtuse* Weyl chamber [A1, p. 936], [Cl2, 2.1]. In particular, for $\mu \in X^*(Y)$ and $m \in M(F)$, we have $\hat{\chi}_N(m\varpi^\mu m^{-1}) = \hat{\tau}_P^G(\mu)$, where we have identified $X^*(Y)$ with a lattice in $\mathbf{a}_{M_0}$. Then

$$(\hat{\chi}_N\phi_\lambda^{(P)})^\wedge(z) = \sum_{w\in W'} \hat{\tau}_P^G({}^w\lambda)\,{}^w\lambda(z),$$

where ${}^w\lambda = w\cdot\lambda$. There are finitely many hyperplanes $X_1,\ldots,X_r$ through the origin of $X^*(Y)\otimes\mathbb{R}$ such that $\hat{\tau}_P^G({}^w\lambda) = \hat{\tau}_P^G({}^w\lambda')$ for all P and all $w \in W'$, whenever λ and λ' belong to the same component of $X^*(Y)\otimes\mathbb{R} \setminus (X_1\cup\cdots\cup X_r)$. (For example, take all singular hyperplanes and all hyperplanes in the W'-orbit of the walls of the obtuse Weyl chambers.) Fix one such component C. Then

$$(\hat{\chi}_N\phi_\lambda^{(P)})^\wedge(z) = \sum_{w\in W''} \lambda(w\cdot z)$$

for λ in C, for some subset $W'' \subset W'$ that depends on C, but not on $\lambda \in C$.

The terms $\mathrm{trace}_c\, \pi'\big(b(f)\big)$ are treated similarly. The map ξ sends nonunitary parameters to nonunitary parameters, and through ξ this compact trace is expressed as a linear combination of terms $\lambda(z)$, again for lattice points λ in a suitable open cone of $X^*(Y)$. By passing to a smaller open cone $C' \subset C$, if necessary, to accommodate the terms $\mathrm{trace}_c\, \pi'\big(b(f)\big)$, the identity then takes the form

$$(*) \qquad \sum_i a_i\lambda(s_i) = \sum_j a_j'\lambda(z_j),$$

for $\lambda \in C'$. The sums are finite, the parameters $s_i \in Y$ are unitary, and the parameters $z_i \in Y$ are nonunitary. The characters s_i, z_i of the lattice may be assumed to be linearly independent, so the only such identities $(*)$ are with both sides zero. Integral combinations of elements in C' span $X^*(Y)$. Thus, if both sides vanish on the cone, then both sides vanish for all $\lambda \in X^*(Y)$. ∎

5. **Local transfer.** To stabilize the simple trace formula of Deligne and Kazhdan, as established in [He], we transfer the κ-orbital integrals of a suitable linear combination of matrix coefficients of supercuspidal representations to the endoscopic group. For this result, we work with fields of sufficiently large residual characteristic. We may assume that our groups are unramified, are defined over the ring of integers O_F, and contain an anisotropic Cartan subgroup. (For the global groups we construct, these conditions will hold locally at infinitely many places.)

The groups G and H give reductive groups G_0 and H_0 over the residue field $\mathbb{F}_q$ of F. Select an elliptic Cartan subgroup T_0 in H_0 and transfer it to G_0. We may assume that T_0 comes from an unramified Cartan subgroup T/O_F and that $T(F) \subset H(O_F)$. We identify T with a Cartan subgroup $T(F) \subset G(O_F)$ by an isomorphism $\psi: T \longrightarrow G$ defined over O_F.

LEMMA 5.1. *In this context, there exists a linear combination of matrix coefficients of supercuspidal representations of G whose orbital integrals are supported on the G(F)-orbit of the set of strongly regular semisimple elements of T(F). The same conclusion holds on H, with the additional property that the orbital integrals of γ and $\gamma^n \in T(F) \subset H(F)$ are equal if $n \in N_G(T,F)$. (By γ^n we mean $\psi^{-1}\big(\psi(\gamma)^n\big)$.)*

PROOF. Consider the Deligne-Lusztig characters $R_{T_0,\theta}$ of $G_0(\mathbb{F}_q)$, where θ is in general position. (See [C] for a definition and the standard facts about these generalized characters.) $\pm R_{T_0,\theta}$ is an irreducible cuspidal character of $G_0(\mathbb{F}_q)$. The characters θ that are in general position correspond to rational regular elements in a torus dual to T_0, and so the number of such characters has leading term q^r, where $r = \dim T_0$. The Deligne-Lusztig characters are supported on elements whose semisimple part is conjugate to $T_0(\mathbb{F}_q)$. The characters $R_{T_0,\theta}$ and $R_{T_0,\theta'}$ are linearly independent if θ and θ' belong to different orbits under the Weyl group. The number of singular elements in $T_0(\mathbb{F}_q)$ is bounded by a constant times q^{r-1}. So for q sufficiently large, an obvious counting argument shows that there exists a nonzero linear combination f_0 of irreducible cuspidal Deligne-Lusztig characters that is supported on the set of strongly regular semisimple elements conjugate to $T_0(\mathbb{F}_q)$.

If σ is an irreducible cuspidal representation of $G_0(\mathbb{F}_q)$, then the representation of $G(F)$ obtained by the inflation of σ to $G(O_F)$ and compact induction to $G(F)$ is supercuspidal. (See [G, 5.2].) Extend f_0 to $G(O_F)$ and then to a function $f \in C_c^\infty(G)$ that is supported on $G(O_F)$. It follows from the results of [G, 5.2] that f is a finite linear combination of matrix coefficients of supercuspidal representations of G.

The orbital integrals of f are supported on the set of elements γ conjugate to elements $g^{-1}\gamma g$ of $G(O_F)$ that are congruent to regular semisimple element of $T_0(\mathbb{F}_q)$. The powers $(g^{-1}\gamma g)^{p^{m\ell}}$ tend to $g^{-1}\gamma_s g$, a regular semisimple element in the $G(O_F)$-orbit of $T(F)$, where γ_s is the absolutely semisimple part of γ (see, for example, [H2, 3]). As γ_s and γ commute and γ_s is regular, we see that γ itself lies in the $G(F)$-orbit of the set of regular semisimple elements of $T(F)$. Thus, the orbital integrals of f are zero except on conjugates of regular semisimple elements of $T(F)$.

The proof of the second claim of the lemma is similar. The irreducible cuspidal Deligne-Lusztig characters on H_0 from T_0 span a vector space whose dimension is asymptotic to q^r/w_0, where w_0 is the cardinality of $W(T_0,H_0) = N_{H_0}(T_0,\mathbb{F}_q)/T_0(\mathbb{F}_q)$, whereas

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the space of invariant functions on $H_0(\mathbb{F}_q)$ that are invariant by the Weyl group of (T_0, G_0) and are supported on regular semisimple elements in the orbit of $T_0(\mathbb{F}_q)$ has dimension $\sim q^r/w_1$, where w_1 is the cardinality of $N_{G_0}(T_0, \mathbb{F}_q)/T_0(\mathbb{F}_q)$. Since these are both subspaces of the space (of dimension $\sim q^r/w_0$) of functions invariant under $W(T_0, H_0)$, they have nontrivial intersection for q sufficiently large. Thus, the desired linear combinations of Deligne-Lusztig characters exist, when q is sufficiently large. ∎

We match functions with regular support on G and H by the following characterization of orbital integrals.

LEMMA 5.2. *Let $C_c^\infty(T'^G)$ be the set of locally constant compactly supported functions on the $G(F)$-orbit of the strongly regular semisimple elements of $T(F)$. Set $\Phi(\gamma) = \Phi(\gamma, f)$, for $f \in C_c^\infty(T'^G)$. Then $\Phi(\gamma)$, for $\gamma \in T(F)$, satisfies*

(i) $\Phi(\gamma^n) = \Phi(\gamma)$, for $n \in N_G(T, F)$,

(ii) $\Phi(\gamma)$ is a locally constant compactly supported function on $T(F)$,

(iii) $\Phi(\gamma)$ is supported on the strongly regular semisimple elements of $T(F)$.

Conversely, a function Φ on $T(F)$ satisfying (i), (ii), and (iii) is realized as the orbital integrals of some function in $C_c^\infty(T'^G)$.

PROOF. This is a special case of [Vi]. ∎

To obtain the simple trace formula for a global group, at some place we take the linear combination of matrix coefficients f on $G(F)$ constructed by Lemma 5.1. Its κ-orbital integrals coincide with its ordinary orbital integrals. By the characterization of Lemma 5.2, there is a matching function f^H on an endoscopic group H. Similarly, at another place, we may select the function f^H constructed by the second part of Lemma 5.1 and find a function on G with matching orbital integrals by the characterization of Lemma 5.2. The hypotheses in [He] for a simple trace formula are then satisfied for G and H.

6. **Global arguments.** This section uses the matching of the unit elements in the Hecke algebra, a global argument, and an inductive hypothesis to produce local data. We assume that G is a basic case and that H is an elliptic endoscopic group of G. We say that the matching of units holds if $\Lambda(\gamma_H, f) = 0$, for all strongly G-regular γ_H, when f is the unit element of the Hecke algebra. A reductive group, defined over a number field, will be associated with each G and endoscopic group H. We are now ready for the main theorem of the paper.

THEOREM 6.1. *Suppose that G is a basic case with elliptic endoscopic group H. Suppose that $\Lambda(\gamma_H, f)$ is zero when γ_H is not elliptic. Suppose that the matching of units holds at almost all unramified places of the global group and corresponding endoscopic group associated with G and H. Then the fundamental lemma holds for (G, H).*

PROOF. To simplify notation in the proof, we now shift notation and add a subscript w to data over the local field F. Thus we have the local field F_w, functions f_w, the group G_w, an endoscopic group H_w, and so forth. The terms without subscripts will be global objects. Thus, f will be a function on the adelic points of a global group G, a function soon to be defined as a product over its local components, and H will be an endoscopic group of G.

For each reductive group G_w and corresponding elliptic endoscopic group H_w, we select quasisplit groups G and H over a global field F that specialize at a given place w to G_w and H_w. We may choose F in such a way that F_v is complex for every archimedean place v. The groups H_w and G_w are unramified and the embedding ξ_w of L-groups may be chosen to factor through a finite unramified extension E_w of F_w of some degree k' (see [H2]). Adjusting F, G, and H if necessary, we may assume that there is a cyclic extension E/F of degree k' that splits H and G, that E_w is a field, and that the natural map $\phi\colon \mathrm{Gal}(E_w/F_w) \longrightarrow \mathrm{Gal}(E/F)$ is an isomorphism. The maps ξ_w and ϕ combine to give a global embedding of L-groups $\xi\colon {}^LH \longrightarrow {}^LG$ that factors through $W_F \longrightarrow \mathrm{Gal}(E/F) \simeq \mathrm{Gal}(E_w/F_w)$. At every archimedean place v, this embedding of L-groups reduces to a product of the inclusion map ${}^LH^0 \subset {}^LG^0$ and the identity map $\xi_v\colon W_{F_v} \longrightarrow W_{F_v}$. This means that ξ_v is of *unitary type* in the sense of Shelstad [Sh3].

By the Tchebotarev density theorem, there are then infinitely many places v at which E_v is a field and $\mathrm{Gal}(E_v/F_v)$ is isomorphic to $\mathrm{Gal}(E/F)$. This means that G and H have the weak approximation property: $G(F)$ is dense in $G(F_S)$ and $H(F)$ is dense in $H(F_S)$ for the completion F_S at any finite set of places S (see [KR]).

Fix a regular elliptic element γ_H in $H_w(F_w)$. Select a strongly regular semisimple element $\gamma \in H(F)$ approximating γ_H at w. More specifically, we demand that $\Lambda(\gamma_w, f) = \Lambda(\gamma_H, f)$, for all f in the Hecke algebra of G_w. Such elements exist by weak approximation and the Howe conjecture. We may also assume, by weak approximation, that γ belongs to an anisotropic unramified Cartan subgroup at some place $w_1 \neq w$, and that γ_v, for v every archimedean place, lies in a given open set U (to be specified below). Let T be the centralizer of γ. The Cartan subgroup T is anisotropic and unramified at w_1, and so by the Tchebotarev density theorem, it is anisotropic and unramified at infinitely many places.

Following Kottwitz [Ko2], we say that a torus of H *transfers* to G if there is an admissible embedding of the torus into G, defined over F. A Cartan subgroup in H transfers to G locally everywhere because G is quasisplit. This fact, combined with a criterion of Kottwitz [Ko2, 9.5] and the results of [S, 1.9], implies that a Cartan subgroup in H transfers to G if it is elliptic. In particular, T transfers to G.

Identify T with a Cartan subgroup in G and take the preimage T_{sc} of T in G_{sc}. Consider a character κ on the image of $H^1(F_v, T_{\mathrm{sc},v})$ in $H^1(F_v, T_v)$ at some nonarchimedean place v. In general, the character κ and T_v do not uniquely determine an endoscopic group H_v. By Tate-Nakayama, the character κ determines a character s on the elements of $X_*(T_{\mathrm{sc},v})$ of norm zero, and a choice is involved in lifting s to a character of $X_*(T_{\mathrm{sc},v})$. But when T_v is elliptic, all elements of $X_*(T_{\mathrm{sc},v})$ have norm zero, and no choice is involved. Thus, the various κ separate the endoscopic groups associated with an elliptic torus. This means that there exist functions supported on the regular elements in the stable orbit of an elliptic Cartan subgroup, whose κ-orbital integrals vanish except when κ is associated with a single prescribed elliptic endoscopic group H_v. Choose such a function at a place v_0. Similarly, at another place v at which T_v is anisotropic and unramified, select matching functions f_v and f_v^H with f_v supported on the $G(F)$-orbit of $T_v(F)$ and f_v^H supported on the stable orbit of $T_v \subset H_v$. We may select f_v^H in such a way that the unstable orbital integrals of f_v^H vanish. Then the only endoscopic group that is relevant for the stabilization of H is H itself.

Let S be a finite set of nonarchimedean places that includes w, all the special places mentioned above, and all the places at which G, H, or T is ramified. There are only finitely many endoscopic groups $H = H_0, H_1, \ldots, H_r$ of G that are quasisplit forms of H, that are equivalent to H at v_0, and that are unramified outside S. For $i > 0$, pick a place $v_i \notin S$ at which H and H_i are inequivalent. Since endoscopic groups are quasisplit, the Tchebotarev density theorem gives infinitely many choices for v_i. Select a function f_{v_i} supported on the orbit of the unramified torus $T_{v_i} \subset G_{v_i}$. Then the only endoscopic groups H that are relevant to the stabilization of G are associated with T_{v_i} at v_i. Since T_{v_i} is unramified at v_i, every global endoscopic group associated with T_{v_i} at v_i is also unramified at v_i. The endoscopic group $H = H_0$ is then the only one relevant to the stabilization of the trace formula, provided the functions f_{v_i} are used at $\{v_i\}$ and the unit element of the Hecke algebra is used at all nonarchimedean places except $S \cup \{v_i\}$ (see [Ko3, 7.5]).

Now we consider the complex reductive group G_∞ at the archimedean places. Fix a maximal compact subgroup K of G_∞. Let B be a Borel subgroup with Langlands decomposition $B = MAN$. This is the only cuspidal parabolic subgroup of the complex group G_∞. Let W be the Weyl group of MA in G. Let H_∞ be a complex endoscopic group of G_∞. Similarly, fix a maximal compact subgroup K_H in H_∞. The global embedding constructed above allows us to assume that the embedding $\xi_\infty\colon {}^L H_\infty \longrightarrow {}^L G_\infty$ is of unitary type.

We recall some facts from the work of Shelstad ([Sh1], [Sh2], [Sh4], and especially [Sh3]). Fix a tempered parameter ϕ' for H_∞. Since H_∞ is quasisplit, ϕ' is relevant in the sense of [B]. We may select ϕ' so that the L-packet is a singleton corresponding to an irreducible principal series representation. The lift of the character to G_∞ is a well-defined invariant distribution on G_∞ [Sh3, 4.0.1]. The lift is, up to a sign, the character of a principal series representation of G_∞.

Fix a regular character $\delta_0 \in \hat{M}$ of M and consider the principal series representation $\pi_{\delta_0,\lambda} = \mathrm{Ind}(\delta \otimes \lambda)$, obtained by unitary induction, for $\lambda \in \mathbf{a}^*$, $\mathbf{a}_0 = \mathrm{Lie}(A)$, and $\mathbf{a} = \mathbf{a}_0 \otimes \mathbb{C}$. We may select δ_0 in such a way that for all $\lambda \in i\mathbf{a}_0^*$, the representation $\pi_{\delta_0,\lambda}$ comes from a parameter ϕ' (depending on λ) for H_∞ in the manner described in the previous paragraph.

Let $C_c^\infty(G_\infty, K)$ denote the space of compactly supported C^∞ functions that are right and left K-finite. Consider the subspace $C_c^\infty(G_\infty, \delta_0)$ of $C_c^\infty(G_\infty, K)$ satisfying the condition

$$\langle \mathrm{trace}\, \pi_{\delta,\lambda}, f \rangle = 0,$$

for all $\lambda \in \mathbf{a}^*$ and all $\delta \in \hat{M} \backslash \{W \cdot \delta_0\}$. By the identities of [Vo, 6.6.7] and the irreducibility of $\pi_{\delta_0,\lambda}$, we find that

$$\langle \operatorname{trace} \pi, f \rangle = 0,$$

for all irreducible admissible π inequivalent to $\pi_{w\delta_0,\lambda}$, for $w \in W$ and $\lambda \in \mathbf{a}^*$.

The invariant Paley-Wiener theorem theorem states that the vector space of functions

$$F_f(\lambda) = \langle \operatorname{trace} \pi_{\delta_0,\lambda}, f \rangle,$$

for $f \in C_c^\infty(G, \delta_0)$, consists of all functions in the Paley-Wiener space (on the complex vector space $\mathbf{a}^*$) and that

$$F_f(w\lambda) = \langle \operatorname{trace} \pi_{w^{-1}\delta_0,\lambda}, f \rangle$$

for $w \in W$ [CD]. Fix a function f for which $F_f(\lambda)$ is not identically zero. By the character formula for principal series representations [Kn, 10.18], there exists an open set U contained in the set of regular semisimple elements on which the orbital integrals f of $\gamma \in U$ are nonzero. Fix a function $f^H \in C_c^\infty(H_\infty)$ that is K_H-finite whose orbital integrals match f. The construction of f^H in [CD, A.4] shows that the invariant distribution attached to a parameter ϕ_H for H_∞ vanishes on f^H except when the parameter $\xi_\infty \circ \phi_H$ for G_∞ gives the L-packet of $\pi_{\delta_0,\lambda}$ for some $\lambda \in \mathbf{a}^*$ (not necessarily in $i\mathbf{a}_0^*$).

Suppose that we have an equality of absolutely convergent sums

$$\sum a(\pi) \operatorname{trace} \pi(f) = \sum b(\pi') \operatorname{trace} \pi'(f^H) \tag{6.2}$$

of characters of irreducible unitary representations of G_∞ and H_∞ that holds whenever $f \in C_c^\infty(G_\infty)$ and $f^H \in C_c^\infty(H_\infty)$ have matching orbital integrals. Inserting the functions f and f^H of the previous paragraph, the sum for H_∞ reduces to a sum over irreducible tempered principal series representations. (Temperedness follows from the characterization of [Kn, 8.53, 16.6].) By the definition of the lift of a tempered distribution, each term trace $\pi'(f^H)$ may be replaced by a term trace $\pi(f)$ for some tempered representation π of G_∞. By our restriction on the function f, the character identity between G_∞ and H_∞ takes the form of an absolutely convergent sum

$$\sum_{\lambda \in \mathbf{a}^*} a(\lambda) F_f(\lambda) = 0, \tag{6.3}$$

for all $f \in C_c^\infty(G, \delta_0)$.

When $\pi_{\delta_0,\lambda}$ is unitary, we must have $\lambda \in i\mathbf{a}_0^*$, so $a(\lambda)F_f(\lambda)$ vanishes off $i\mathbf{a}_0^*$ (see [Kn, 16.6]). Fix $f \in C_c^\infty(G, \delta_0)$. We claim that $a(\lambda)F_f(\lambda) = 0$, for all λ. Otherwise, there exists a nonzero constant $c = |a(\lambda_0)F_f(\lambda_0)|$ for some λ_0. The sum (6.3) may be broken into the term $a(\lambda_0)F_f(\lambda_0)$, a sum over a finite set $S_0 \subset i\mathbf{a}_0^*$, and a sum over the remaining terms. By choosing S_0 large enough, we may assume that

$$\sum_{\lambda \in \mathbf{a}^* \setminus S_0} |a(\lambda) F_f(\lambda)| < c.$$

Pick a Paley-Wiener function h on $\mathbf{a}^*$ such that $h(\lambda_0) = 1$, $h(\lambda) = 0$, for $\lambda \in S_0$, and $|h(\lambda)| \leq 1$, for all $\lambda \in i\mathbf{a}_0^*$. $h(\lambda)F_f(\lambda)$ is a Paley-Wiener function, so there exists $f_1 \in C_c^\infty(G_\infty, K)$ such that

$$F_{f_1}(\lambda) = h(\lambda)F_f(\lambda) = \langle \operatorname{trace} \pi_{\delta_0,\lambda}, f_1 \rangle,$$

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and whose trace vanishes on the other components of the admissible dual of G_∞. Apply equation (6.3) to f_1 to conclude that $\sum a(\lambda)h(\lambda)F_f(\lambda) = 0$, with absolute convergence. We then obtain the contradiction

$$c = |a(\lambda_0)h(\lambda_0)F_f(\lambda_0)| = \Big|\sum_{\mathbf{a}^*\setminus\{\lambda_0\}} a(\lambda)h(\lambda)F_f(\lambda)\Big| < c.$$

The simple form of the trace formula gives a formula for the trace of the operator $R(f)$, when f is supercuspidal and R is the right-regular representation of $G(\mathbb{A})$ on $L^2\big(G(F)\setminus G(\mathbb{A})\big)$. (In our context, $Z(G)(F)\setminus Z(G)(\mathbb{A})$ is compact.) When f is supercuspidal, the image of $R(f)$ lies in the space of cusp forms and $R(f)$ is of trace class.

Kottwitz has stabilized the elliptic part of the trace formula. We will only use the elliptic regular part, stabilized by Langlands, obtained by requiring the support of the function f on the adelic points of G to be supported on the regular elliptic set at some place. To compare the trace formulas on G and H we use the main identity from Kottwitz [Ko3], for both G and H. The stabilization in [Ko3] assumes that the derived group of G is simply connected. But, as Kottwitz points out, this assumption may be avoided; the treatments in [L] and [KS2] do not make this assumption. Kottwitz writes $T_e^{**}(f)$ for the elliptic term of the trace formula, for a function f. The superscript $**$ indicates that the sum extends only over the (G,H)-regular terms of the trace formula. By our support conditions on the functions f and f^H, the omitted terms do not belong to the support of f anyway. Similarly, the expression ST_e^{**} stands for the stable elliptic term of the trace formula. The main identity of Kottwitz, applied to both G and H, becomes

$$T_e^{**}(f) = i(G,H)ST_e^{**}(f^H), \quad \text{and}$$
$$T_e^{**}(f^H) = i(H,H)ST_e^{**}(f^H),$$

where $i(\cdot,\cdot)$ are nonzero constants. We take f and f^H to be products of compactly supported smooth functions at all the places. The functions f and f^H must have matching orbital integrals locally everywhere for these identities to hold. Combining the identities, we find a nonzero constant c for which $T_e^{**}(f) = cT_e^{**}(f^H)$.

The existence of local data at the place w is now established by Clozel's arguments. We assume that f and f^H have matching orbital integrals everywhere, except possibly at w, and that the w-components of f and f^H are f_w and $b(f_w)$ in the Hecke algebra. Let f_e^H be the function obtained from f^H by replacing f_w^H with the characteristic function of a compact set that meets all elliptic conjugacy classes in H_w. The support of f_e^H meets only finitely many $H(\mathbb{A})$-conjugacy classes in $H(\mathbb{A})$ that come from global elements in $H(F)$ [Ko3, 8.2]. Shrink the support of the function f_v^H at some place v so the only $H(\mathbb{A})$-conjugacy classes in $H(\mathbb{A})$ meeting the support of f_e^H come from γ. The transfer of T to G gives a corresponding global element $\gamma \in T(F) \subset G(F)$. Every $G(\mathbb{A})$-conjugacy class in $G(\mathbb{A})$ that comes from a global element other than $\gamma \in G(F)$ and that is elliptic at w has vanishing κ-orbital integrals at some place other than w. By the choices made above, we may arrange that the κ-orbital integrals of f on γ are nonzero at all nonarchimedean places except possibly w.

Suppose first that the Hecke functions f_w and $b(f_w)$ have matching orbital integrals at w. Viewed as an identity in f_∞ and f_∞^H, the spectral side of the identity $T_e^{**}(f) = cT_e^{**}(f^H)$ takes the form of Equation 6.2. Set $F(\lambda) = F_{f_\infty}(\lambda)$. The argument of 6.3 shows that $a(\lambda)F(\lambda) = 0$ for all λ. Each term $a(\lambda)F(\lambda)$, viewed as a function $a(\lambda, f_v)F(\lambda)$ of f_v in the Hecke algebra of G_v, is linear. By Harish-Chandra's finiteness theorem, applied to both G and H, each identity $a(\lambda, f_v)F(\lambda) = 0$ is a finite sum of the form of Condition 4.1.2.A [BJ]. This is the implication (Condition 2.B implies Condition 2.A) in the definition of local data. (It is necessary to vary the elliptic element γ_H, to obtain a collection of character identities for each γ_H.)

Conversely, if the character identities $a(\lambda, f_v)F(\lambda) = 0$ hold for f_v and all λ, then we have an equality on the spectral side of the trace formula. The identity $T_e^{**}(f) - cT_e^{**}(f^H) = 0$ then holds. The κ-orbital integrals of γ are nonzero away from w. Since, up to stable conjugacy, the support of f contains only one global element γ that is elliptic at w, and since the fundamental lemma is assumed on nonelliptic elements, this identity simply becomes $\Lambda(\gamma, f_v) = 0$. This is the implication (Condition 2.A implies Condition 2.B) in the definition of local data. Note that the constant in the normalization of the transfer factor at v is fixed by the condition that $\Lambda(\gamma_H, f_v) = 0$ when γ_H is not elliptic. ∎

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