



Moments of the central L -values of the Asai lifts

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Abstract. We study some analytic properties of the Asai lifts associated with cuspidal Hilbert modular forms, and prove sharp bounds for the second moment of their central L -values.

1 Introduction

Let \mathbf{F} be a fixed real quadratic field over \mathbf{Q} , with ring of integers $O = O_{\mathbf{F}}$ and the real imbeddings $\sigma_1 = 1, \sigma_2$. For simplicity, we assume the narrow class number of \mathbf{F} is 1, so the totally positive units are squares of units and every ideal has a totally positive generator. Let $SL(2, O)$ be the Hilbert modular group. For any ideal $\mathcal{C} \subset O$, the Hecke congruence subgroups $\Gamma_0(\mathcal{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, O), \quad c \equiv 0 \pmod{\mathcal{C}} \right\}$, act discontinuously on the upper half-space \mathbf{H}^2 in the usual way with finite co-volumes, i.e., for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathcal{C}) \text{ and } z = (z_1, z_2) \in \mathbf{H}^2,$$

we have

$$\gamma(z) = \left(\frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \frac{\sigma_2(a)z_1 + \sigma_2(b)}{\sigma_2(c)z_1 + \sigma_2(d)} \right).$$

Denote by $M_k(\Gamma_0(\mathcal{C}))(k \in 2\mathbf{Z} \text{ and } \geq 2)$, the space of Hilbert modular forms of parallel even weight (k, k) , level \mathcal{C} with trivial character, i.e., the space of holomorphic functions $f(z)$ on \mathbf{H}^2 such that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathcal{C})$, $f(\gamma(z)) = N(cz + d)^k f(z)$, where for $z = (z_1, z_2) \in \mathbf{H}^2$,

$$N(cz + d)^k = (\sigma_1(c)z_1 + \sigma_1(d))^k \cdot (\sigma_2(c)z_2 + \sigma_2(d))^k.$$

Any $f(z)$ in $M_k(\Gamma_0(\mathcal{C}))$ has the following Fourier expansion (we assume that the different of \mathbf{F} is generated by $\delta = \delta_{\mathbf{F}} > 0$, where and henceforth $\xi > 0$ for $\xi \in \mathbf{F}$ means that ξ is a totally positive element in \mathbf{F} , and denote $v^{(i)} = \sigma_i(v)$, the i th conjugate of v

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for $i = 1, 2$):

$$(1) \quad f(z) = \sum_{v \in O, v \geq 0} a(v) \exp(2\pi i \operatorname{Tr}(vz)),$$

where

$$\operatorname{Tr}(vz) = \sum_{i=1}^2 v^{(i)} z_i \delta^{(i)-1}.$$

Since any $f(z)$ in $M_k(\Gamma_0(\mathcal{C}))$ is invariant under $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, where ε is a unit in O , we have $a(\varepsilon^2 v) = a(v)$.

$f(z) \in M_k(\Gamma_0(\mathcal{C}))$ is called a Hilbert modular cusp form if the Fourier expansion of $f(g(z))N(cz + d)^{-k}$ (see [Lu, p. 130]) has no constant term for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{F})$. Space of all such cusp forms is denoted by $S_k(\Gamma_0(\mathcal{C}))$.

It is well-known (see [Ga]) that $\dim_{\mathbf{C}} S_k(\Gamma_0(\mathcal{C}))$ is finite, and (see [Sh]) $J =: \dim_{\mathbf{C}} S_k(\Gamma_0(\mathcal{C})) \sim \frac{\operatorname{vol}(\Gamma_0(\mathcal{C}) \backslash \mathbf{H}^2)}{(4\pi)^2} (k-1)^2$ as $k \rightarrow \infty$. Moreover,

$$\begin{aligned} \operatorname{vol}(\Gamma_0(\mathcal{C}) \backslash \mathbf{H}^2) &= [SL(2, O) : \Gamma_0(\mathcal{C})] \operatorname{vol}(SL(2, O) \backslash \mathbf{H}^2) \\ &= 2N(\mathcal{C}) \prod_{\mathcal{P} | \mathcal{C}} (1 + N(\mathcal{P})^{-1}) \times \pi^{-2} \zeta_{\mathbf{F}}(2) D^{3/2}, \end{aligned}$$

where $\zeta_{\mathbf{F}}(s)$ is the Dedekind zeta-function of \mathbf{F} and $D = D_{\mathbf{F}}$ is the discriminant. The Petersson inner product on $S_k(\Gamma)$ is defined by

$$\langle g_1, g_2 \rangle = \int_{\Gamma \backslash \mathbf{H}^2} g_1(z) \overline{g_2(z)} \prod_{i=1}^2 y_i^{k-2} dx_i dy_i,$$

where $z = (z_1, z_2)$ with $z_i = x_i + y_i \sqrt{-1}$, $i = 1, 2$.

Now, let f be a cuspidal Hilbert modular form of parallel weight (k, k) for even $k \geq 2$ and with respect to $GL^+(2, O) \supset SL(2, O)$. We assume f is a normalized Hecke eigenform with Fourier coefficients $a_f(v) = a_f(1) \lambda_f(v) N(v)^{(k-1)/2}$, $v \in O$, where $\lambda_f(\mu)$ is the eigenvalue of $f(z)$ for the Hecke operator $T_{(\mu)}$ (see, e.g., [Ga]). We have

$$\lambda_f(\mu) \lambda_f(v) = \sum_{(d), d | (\mu, v), d > 0} \lambda_f\left(\frac{\mu v}{d^2}\right).$$

The standard L -function associated with f is defined, for $\Re(s) > 1$, by

$$L(s, f) = \sum_{(\mu), \mu > 0} \lambda_f(\mu) N(\mu)^{-s},$$

which has Euler product

$$\prod_{(\pi), \pi > 0} (1 - \lambda_f(\pi) N(\pi)^{-s} + N(\pi)^{-2s})^{-1},$$

where π stands for prime element of O . It is well-known that $L(s, f)$ has analytic continuation to the whole complex plane as an entire function. Let

$$\Lambda(s, f) = (2\pi)^{-2s} \Gamma^2(s + (k-1)/2) L(s, f).$$

We then have the functional equation

$$\Lambda(s, f) = \varepsilon_f D^{1-2s} \Lambda(1-s, f),$$

where ε_f is the root number of absolute value 1.

Asai [As] defined a new Dirichlet series by restricting the coefficients on rational integers,

$$L(s, \text{As}(f)) = \zeta(2s) \sum_{m=1}^{\infty} \lambda_f(m) m^{-s}, \quad \Re(s) > 1.$$

He showed that the function

$$\Lambda(s, \text{As}(f)) = D^{s/2} (2\pi)^{-2s} \Gamma(s + k-1) \Gamma(s) L(s, \text{As}(f))$$

admits analytic continuation to the whole s -plane with possible simple poles at $s = 0, 1$, and satisfies the functional equation

$$\Lambda(s, \text{As}(f)) = \Lambda(1-s, \text{As}(f)).$$

Moreover, if

$$\begin{aligned} L(s, f) &= \prod_{(\pi), \pi > 0} (1 - \lambda_f(\pi) N(\pi)^{-s} + N(\pi)^{-2s})^{-1} \\ &= \prod_{(\pi), \pi > 0} [(1 - \alpha_f(\pi) N\pi^{-s})(1 - \beta_f(\pi) N\pi^{-s})]^{-1}, \end{aligned}$$

then we have

$$L(s, \text{As}(f)) = \prod_p L_p(s),$$

where

$$L_p^{-1}(s) = \begin{cases} (1 - \alpha_f(\pi_1) \alpha_f(\pi_2) p^{-s})(1 - \alpha_f(\pi_1) \beta_f(\pi_2) p^{-s}) \\ (1 - \beta_f(\pi_1) \alpha_f(\pi_2) p^{-s})(1 - \beta_f(\pi_1) \beta_f(\pi_2) p^{-s}), & \text{if } p = \pi_1 \pi_2, \pi_1 \neq \pi_2; \\ (1 - \alpha_f(\pi) p^{-s})(1 - \beta_f(\pi) p^{-s})(1 - p^{-2s}), & \text{if } p = \pi; \\ (1 - \alpha_f^2(\pi) p^{-s})(1 - \beta_f^2(\pi) p^{-s})(1 - p^{-s}), & \text{if } p = \pi^2. \end{cases}$$

Ramakrishnan [Ra] and Krishnamurthy [Kr] proved that $\Lambda(s, \text{As}(f))$ is in fact the L -function associated with an automorphic form on $GL(4, A_Q)$, the Asai lift $\text{As}(f)$ of f . Then, in view of the Splitting Formula in [As] and assuming $D = D_F$ is odd, we have

$$L(s, f \otimes f^t) = L(s, \text{As}(f)) L(s, \text{As}(f) \otimes \chi_D),$$

where

$$\chi_D(\cdot) = \left(\frac{D}{\cdot} \right)$$

is the Kronecker symbol, and

$$f^t(z_1, z_2) = f(z_2, z_1).$$

If f is a base change from an Hecke eigenform $h \in S_k(SL_2(\mathbf{Z}))$, then f is symmetric, i.e., $f = f^t$, and

$$L(s, \text{As}(f)) = L(s, \text{sym}^2(h)) L(s, \chi_D),$$

while if f is a base change from an Hecke eigenform $h \in S_k(\Gamma_0(D), \chi_D)$, then also $f = f^t$, and

$$L(s, \text{As}(f)) = L(s, \text{sym}^2(h)) \zeta(s)$$

(see [As, Section 5]).

Moreover, Prasad and Ramakrishnan [PR] established the following (special case of) cuspidal criterion for $\text{As}(f)$.

Theorem 1.1 (Prasad and Ramakrishnan) *With the same notation as above. If f is non-dihedral, then $\text{As}(f)$ is non-cuspidal iff f and f^t are twist-equivalent; if f is dihedral, then $\text{As}(f)$ is non-cuspidal iff f is induced from a quadratic extension K of F which is biquadratic over \mathbf{Q} .*

Choosing an orthonormal basis $\{f_j(z)\}_{j=1}^J$ of $S_k(\Gamma_0(\mathcal{C}))$ and denote the Fourier coefficients of $f_j(z)$ by $a_j(\cdot)$. We normalize the Fourier coefficients $a_j(\mu)$ by

$$\psi_j(\mu) = \left(\frac{N(\mathcal{C})((k-1)!)^2 D^{k+1}}{((4\pi)^2 N(\mu))^{k-1}} \right)^{1/2} a_j(\mu).$$

We then have the Petersson formula for Hilbert modular forms as proved in [Lu],

$$\sum_{j=1}^J \bar{\psi}_j(v) \psi_j(\mu) = \chi_v(\mu) D^{3/2} N(\mathcal{C}) (k-1)^2 + N(\mathcal{C}) (k-1)^2 D (2\pi)^2 \sum_{\varepsilon \in U} \sum_{c \in \mathcal{C}^\times / U} \frac{1}{|N(c)|} S(v, \mu \varepsilon^2; c) N J_{k-1}(4\pi \sqrt{\mu v} |\varepsilon|/|c|),$$

where χ_v is the characteristic function of the set $\{v\varepsilon^2, \varepsilon \in U\}$, U is the unit group of \mathbf{F} ,

$$S(v, \mu; c) = \sum_{h \pmod{c}}^* e\left(\frac{vh + \mu \bar{h}}{c}\right)$$

is the generalized Kloosterman sum, and $e(x) = \exp(2\pi i \text{Tr}(x))$ for $x \in \mathbf{F}$. We will assume that in the above formula, the c 's are chosen among their associates the representatives satisfying $|N(c)|^{1/2} \ll |c^{(i)}| \ll |N(c)|^{1/2}$, $i = 1, 2$.

If the L^2 -normalized basis element $f_j = \tilde{f}_j/|\tilde{f}_j|$ is a newform, where \tilde{f}_j is the corresponding arithmetically normalized newform with the first Fourier coefficient 1, then $\psi_j(\mu) = \psi_j(1) \lambda_j(\mu)$, where $\lambda_j(\cdot)$ denotes the (normalized) Hecke eigenvalues of f_j as noted above. For $\mathcal{C} = (1)$, from the integral representation for $L(s, \tilde{f}_j \otimes \overline{\tilde{f}_j})$,

and the factorization $L(s, \tilde{f}_j \otimes \overline{\tilde{f}_j}) = \zeta_{\mathbf{F}}(s) L(s, \text{ad}(\tilde{f}_j))$, we have

$$|a_j(1)|^{-2} = \|\tilde{f}_j\|^2 = 16D^{1+k} (4\pi)^{-2k-2} \Gamma^2(k) L(1, \text{ad}(\tilde{f}_j)) / L(1, \chi_D).$$

Thus for $\mathcal{C} = (1)$,

$$\bar{\psi}_j(v) \psi_j(\mu) = \frac{(4\pi)^4 L(1, \text{ad}(\tilde{f}_j))}{16L(1, \chi_D)} \lambda_j(v) \lambda_j(\mu).$$

For each j , $1 \leq j \leq J$ and any $\varepsilon > 0$, we have (see [Ta])

$$\lambda_j(\mu) \ll N(\mu)^\varepsilon,$$

and by a straightforward extension of results of [Iw] and [HL] that

$$k^{-\varepsilon} \ll L(1, \text{ad}(\tilde{f}_j)) \ll k^\varepsilon.$$

In [Lu], we proved an asymptotic formula for the mean value of the linear form in $\psi_j(\cdot)$ in the level aspect. In this paper, we establish an analogous result for the weight aspect as well in the context of the quadratic field \mathbf{F} , with an application to the second moment of $L(1/2, \text{As}(f))$. The generalization of Theorem 1.2 to the general totally real fields is straightforward.

Theorem 1.2 *Let $b(\cdot)$ be an arbitrary complex numbers such that $b(\varepsilon^2 \mu) = b(\mu)$ for $\varepsilon \in U$, and $\eta > 0$. Then for $S_k(\Gamma_0(\mathcal{C}))$, we have as $k \rightarrow \infty$,*

$$\sum_{j=1}^J \left| \sum_{\mu} b(\mu) \psi_j(\mu) \right|^2 \ll (N(\mathcal{C})k^2 + X)(kXN(\mathcal{C}))^\eta \sum_{\mu} |b(\mu)|^2,$$

where the summation over μ 's is restricted to $\mu \in O^\times / U^2$, $\mu > 0$, $N(\mu) \leq X$, and the implicit constant only depends on the quadratic field \mathbf{F} and η .

Assume $\text{As}(f)$ is cuspidal. From [IK, p. 98], we have a series representation for the central L -value of $L(s, \text{As}(f))$,

$$(3) \quad L(1/2, \text{As}(f)) = 2 \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1/2}} V_{1/2} \left(\frac{n}{\sqrt{D}} \right),$$

where

$$V_{1/2}(y) = \frac{1}{2\pi i} \int_{(2)} (4\pi^2 y)^{-u} \zeta(1+2u) \frac{\Gamma(1/2+u) \Gamma(k+u-1/2)}{\Gamma(1/2) \Gamma(k-1/2)} \frac{du}{u}.$$

Since

$$\frac{\Gamma(k+u-1/2)}{\Gamma(k-1/2)} \ll k^{\Re(u)}$$

by Stirling's formula, we see that $V_{1/2}(y) \ll k^{-A}$ for any $A \geq 1$, if $y > k^{1+\eta}$ for any $\eta > 0$. Thus, we have

$$L(1/2, \text{As}(f)) = 2 \sum_{n \leq k^{1+\eta}} \frac{\lambda_f(n)}{n^{1/2}} V_{1/2} \left(\frac{n}{\sqrt{D}} \right) + O(1).$$

From Theorem 1.2 and the above formula for $L(1/2, \text{As}(f))$, and by extending the orthonormal Hecke basis of $S_k(\text{GL}_2^+(\mathcal{O}))$ to an orthonormal (Hecke) basis of $S_k(\text{SL}(2, \mathcal{O}))$ and the positivity, we obtain the following theorem.

Theorem 1.3 For the orthonormal Hecke basis $\{f_j\}$ of $S_k(\text{GL}_2^+(\mathcal{O}))$ and any $\eta > 0$, we have

$$\sum_{1 \leq j \leq J}^* |L(1/2, \text{As}(f_j))|^2 \ll k^{2+\eta},$$

where the $*$ means that the summation is restricted to cuspidal Asai lifts $\text{As}(f_j)$, and the constant implicit only depends on the quadratic field \mathbf{F} and η .

It remains to prove Theorem 1.2, which is the goal of the next section.

2 Proof of the Theorem 1.2

From the Poisson integral representation [GR, p. 953, (8)], we have

$$\begin{aligned} J_{k-1}(x) &= \frac{\left(\frac{x}{2}\right)^{k-1}}{\sqrt{\pi} \Gamma(k-1/2)} \int_{-1}^1 (1-t^2)^{k-3/2} \cos(xt) dt \\ (4) \quad &\ll \left(\frac{ex}{2k}\right)^{k-1}, \end{aligned}$$

where the implicit constant is absolute.

To prove Theorem 1.2, we may assume that μ 's are chosen among their associates mod U^2 the representatives satisfying $N(v)^{1/2} \ll v^{(i)} \ll N(v)^{1/2}$, $i = 1, 2$. We have by the Petersson formula (2),

$$\begin{aligned} &\sum_{j=1}^J \left| \sum_{\mu} b(\mu) \psi_j(\mu) \right|^2 \\ &= \sum_{\mu, v} b(\mu) \bar{b}(v) \sum_{j=1}^J \psi_j(\mu) \bar{\psi}_j(v) \\ &= \sum_{\mu} |b(\mu)|^2 D^{3/2} (k-1)^2 N(\mathcal{C}) \\ &\quad + (k-1)^2 DN(\mathcal{C}) (2\pi)^2 \sum_{\varepsilon \in U} \sum_{c \in \mathcal{C}^\times / U} \\ &\quad \times \frac{1}{|N(c)|} \sum_{\mu, v} b(\mu) \bar{b}(v) S(v, \mu \varepsilon^2; c) NJ_{k-1}(4\pi \sqrt{\mu v} |\varepsilon| / |c|) \\ &= \sum_1 + \sum_2, \text{ say.} \end{aligned}$$

We first prove Theorem 1.2 under the condition that $k^2 N(\mathcal{C}) \geq 8(4\pi)^2 X$. In view of (4) and bound $|J_{k-1}(y)| \leq 1$, we have $J_{k-1}(y) \ll \left(\frac{ey}{2k}\right)^{k-1-\eta'} \ll \left(\frac{2y}{k}\right)^{k-1-\eta'}$, for $y > 0$ and $0 \leq \eta' < 1/2$, we have (choosing η' to be 0 or η , $0 < \eta < 1/2$ depending upon

whether $|\varepsilon^{(i)}| \geq 1$ or not)

$$NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|) \ll (4(4\pi)^2\sqrt{(N\mu)(N\nu)}/k^2|N(c)|)^{k-1}(k^2|N(c)|)^\eta \prod_{1 \leq j \leq 2, |\varepsilon^{(j)}| \geq 1} |\varepsilon^{(j)}|^{-\eta} \\ \ll \left(\frac{1}{2|N(c_1)|}\right)^{k-1} (k^2|N(c)|)^\eta \prod_{1 \leq j \leq 2, |\varepsilon^{(j)}| \geq 1} |\varepsilon^{(j)}|^{-\eta},$$

where we write $c = c_1\mathcal{C}$.

Also we have trivially

$$|S(v, \mu\varepsilon^2; c)| \leq N(c).$$

Hence, the partial sum of Σ_2 with the condition $*$ on U that $\varepsilon^{(0)} =: \max(|\varepsilon^{(1)}|, |\varepsilon^{(2)}|) \geq \exp(\log^2 N(\mathcal{C}))$, is bounded by

$$k^{2+2\eta}(N(\mathcal{C}))^{1+\eta} \sum_{\varepsilon \in U}^* |\varepsilon^{(0)}|^{-\eta} \sum_{c_1 \in O^\times/U} \frac{2^{-k}X}{|N(c_1)|^{k-1-\eta}} \sum_{\mu} |b(\mu)|^2 \ll X \sum_{\mu} |b(\mu)|^2,$$

where we use the fact that the number of units ε satisfying $x \leq \log \varepsilon^{(0)} < 2x$, is $O(x)$ since U is cyclic and generated by a fundamental unit of O .

It remains to deal with the remaining sum Σ_2' with the sum over the units ε in U satisfying the condition $\#$: $\log \varepsilon^{(0)} < \log^2 N(\mathcal{C})$. Note the above method clearly also works in this case if $N(\mathcal{C}) \leq 2^{k/2}$. Hence, we may assume $N(\mathcal{C}) > 2^{k/2}$ and thus $k \ll \log N(\mathcal{C})$. We will apply the following lemma proved in [Lu].

Lemma Let $c_1, c_2 > 0$ be constants, $X \geq 1$, $d(\cdot)$ arbitrary complex numbers, and $c \in O$. Then we have

$$\sum_{a \pmod{c}} \left| \sum_{N(v) \leq X, v \in O} ' d(v) e\left(\frac{va}{c}\right) \right|^2 = (|N(c)| + O(X)) \sum_{N(v) \leq X, v \in O} ' |d(v)|^2,$$

where “ $'$ ” means that the summation is restricted to those v 's such that $v > 0$, $c_1 N(v)^{1/2} \leq v^{(i)} \leq c_2 N(v)^{1/2}$.

Using the Mellin–Barnes integral representation [MOS, Section 3.6.3, p. 82],

$$J_{k-1} \left(\frac{4\pi\sqrt{\mu^{(i)}v^{(i)}}|\varepsilon^{(i)}|}{|c^{(i)}|} \right) \\ = \frac{1}{4\pi i} \int_{(2+\eta)} \left(\frac{2\pi\sqrt{\mu^{(i)}v^{(i)}}|\varepsilon^{(i)}|}{|c^{(i)}|} \right)^s \Gamma\left(\frac{k-1}{2} - \frac{s}{2}\right) \left[\Gamma\left(1 + \frac{k-1}{2} + \frac{s}{2}\right) \right]^{-1} ds,$$

opening the Kloosterman sum, and by Cauchy's inequality, we infer that for $c \in \mathcal{C}^\times/U$ and with $s_i = 2 + \eta + \sqrt{-1}t_i$ ($i = 1, 2$) and $0 < \eta < 1/2$,

$$\sum_{\mu, v} b(\mu)\bar{b}(v)S(v, \mu\varepsilon^2; c) NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|) \\ \ll \int_{(2+\eta)} |ds_1| \int_{(2+\eta)} |ds_2| \left| \frac{\Gamma\left(\frac{k-1}{2} - \frac{s_1}{2}\right)}{\Gamma\left(1 + \frac{k-1}{2} + \frac{s_1}{2}\right)} \right| \cdot \left| \frac{\Gamma\left(\frac{k-1}{2} - \frac{s_2}{2}\right)}{\Gamma\left(1 + \frac{k-1}{2} + \frac{s_2}{2}\right)} \right|$$

$$\begin{aligned}
 & \times \max_{s_1, s_2} \sum_{h \pmod{c}} \left| \sum_{\mu, \nu} b(\mu) \bar{b}(\nu) \left(4\pi^2 \sqrt{N(\mu)N(\nu)} / |N(c)| \right)^{2+\eta} \prod_{i=1}^2 (\sqrt{\mu^{(i)} \nu^{(i)}})^{\sqrt{-1}t_i} e\left(\frac{\mu h}{c}\right) \right| \\
 & \ll N(c)^{-(2+\eta)} \int_{(2+\eta)} \frac{|ds_1|}{k + |s_1|} \int_{(2+\eta)} \frac{|ds_2|}{k + |s_2|} \left| \frac{\Gamma\left(\frac{3}{2} - \frac{s_1}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{s_1}{2}\right)} \right| \left| \frac{\Gamma\left(\frac{3}{2} - \frac{s_2}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{s_2}{2}\right)} \right| \\
 & \times \max_{s_1, s_2} \sum_{h \pmod{c}} \left| \sum_{\mu} b(\mu) (N(\mu))^{1+\eta/2} \prod_{i=1}^2 (\mu^{(i)})^{\sqrt{-1}t_i/2} e\left(\frac{\mu h}{c}\right) \right|^2 \\
 & \ll N(c_1)^{-(2+\eta)} (|N(c)| + X) (N(\mathcal{C}))^\eta \sum_{\mu} |b(\mu)|^2,
 \end{aligned}$$

since $k \ll \log N(\mathcal{C})$, where as before, we write $c = c_1 \mathcal{C}$.

Thus the partial sum Σ'_2 is bounded by

$$\begin{aligned}
 & k^2 (N(\mathcal{C}))^\eta \sum_{\varepsilon \in U}^\# \sum_{c_1 \in O^\times / U} \frac{1}{|N(c_1)|^{2+\eta}} (|N(c_1 \mathcal{C})| + X) \sum_{\mu} |b(\mu)|^2 \\
 & \ll (N(\mathcal{C}) + X) N(\mathcal{C})^\eta \sum_{\mu} |b(\mu)|^2,
 \end{aligned}$$

since

$$\sum_{\varepsilon \in U}^\# 1 \ll \log^2 N(\mathcal{C}).$$

Hence, Theorem 1.2 is true if $k^2 N(\mathcal{C}) \geq 8(4\pi)^2 X$.

In the case $k^2 N(\mathcal{C}) < 8(4\pi)^2 X$, we reduce it to the previous case by the famous embedding trick of Iwaniec. Choosing a prime ideal $\mathcal{P} \subset O$ such that $N(\mathcal{P}) k^2 N(\mathcal{C}) \asymp X$ and $N(\mathcal{P}) k^2 N(\mathcal{C}) \geq 8(4\pi)^2 X$. Note that $[\Gamma_0(\mathcal{C}) : \Gamma_0(\mathcal{P}\mathcal{C})] \leq N(\mathcal{P}) + 1$. Let $H_k(\mathcal{C})$ denote an orthonormal basis of $S_{2k}(\Gamma_0(\mathcal{C}))$, and write

$$S_{\mathcal{C}}(b) = \sum_{f \in H_k(\mathcal{C})} \left| \sum_{\mu} b(\mu) \psi_f(\mu) \right|^2.$$

We deduce that

$$\begin{aligned}
 S_{\mathcal{C}}(b) & \leq (1 + N(\mathcal{P})^{-1}) S_{\mathcal{P}\mathcal{C}}(b) \\
 & \ll (N(\mathcal{P}\mathcal{C}) k^2 + X) (k X N(\mathcal{C}))^\eta \sum_{\mu} |b(\mu)|^2 \\
 & \ll X (k X N(\mathcal{C}))^\eta \sum_{\mu} |b(\mu)|^2,
 \end{aligned}$$

and this completes our proof.

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