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ON THE ℓ-ADIC REPRESENTATIONS ATTACHED TO SIMPLE ABELIAN VARIETIES OF TYPE IV

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The ℓ-adic representations associated to prime dimensional type IV absolutely simple abelian varieties over number fields are studied. The image of such a representation was computed. The results coincide with the well-known conjectures of Mumford and Tate.

1. INTRODUCTION

Let K be an algebraic number field and let $\overline{K}$ be an algebraic closure of K. Let $G_K = \operatorname{Gal}(\overline{K}/K)$. For an abelian variety A defined over K, we denote by $\operatorname{End}^\circ(A)$ the endomorphism algebra $\operatorname{End}_{\overline{K}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ of A. For each prime number ℓ, let T_ℓ be the Tate module of A and let $V_\ell = T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. The Galois group G_K acts continuously on T_ℓ. One has the ℓ-adic representation $\rho_\ell \colon G_K \to \operatorname{Aut}(V_\ell)$.

According to Albert's classification of division algebras with positive involutions, the so-called type IV absolutely simple abelian varieties over K are those abelian varieties A with $D = \operatorname{End}^\circ(A)$ is a division algebra over its centre E, where E is a CM-field. Let E^+ be the maximal totally real subfield of E. If $[D : E] = f^2$ and $[E^+ : \mathbb{Q}] = e$, then ef^2 divides $\dim A$ (see [6], Section 21). In particular, when $\dim A = p$ is a prime number, it is easy to see that $D = E$ and E is a CM-field of degree $2p$ or an imaginary quadratic field. Existence of such type of abelian varieties over number fields except the case where $\dim A = 2$ and E is an imaginary quadratic field was proved by Shimura in [**12**].

In this paper, we are interested in the ℓ-adic representations associated to the above prime dimensional type IV absolutely simple abelian varieties over number fields. For the case where E is a CM-field of $2\dim A$, the $\mathbb{Q}_\ell$-Lie algebra $\mathcal{G}_\ell$ of the image of the ℓ-adic representation is well-known to be equal to $\mathcal{M}_\ell = \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, where $\mathcal{M}$ is the $\mathbb{Q}$-Lie algebra of the Mumford–Tate group associated to A (thought of as over $\mathbb{C}$). This is due to Taniyama and Shimura in [**11**]. In the sequel, we shall study the remaining cases. Namely, $\dim A = p$ is an odd prime number and $\operatorname{End}^\circ(A) = E$ is an imaginary quadratic field.

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Fix a K-polarisation on A once for all. Let ψ be the associated Riemann form on V_ℓ. The induced Rosati involution on E is the complex conjugation. One has $\psi(\alpha v, w) = \psi(v, \overline{\alpha} w)$ for α in E and v, w in V_ℓ. The Tate module V_ℓ is a free $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank p. Let G_{V_ℓ} be the algebraic envelope of the ℓ-adic Lie group $G_\ell = \operatorname{Im} \rho_\ell$. By the theorem of Faltings ([4], Section 5, Satz 3), G_{V_ℓ} is a reductive algebraic group over $\mathbb{Q}_\ell$. Let S_{V_ℓ} be the connected component of the identity of $G_{V_\ell} \cap SL_{V_\ell}$ and let $\mathcal{S}_\ell$ be its Lie algebra. By replacing the base field K by a finite extension, we may assume that $\operatorname{End}_{\overline{K}}(A) = \operatorname{End}_K(A)$. Then $G_{V_\ell}(\mathbb{Q}_\ell)$ is contained in the commutant of E_ℓ in the symplectic similitudes $GSp(V_\ell, \psi)$. On the other hand, let α be a nonzero element in E such that $\overline{\alpha} = -\alpha$. It can be shown that there is a unique E_ℓ-Hermitian form ϕ on V_ℓ such that

$$\psi(v, w) = Tr_{E_\ell/\mathbb{Q}_\ell}(\alpha\phi(v, w)) \text{ for all } v, w \text{ in } V_\ell.$$

The commutant of E_ℓ in the symplectic group $Sp(V_\ell, \psi)$ is easily seen to be the unitary group $U(V_{\ell/E_\ell}, \phi)$, which can be regarded as an algebraic group over $\mathbb{Q}_\ell$.

By an ℓ-adic analogy to the method in [7, 13], we shall prove that, for all prime dimensional absolutely simple abelian varieties of type IV over number fields, the reductive Lie algebra $\mathcal{S}_\ell$ is equal to the Lie algebra of $U(V_{\ell/E_\ell}, \phi)$. In particular, $\mathcal{G}_\ell = \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ as was conjectured in [5]. Consequently, the conjecture of Tate on algebraic cycles (see [15]) is true for all prime dimensional absolutely simple abelian varieties of type IV over number fields.

2. Preliminaries

2.1. The algebraic envelope G_{V_ℓ}.

Let A be an abelian variety defined over a number field K. For each prime number ℓ, let $\rho_\ell \colon G_K \to \operatorname{Aut}(V_\ell)$ be the associated ℓ-adic representation. The image G_ℓ of ρ_ℓ is then an ℓ-adic Lie group. Let $\mathcal{G}_\ell$ be the Lie algebra of G_ℓ. It is easily seen that $\mathcal{G}_\ell$ is invariant under finite extensions of the base field K.

Let G_{V_ℓ} be the algebraic envelope of G_ℓ, that is, G_{V_ℓ} is the smallest algebraic subgroup of GL_{V_ℓ} defined over $\mathbb{Q}_\ell$ such that G_ℓ is contained in $G_{V_\ell}(\mathbb{Q}_\ell)$. By the Theorems of Faltings ([4], Section 5, Satz 3, 4), $\mathcal{G}_\ell$ (respectively G_{V_ℓ}) is reductive and $\operatorname{End}_{\mathcal{G}_\ell}(V_\ell) = \operatorname{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ (respectively $\operatorname{End}_{G_{V_\ell}(\mathbb{Q}_\ell)}(V_\ell) = \operatorname{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$). On the other hand, Bogomolov ([1], Corollary 1) proved that $\mathcal{G}_\ell$ (respectively G_{V_ℓ}) contains the homotheties $\mathbb{Q}_\ell$ (respectively G_m). Replacing the base field K by a finite extension of K, we may assume that G_{V_ℓ} is connected (see [2], Section 3.3). Let S_{V_ℓ} be the connected component of the identity of $G_{V_\ell} \cap SL_{V_\ell}$. Then S_{V_ℓ} is again a connected reductive algebraic subgroup of GL_{V_ℓ} defined over $\mathbb{Q}_\ell$. Then $G_{V_\ell} = S_{V_\ell}.G_m$. Let $\mathcal{S}_\ell$ be the Lie algebra of S_{V_ℓ}. Then $\mathcal{S}_\ell$ is a reductive Lie algebra over $\mathbb{Q}_\ell$ and $\mathcal{G}_\ell = \mathcal{S}_\ell \oplus \mathbb{Q}_\ell$.

2.2 THE HODGE–TATE DECOMPOSITION OF V_ℓ. (see [**8**, **10**])

Let C_ℓ be the completion of a fixed algebraic closure of $\mathbb{Q}_\ell$ and let S_ℓ be the set of all finite places of K dividing ℓ. For each $v \in S_\ell$, let $\overline{K}_v$ be the algebraic closure of K_v in C_ℓ. As a $\mathrm{Gal}(\overline{K}_v/K_v)$-module, it is well-known that V_ℓ is a Hodge–Tate module of weights 0 and 1, each of them with multiplicity $\dim A$ (due to Tate and Raynaud, see [**10**], p.157). Denote the Hodge–Tate decomposition of V_ℓ by $V_\ell \otimes_{\mathbb{Q}_\ell} C_\ell = V_{C_\ell}(0) \oplus V_{C_\ell}(1)$. More precisely, $V_{C_\ell}(0)$ is the cotangent space (over C_ℓ) to the dual abelian variety A' of A at its origin and $V_{C_\ell}(1)$ is the 1-fold Tate twist of the tangent space (over C_ℓ) to A at its origin (see [**16**], Corollary 2 of Theorem 3).

For each $v \in S_\ell$, let $\overline{v}$ be an extension of v to $\overline{K}$. Then the local Galois group $\mathrm{Gal}(\overline{K}_v/K_v)$ can be identified with the decomposition group $D_{\overline{v}}$ for $\overline{v}$ in $\mathrm{Gal}(\overline{K}/K)$. Let $I_{\overline{v}}$ be the inertia subgroup of $D_{\overline{v}}$. Then the algebraic envelope of $\rho_\ell(I_{\overline{v}})$ is an algebraic subgroup of G_{V_ℓ}. By a theorem of Sen ([**8**], Section 6), the one-parameter subgroup h_{V_ℓ} of $GL_{V_{\ell/C_\ell}}$ defined by

$$h_{V_\ell}(c)(x) = \begin{cases} x, & \text{if } x \in V_{C_\ell}(0) \\ cx, & \text{if } x \in V_{C_\ell}(1), \end{cases}$$

maps G_{m/C_ℓ} into the algebraic envelope of $\rho_\ell(I_{\overline{v}})$ over C_ℓ. So h_{V_ℓ} is a one-parameter subgroup of G_{V_ℓ} defined over C_ℓ.

2.3. THE UNITARY GROUP $U(V_{\ell/E_\ell}, \phi)$.

For our purpose, we now assume that $E = \mathrm{End}^\circ(A)$ is an imaginary quadratic field. For a fixed K-polarisation on A, let ψ be the associated Riemann form on V_ℓ. The induced Rosati involution on E is the complex conjugation. Let $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. The Tate module V_ℓ is then a free E_ℓ-module of rank $\dim A$. By $\mathbb{Q}_\ell$-linearity, the complex conjugation on E extends to an involution on the $\mathbb{Q}_\ell$-algebra E_ℓ. We denote it again by $^-$. Then

$$\psi(\alpha v, w) = \psi(v, \overline{\alpha} w) \text{ for all } v, w \text{ in } V_\ell; \alpha \text{ in } E_\ell.$$

Let $\mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}$ be the regular trace of E_ℓ over $\mathbb{Q}_\ell$. The following results are an analogy of Lemmas 4.6, 4.7 in [**3**].

LEMMA 2.1. *Let V and W be free E_ℓ-modules of finite rank and let $\psi: V \times W \to \mathbb{Q}_\ell$ be a $\mathbb{Q}_\ell$-bilinear form such that $\psi(ev, w) = \psi(v, ew)$ for all e in E_ℓ, v in V, and w in W. Then there exists a unique E_ℓ-bilinear form ϕ such that*

$$\psi(v, w) = \mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\phi(v, w)) \text{ for all } v \text{ in } V, w \text{ in } W.$$

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PROOF: ψ defines a $\mathbb{Q}_\ell$-linear map $V \otimes_{E_\ell} W \to \mathbb{Q}_\ell$, that is, an element of the $\mathbb{Q}_\ell$-linear dual of $V \otimes_{E_\ell} W$. But $\mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}$ identifies the $\mathbb{Q}_\ell$-linear dual of $V \otimes_{E_\ell} W$ with the E_ℓ-linear dual, and ψ with ϕ. ∎

LEMMA 2.2. *Let $\alpha \in E^*$ be such that $\overline{\alpha} = -\alpha$. Then there exists a unique E_ℓ-Hermitian form ϕ on V_ℓ such that $\psi(v, w) = \mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha\phi(v, w))$ for all v, w in V_ℓ.*

PROOF: Take V to be V_ℓ and W to be V_ℓ with E_ℓ acting through the involution $^-$. Then, by Lemma 2.1, there exists a unique E_ℓ-sesquilinear form ϕ_1 on V_ℓ such that $\psi(v, w) = \mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\phi_1(v,w))$.

Let $\phi = \alpha^{-1}\phi_1$ be such that $\psi(v, w) = \mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha\phi(v, w))$. Since ϕ is sesquilinear, it remains to show that $\phi(v, w) = \overline{\phi(w, v)}$.

By $\psi(v, w) = -\psi(w, v)$ for all v, w in V_ℓ, $\mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha\phi(v, w)) = -\mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha\phi(w, v)) = \mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\overline{\alpha}\phi(w, v))$.

Replacing v by ev with e in E_ℓ, one finds that $\mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha e\phi(v, w)) = \mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\overline{\alpha e}\phi(w, v))$. On the other hand, $\mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha e\,\phi(v, w)) = \mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}\left(\overline{\alpha e}\,\overline{\phi(v, w)}\right)$ and as $\overline{\alpha e}$ is an arbitrary element of E_ℓ, the non-degeneracy of the trace implies that $\overline{\phi(v, w)} = \phi(w, v)$. The uniqueness of ϕ is obvious from Lemma 2.1. ∎

LEMMA 2.3. *The commutant of E_ℓ in $Sp(V_\ell, \psi)$ is equal to $U(V_{\ell/E_\ell}, \phi)$.*

PROOF: Let $T \in \mathrm{Sp}(V_\ell, \psi)$ be such that $T\alpha = \alpha T$ for all α in E_ℓ. Then T can be thought of as an element in $\mathrm{Aut}_{E_\ell}(V_\ell)$. It is easy to check that the map $(v, w) \mapsto \phi(Tv, Tw)$ is an E_ℓ-Hermitian form.

On the other hand, $\psi(Tv, Tw) = \psi(v, w)$ is equivalent to $\mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha\phi(Tv, Tw)) = \mathrm{Tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha\phi(v, w))$. By the uniqueness of ϕ, this amounts to saying that $\phi(Tv, Tw) = \phi(v, w)$. ∎

REMARK. For those ℓ which remain prime in E, $U(V_{\ell/E_\ell}, \phi)$ is an algebraic group over the field E_ℓ. By Weil's restriction of scalars, it can be thought of as a connected algebraic group over $\mathbb{Q}_\ell$. For those ℓ such that $E_\ell = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$, although $U(V_{\ell/E_\ell}, \phi)$ is an algebraic group over $\mathbb{Q}_\ell$, it doesn't seem to be obvious that $U(V_{\ell/E_\ell}, \phi)$ is a connected algebraic group.

3. PROOF OF THE MAIN RESULT

In this section, let A be an abelian variety defined over a number field K where $\dim A = p$ is an odd prime number and $E = \mathrm{End}^\circ(A)$ is an imaginary quadratic field. For simplicity, we shall assume the following conditions (by extending the base field K):

(i) $\mathrm{End}_K(A) = \mathrm{End}_{\overline{K}}(A)$.

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(ii) $E \subseteq K \subseteq \overline{K}$ (identifying E as a subfield of $\overline{K}$).

(iii) The algebraic envelope G_{V_ℓ} is connected.

Fix a non-zero element α in E such that $\overline{\alpha} = -\alpha$. Then as in Section 2.3, let $U(V_{\ell/E_\ell}, \phi)$ be the unitary group with respect to the E_ℓ-Hermitian form ϕ associated with the Riemann form ψ on the free E_ℓ-module V_ℓ of rank p.

We now prove the main theorem.

Theorem 3.1. *The reductive Lie algebra S_ℓ is equal to the Lie algebra of $U(V_{\ell/E_\ell}, \phi)$.*

Proof: By Lemma 2.3, it is clear that S_ℓ is contained in $\mathrm{Lie}(U(V_{\ell/E_\ell}, \phi))$. On the other hand, it is easily seen that $\dim_{\mathbb{Q}_\ell} \mathrm{Lie}(U(V_{\ell/E_\ell}, \phi)) = p^2$. It suffices to show that $\dim S_{V_\ell}$ is at least $p^2 - 1$ and the centre of G_{V_ℓ} is at least of dimension 2.

Now, we divide the rest of the proof into the following steps:

Step 1. Decomposition of V_ℓ by the action of E_ℓ.

Let $\overline{V}_\ell = V_\ell \otimes_{\mathbb{Q}_\ell} C_\ell$, $\overline{E}_\ell = E_\ell \otimes_{\mathbb{Q}_\ell} C_\ell = E \otimes_{\mathbb{Q}} C_\ell$, and let $\{\sigma, \tau\}$ be the two embeddings of E into C_ℓ. Corresponding to σ, τ, one has an $\overline{E}_\ell[G_{V_\ell}]$-module decomposition $\overline{V}_\ell = X \oplus Y$. Namely, $X = \{v \in \overline{V}_\ell \mid e \cdot v = \sigma(e)v \text{ for all } e \text{ in } E\}$ and $Y = \{v \in \overline{V}_\ell \mid e \cdot v = \tau(e)v \text{ for all } e \text{ in } E\}$. Since V_ℓ is a free E_ℓ-module, both of X and Y are p-dimensional C_ℓ-vector spaces.

Let H be the image of the representation $\rho_\ell \colon G_{V_{\ell/C_\ell}} \to GL_X$ given by the action of $G_{V_{\ell/C_\ell}}$ on X.

Lemma 3.1.1. *H is a reductive connected algebraic subgroup of GL_X and $\mathrm{End}_H(X) = C_\ell$. In particular X is an irreducible H-module.*

Proof: By the theorems of Faltings ([4], Section 5, Satz 3, 4) $G_{V_{\ell/C_\ell}}$ acts on $\overline{V}_\ell$ and hence on X semisimply. Moreover, one has $\mathrm{End}_{G_{V_\ell}(\mathbb{Q}_\ell)}(V_\ell) = E_\ell$. By $E_\ell = C_\ell \times C_\ell$, one concludes that $\mathrm{End}_H(X) = C_\ell$.

Step 2. The Hodge–Tate decomposition of V_ℓ.

As in Section 2.2, $\overline{V}_\ell$ has a Hodge–Tate decomposition $\overline{V}_\ell = V_{C_\ell}(0) \oplus V_{C_\ell}(1)$ with $\dim V_{C_\ell}(0) = \dim V_{C_\ell}(1) = p$. Here $V_{C_\ell}(0)$ is the cotangent space (over C_ℓ) to the dual abelian variety $\hat{A}_{/C_\ell}$ at its origin. Let $M = V_{C_\ell}(0)$ and $N = V_{C_\ell}(1)$. From condition (ii) of our assumption, both M, N are $\overline{E}_\ell$-modules. Accordingly, $M = M_\sigma \oplus M_\tau$, where E acts via σ on the former space and via τ on the latter. Similarly, one has $N = N_\sigma \oplus N_\tau$. Let $\dim M_\sigma = n_\sigma$ and $\dim M_\tau = n_\tau$. Then $n_\sigma + n_\tau = p$, where $p \geqslant 3$.

Fix an isomorphism between C_ℓ and $\mathbb{C}$. Consider the dual module of the $\overline{E}_\ell$-module $\mathrm{Lie}(\hat{A}_{/C_\ell})$ (that is the tangent space of $\hat{A}_{/C_\ell}$ at its origin). By a result of Shimura ([**12**], Theorem 5), one concludes that both n_σ and n_τ are positive. ∎

Lemma 3.1.2. *$X = M_\sigma \oplus N_\sigma$ and $\dim M_\sigma$, $\dim N_\sigma$ are relatively prime.*

Proof: $\overline{V}_\ell = X \oplus Y = (M_\sigma \oplus M_\tau) \oplus (N_\sigma \oplus N_\tau)$. One sees easily that $X = (X \cap M) \oplus (X \cap N) = M_\sigma \oplus N_\sigma$. In particular, $\dim N_\sigma = n_\tau$. Since $n_\sigma + n_\tau = p$ (odd prime), so n_σ, n_τ are relatively prime.

Note that Lemmas 3.1.1, 3.1.2 verify the hypotheses of a theorem of Serre ([9], Theorem 3). So we conclude that $H = GL_X$. In particular, ρ_ℓ maps the commutator subgroup of $G_{V_{\ell/C_\ell}}$ onto SL_X. This shows that $\dim S_{V_\ell}$ is at least $p^2 - 1$.

Step 3. The 2-dimensional C_ℓ-torus $T_{E_{\ell/C_\ell}} \simeq G_{m/C_\ell} \times G_{m/C_\ell}$. Let $T_{E_{\ell/C_\ell}} \simeq G_{m/C_\ell} \times G_{m/C_\ell}$ be the 2-dimensional torus $\overline{E}_\ell^*$ over C_ℓ. Recall that $G_{V_\ell}(C_\ell)$ is contained in $\mathrm{Aut}_{\overline{E}_\ell}(\overline{V}_\ell) = GL_X \oplus GL_Y$. Let $\theta\colon G_{V_\ell}(C_\ell) \subseteq \mathrm{Aut}_{\overline{E}_\ell}(\overline{V}_\ell) \xrightarrow{\det} T_{E_{\ell/C_\ell}}$ be the determinant map. Bogomolov ([1], Corollary 1) asserts that $G_{V_{\ell/C_\ell}}$ contains the homotheties G_{m/C_ℓ}. So, the image of θ contains the diagonal of $G_{m/C_\ell} \times G_{m/C_\ell}$. On the other hand, the map $\theta \circ h_{V_\ell}\colon G_{m/C_\ell} \overset{h_{V_\ell}}{\mapsto} G_{V_{\ell/C_\ell}} \xrightarrow{\theta} T_{E_{\ell/C_\ell}}$ gives $(\theta \circ h_{V_\ell})(c) = (c^{n_\tau}, c^{n_\sigma})$ for all c in G_{m/C_ℓ}. Since $n_\sigma \neq n_\tau$, the image of $\theta \circ h_{V_\ell}$ is distinct from the diagonal of $G_{m/C_\ell} \times G_{m/C_\ell}$. It follows that θ is surjective.

So the 2-dimensional torus $T_{E_{\ell/C_\ell}}$ is a quotient of $G_{V_{\ell/C_\ell}}$. We conclude that the centre of G_{V_ℓ} has dimension at least 2.

This completes the proof of Theorem 3.1. ∎

Corollary 3.2. *For all prime dimensional absolutely simple abelian varieties of type IV over number fields,*

$$\mathcal{G}_\ell = \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

Proof: This follows immediately from Theorem 3.1, Theorem 2 in [7], and the result of Taniyama and Shimura in [11] ∎

Corollary 3.3. *The Tate conjecture is true for all prime dimensional absolutely simple abelian varieties of type IV over number fields.*

Proof: After Faltings proved his theorems ([4], Section 5, Satz 3, Satz 4), it is well-known that if $\mathcal{G}_\ell = \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, then the conjectures of Hodge, Tate on algebraic cycles (see [7, 15]) over $A(\mathbb{C})$, A respectively are equivalent. On the other hand, Hodge's conjecture for all prime dimensional absolutely simple abelian varieties of type IV was proved in [14]. ∎

Concluding Remark. Let A be a prime dimensional absolutely simple abelian variety over a number field K. According to Theorem 2 of Section 21 in [6], one has the following possibilities:

Type I. $\dim A$ is a prime number and $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$.

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Type II. $\dim A = 2$ and $\mathrm{End}^\circ(A)$ is an indefinite quaternion algebra over $\mathbb{Q}$.
Type III. $\dim A = 2$ and $\mathrm{End}^\circ(A)$ is a definite quaternion algebra over $\mathbb{Q}$.
Type IV. A is as in Section 1.

In his 1984–85 course at Collège de France, J-P. Serre has proved $\mathcal{G}_\ell \simeq sp(2d, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell$, where $d = \dim A$ is odd and $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$. For $\dim A = 2$ and A of type II, $\mathcal{G}_\ell = \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is well-known (see [**2**], Corollary 4.2). On the other hand, according to a result of Shimura ([**12**], Theorem 5), the above Type III abelian variety of dimension 2 doesn't exist. Taking all of these into account, Corollaries 3.2, 3.3 are true for all prime dimensional absolutely simple abelian varieties over number fields.

References

[1] F.A. Bogomolov, 'Sur l'algébricité des représentations ℓ-adiques', *C.R. Acad. Sc. Paris,* **290** (1980), 701–703.

[2] W. Chi, 'On the ℓ-adic representations attached to some absolutely simple abelian varieties of type II', *J. Fac. Sci. Univ. Tokyo. Sect. IA Math.* (1990) (to appear).

[3] P. Deligne, *Hodge cycles on Abelian varieties (notes by J.S. Milne)*: *Lecture Notes in Math.* 900, pp. 9–100 (Springer-Verlag, Berlin, Heidelberg, New York, 1982).

[4] G. Faltings, 'Endlichkeitssätze fur abelsche varietäten uber zahlkörpern', *Invent. Math.* **73** (1983), 349–366.

[5] D. Mumford, 'Families of abelian varieties', in *Proc. of Symposia in Pure Math.* IX, pp. 347–351 (A.M.S., 1966).

[6] D. Mumford, *Abelian varieties* (Oxford University Press, 1974).

[7] K. Ribet, 'Hodge classes on certain types of abelian varieties', *Amer. J. Math.* **105** (1983), 523–538.

[8] S. Sen, 'Lie algebras of Galois groups arising from Hodge–Tate modules', *Ann. of Math.* **97** (1973), 160–170.

[9] J–P. Serre, 'Sur les groupes de Galois attachés aux groupes p-divisibles', in *Proceedings of a Conference on Local Fields*, pp. 118–131 (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[10] J–P. Serre, 'Groupes algébriques associés aux modules de Hodge–Tate', *Astérisque* **65** (1979), 155–188.

[11] G. Shimura and Y. Taniyama, 'Complex multiplication of abelian varieties and its applications to number theory', *Publ. Math. Soc. Japan* **6** (1961).

[12] G. Shimura, 'On analytic families of polarized abelian varieties and automorphic functions', *Ann. of Math.* **78** (1963), 149–192.

[13] S.G. Tanke'ev, 'Algebraic cycles on simple 5-dimensional abelian varieties', *Math. USSR Izv.* **19** (1982), 95–123.

[14] S.G. Tanke'ev, 'Cycles on simple abelian varieties of prime dimension', *Math. USSR Izv.* **20** (1983), 157–171.

[15] J. Tate, *Algebraic cycles and poles of zeta functions, Arithmetical algebraic geometry*, pp. 93–110 (Harper and Row, New York, 1965).

[16] J. Tate, 'p-divisible groups', in *Proc. of a Conference on Local Fields*, pp. 158–183 (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

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