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ON THE LOCATION OF THE MAXIMUM OF A CONTINUOUS STOCHASTIC PROCESS

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Abstract

In this short article we will provide a sufficient and necessary condition to have uniqueness of the location of the maximum of a stochastic process over an interval. The result will also express the mean value of the location in terms of the derivative of the expectation of the maximum of a linear perturbation of the underlying process. As an application, we will consider a Brownian motion with variable drift. The ideas behind the method of proof will also be useful to study the location of the maximum, over the real line, of a two-sided Brownian motion minus a parabola and of a stationary process minus a parabola.

Keywords: Sample path properties; maxima; argmax; Brownian motion; stationary process; parabolic drift

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1. Introduction

Let $(X(z),\ z \in [s,t])$ be a stochastic process with continuous paths on $[s,t] \subseteq \mathbb{R}$. The maximum of X on $[s,t]$ is defined as

$$M(X) := \max_{z\in[s,t]} X(z),$$

and the set of locations of the maximum (or arg max) is defined as

$$\arg\max(X) := \{z \in [s,t]\colon X(z) = M\}.$$

By continuity, M is well defined and $\arg\max(X)$ is a nonempty compact subset of $[s,t]$. In many situations, we do expect that the maximum is actually attained almost surely (a.s.) at a unique location Z, so that

$$\arg\max(X) = \{Z\} \quad \text{a.s.} \tag{1}$$

In this article, we will prove a sufficient and necessary condition such that (1) holds. The main result is stated below.

Theorem 1. *Let $(X(z),\ z \in [s,t])$ be a stochastic process with continuous paths on $[s,t]$ and assume that $\mathbb{E}|M| < \infty$. For $a \in \mathbb{R}$ let*

$$X^a(z) := X(z) + az, \qquad z \in [s,t],$$

and define

$$M^a := M(X^a) \quad \textit{and} \quad m(a) := \mathbb{E}M^a.$$

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Then $a \mapsto m(a)$ is differentiable at $a = 0$ if and only if the location of the maximum is almost surely unique (i.e. (1) holds). In the latter case we have

$$\mathbb{E}Z = m'(0), \tag{2}$$

where $m'(0)$ is the derivative of m at $a = 0$.

The proof of Theorem 1 is based on a simple nonprobabilistic result (Lemma 1 below), which roughly states that the left- and right-directional derivatives of the functional M, with respect to the identity function, are given by the left-most and the right-most locations of the maximum, respectively. (Note that if $X(z) = 0$ for all $z \in [0, t]$ then $m(a)$ is not differentiable at $a = 0$, since $m(a) = 0$ for $a \leq 0$ and $m(a) = at$ for $a > 0$.)

An example where we can apply Theorem 1 is given by $X = B + f$, where B is a Brownian motion and f is a deterministic continuous function.

Theorem 2. *Let*

$$X(z) = B(z) + f(z), \qquad z \in [0, t],$$

where B is a standard Brownian motion process and f is a deterministic continuous function. Then the location of the maximum is almost surely unique (i.e. (1) holds) and

$$\mathbb{E}Z = \operatorname{cov}(M, B(t)). \tag{3}$$

Theorem 2 implies a similar result when f is a continuous process that is independent of B. An interesting aspect of (3) is that it gives the same result as if Z were independent of B. (If $U \in [0, t]$ is independent of B then $\operatorname{cov}(B(U) + f(U), B(t)) = \operatorname{cov}(B(U), B(t)) = \mathbb{E}U$. Can we understand this behavior of Z, for $f \equiv 0$, in light of Lévy's $M - B$ theorem, or Pitman's $2M - B$ theorem?) Analogous identities have also appeared in particle systems and percolation models; see [1], [2], [5], and [14]. The uniqueness of the location of the maximum for a continuous Gaussian process was proved in [11], and it can certainly be used in our context. The author has tried to compute the derivative of m for a Gaussian process X, in order to provide an alternative proof of uniqueness based on Theorem 1, but with no success so far.

In the previous situation we considered the expectation of the maximum of a linear perturbation of the process X and computed its derivative at zero. Other types of perturbation can also provide useful information about the location of the maximum. For instance, consider $X = B + f$, where now B denotes a standard two-sided Brownian motion, f is again a continuous function, and $z \in [-t, t]$. By taking a perturbation with respect to $z \mapsto z_+ := \max\{z, 0\}$, we see that

$$\mathbb{E}Z_+ = \operatorname{cov}(M, B(t)). \tag{4}$$

The maximum, over the real line, of a two-sided Brownian motion minus a parabola, and its location, arises as a limit object in many different statistical problems. Theorem 1 can be used in this context as well to ensure uniqueness, since a.s. the arg max will be compact (due to the negative parabolic drift). For many examples and various results, see [6]. By symmetry, it is not hard to see that the location has zero mean. The expectation of the maximum and the variance of the location can be expressed in terms of integrals involving the Airy function; see [4], [7], [8], and [9]. Relying on those expressions, Groeneboom [7] and Janson [8] remarked that the variance of the location is equal to one third of the expectation of the maximum. By adding a quadratic perturbation (i.e. az^2), and computing the derivative of the expected maximum, we can directly prove the Groeneboom–Janson relation. We also note that the probability that the maximum over the real line differs from the maximum over $[-t, t]$ decays exponentially fast to zero; thus, (4) can be extended to the limiting behavior.

Theorem 3. *Let*

$$X(z) = B(z) - z^2, \qquad z \in \mathbb{R},$$

where B is a standard two-sided Brownian motion process. Then the location of the maximum is almost surely unique (i.e. (1) holds) and

$$\mathbb{E}Z = 0, \qquad \mathbb{E}Z_+ = \lim_{t\to\infty} \operatorname{cov}(M, B(t)), \qquad \mathbb{E}Z^2 = \tfrac{1}{3}\mathbb{E}M.$$

Another situation where uniqueness can be proved by the same methodology is when X is a stationary process minus a parabola.

Theorem 4. *Let*

$$X(z) = A(z) - z^2, \qquad z \in \mathbb{R},$$

where A is a stationary process with continuous paths. Assume that

$$\mathbb{E}|M| < \infty \quad \text{and} \quad \int_0^\infty \mathbb{P}(\arg\max(X) \not\subseteq [-u, u])\,\mathrm{d}u < \infty. \tag{5}$$

Then the location of the maximum is almost surely unique (i.e. (1) holds) and

$$\mathbb{E}Z = 0. \tag{6}$$

It is surprising that (6) holds for any stationary process minus a parabola. However, as we shall see, the derivative of $m(a)$ can be easily computed in this case. The Airy process [13] is an example where Theorem 4 can be used. It is a one-dimensional stationary process with continuous paths, whose finite-dimensional distributions are described by Fredholm determinants. The interest in this process is mainly due to the fact that it gives the limit fluctuations of a number of processes appearing in statistical mechanics. Under the assumption that the maximum is indeed attained at a unique location, Johansson [10] was able to prove that the law of the location describes the limit transversal fluctuations of maximal paths in last passage percolation models. This assumption was proved to be true by Corwin and Hammond [3] and by Moreno Flores *et al.* [12]. Both proofs used very strong results that depend on particular features of the Airy process. Theorem 4 is an alternative way to get uniqueness.

2. Proofs

2.1. Proof of Theorem 1

Let $h : [s, t] \to \mathbb{R}$ be a continuous real function and let

$$Z_1(h) := \inf \arg\max(h) \quad \text{and} \quad Z_2(h) := \sup \arg\max(h).$$

We start with the analytic counterpart of the proof that is given by Lemma 1, below. It shows that the left- and right-directional derivatives of the functional M, with respect to the identity function, are given by Z_1 and Z_2, respectively.

Lemma 1. *Let*

$$h^a(z) := h(z) + az.$$

Then

$$\lim_{a\to 0^-} Z_1(h^a) = Z_1(h) \quad \text{and} \quad \lim_{a\to 0^+} Z_2(h^a) = Z_2(h). \tag{7}$$

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Furthermore,

$$\lim_{a\to 0^-}\frac{M(h^a)-M(h)}{a}=Z_1(h) \quad \textit{and} \quad \lim_{a\to 0^+}\frac{M(h^a)-M(h)}{a}=Z_2(h). \tag{8}$$

Proof. For simple notation, put $M^a = M(h^a)$ and $Z_i^a = Z_i(h^a)$. By continuity of h, we have

$$M + aZ_i = h(Z_i) + aZ_i \le M^a = h(Z_i^a) + aZ_i^a \le M + aZ_i^a. \tag{9}$$

This implies that

$$0 \le (M^a - M) - aZ_i \le a(Z_i^a - Z_i). \tag{10}$$

The left-hand inequality in (10) is equivalent to

$$0 \le a(Z_i^a - Z_i) - (h(Z_i) - h(Z_i^a)). \tag{11}$$

Since $h(Z_i) \ge h(Z_i^a)$, (11) yields

$$Z_i^a \le Z_i \quad \text{for } a < 0, \qquad \text{and} \qquad Z_i^a \ge Z_i \quad \text{for } a > 0. \tag{12}$$

By (12), if (7) is not true for $i = 1$ then there exist $\delta > 0$ and a sequence $a_n \to 0^-$ such that $Z_1^{a_n} \le Z_1 - \delta$ for all $n \ge 1$. By compactness of $[s, t]$, we can find a subsequence $a_{n_k} \to 0^-$ and $\tilde{Z}_1 \in K$ such that $\tilde{Z}_1 = \lim_{k\to\infty} Z_1^{a_{n_k}} \le Z_1 - \delta$. By (11) (and continuity of h), this implies that $h(\tilde{Z}_1) \ge h(Z_1)$, which leads to a contradiction since Z_1 is the left-most location of the maximum. The proof for $i = 2$ is analogous.

Now, by (10),

$$0 \ge \frac{M^a - M}{a} - Z_1 \ge Z_1^a - Z_1 \ge s - t \quad \text{for } a < 0, \tag{13}$$

and

$$0 \le \frac{M^a - M}{a} - Z_2 \le Z_2^a - Z_2 \le t - s \quad \text{for } a > 0. \tag{14}$$

Together with (7), (13), and (14), this implies (8).

Proof of Theorem 1. Note that (9) implies that $|M^a|$ has finite expectation, since we assume that $\mathbb{E}|M| < \infty$, and $Z_i, Z_i^\epsilon \in [s, t]$. Also, the distance between Z_i and Z_i^a is always bounded by $t - s$. This will be important in the probabilistic counterpart of the proof, in order to use dominated convergence, as follows. If $m(a)$ is differentiable at $a = 0$ then

$$m'(0) = \lim_{a\to 0^-}\frac{m(a)-m(0)}{a} = \lim_{a\to 0^+}\frac{m(a)-m(0)}{a}.$$

Together with (8), and dominated convergence, this proves that $\mathbb{E}Z_1 = \mathbb{E}Z_2$. Since $Z_1 \le Z_2$, we must have $Z_1{=}Z_2$ a.s., which yields (1) and (2). Reciprocally, if (1) is true then, by Lemma 1,

$$\lim_{a\to 0^-}\frac{M^a - M}{a} = \lim_{a\to 0^+}\frac{M^a - M}{a} \quad \text{a.s.}$$

Thus, dominated convergence implies that

$$\lim_{a\to 0^-}\frac{m(a)-m(0)}{a} = \lim_{a\to 0^+}\frac{m(a)-m(0)}{a},$$

which shows that $m(a)$ is differentiable at $a = 0$, and the proof is complete.

2.2. Proof of Theorem 2

In Lemma 2 below we take $X = B + f$ and compute the derivative of $m(a)$ in a different way. This derivative can be computed by using the Cameron–Martin theorem. For simplicity, we will present an alternative proof requiring only basic knowledge of Brownian motion.

Lemma 2. *Let $Y = Y(B)$ be a (measurable) functional of a standard Brownian motion B on $[0, t]$ satisfying $\mathbb{E}Y^2 < \infty$. Define*

$$y(a) := \mathbb{E}Y^a, \quad \text{where } Y^a := Y(B^a),$$

and $B^a(z) := az + B(z)$. Assume that $y(\cdot)$ is well defined in a neighborhood of $a = 0$. Then

$$y'(0) = \operatorname{cov}(Y, B(t)). \tag{15}$$

Proof. Without loss of generality, we assume that $t = 1$. The Brownian motion can be decomposed into

$$B(z) \overset{\mathrm{D}}{=} Nz + B_0(z)$$

(as processes), where B_0 is a standard Brownian bridge with $B_0(0) = B_0(1) = 0$, and N is an independent Normal random variable with mean 0 and variance 1. Thus,

$$B^a(z) = az + B(z) \overset{\mathrm{D}}{=} (a + N)z + B_0(z).$$

Since $B^a(1) = u$ if and only if $a + N = u$, we have

$$B^a(z) \overset{\mathrm{D}}{=} uz + B_0(z), \tag{16}$$

conditioned on the event that $B^a(1) = u$. By (16), the conditional expectation of Y^a, given that $B^a(1) = u$, does not depend on $a \in \mathbb{R}$. Specifically, denote by B_u the process on the right-hand side of (16). Then we obtain

$$\mathbb{E}(Y^a \mid B^a(1) = u) = \mathbb{E}(Y(B_u)).$$

Therefore, by writing

$$\rho_u(a) := \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(u-a)^2}{2}\right\},$$

we have

$$y(a) = \int \mathbb{E}(Y^a \mid B^a(1) = u)\rho_u(a)\,\mathrm{d}u = \int \mathbb{E}(Y(B_u))\rho_u(a)\,\mathrm{d}u.$$

Hence, by interchanging the derivative with the integral, we obtain

$$\begin{aligned} y'(a) &= \int \mathbb{E}(Y^a \mid B^a(1) = u)\rho_u'(a)\,\mathrm{d}u \\ &= \int \mathbb{E}(Y^a \mid B^a(1) = u)(u - a)\rho_u(a)\,\mathrm{d}u \\ &= \mathbb{E}(Y^a B^a(1)) - a\mathbb{E}(Y^a), \end{aligned}$$

which proves (15).

Proof of Theorem 2. Take $Y(B) := M(B + f)$. By Lemma 2,

$$m'(0) = \operatorname{cov}(M, B(t));$$

hence, Theorem 1 implies Theorem 2.

Proof of (4). Put $X^{a,+}(z) := X(z) + az_+$ for $z \in \mathbb{R}$, and $M^{a,+} = M(X^{a,+})$. Then

$$\lim_{a\to 0^+} \frac{M^{a,+} - M}{a} = Z_+,$$

which implies that

$$\lim_{a\to 0^+} \frac{m^+(a) - m^+(0)}{a} = \mathbb{E}Z_+,$$

where $m^+(a) = \mathbb{E}M^{a,+}$. By conditioning on $B(t)+at$, this derivative is equal to $\operatorname{cov}(M, B(t))$, which shows that (4) holds.

Remark 1. Given a square integrable function ϕ on $[0, t]$, define the function ψ on $[0, t]$ by

$$\psi(z) := \int_0^z \phi(u)\,\mathrm{d}u.$$

By the Cameron–Martin theorem, if $Y = Y(B)$ is a (measurable) functional of standard Brownian motion B on $[0, t]$ satisfying $\mathbb{E}Y^2 < \infty$ then

$$\lim_{a\to 0} \frac{\mathbb{E}Y(B + a\psi) - \mathbb{E}Y}{a} = \mathbb{E}\biggl(Y \int_0^t \phi(z)\,\mathrm{d}B(z)\biggr).$$

If ψ is increasing, then the same reasoning that we used to prove Lemma 1 yields

$$\lim_{a\to 0} \frac{M(h^{a,\psi}) - M(h)}{a} = \psi(Z),$$

where $h^{a,\psi}(z) := h(a) + a\psi(z)$. Therefore,

$$\mathbb{E}\psi(Z) = \mathbb{E}\biggl(M \int_0^t \phi(u)\,\mathrm{d}B(u)\biggr).$$

By the chain rule, we also have

$$\mathbb{E}(H'(M)Z) = \mathbb{E}(H(M)B(t)).$$

2.3. Proof of Theorem 3

As we mentioned before, in this case (1) can be obtained from Theorem 2 by using a.s. compactness of the arg max, and $\mathbb{E}Z = 0$ follows easily by symmetry. The uniqueness also follows from [11]. For the sake of completeness, we also present an alternative proof which uses similar ideas as before. We start with a key lemma, which contains well-known facts; see, for instance, [6].

Lemma 3. *Let B be a two-sided Brownian motion, and for $\beta \in \mathbb{R}$ define*

$$M(\beta) = M(B, \beta) := \max_{z\in\mathbb{R}}\{B(z) - (z-\beta)^2\}.$$

Let $Z_1(\beta) = Z_1(B, \beta)$ and $Z_2(\beta) = Z_2(B, \beta)$ denote the left-most and right-most locations of the maximum of $B(z) - (z-\beta)^2$, respectively. Then

$$\mathbb{E}M(\beta) = \mathbb{E}M(0) \quad \textit{and} \quad \mathbb{E}Z_i(\beta) = \beta + \mathbb{E}Z_i(0).$$

Proof. By the almost-sure compactness of the arg max, $Z_i(\beta)$ is well defined for $i = 1, 2$. Note that $\bar{B} \stackrel{\mathrm{D}}{=} B$, where $\bar{B}(x) := B(x + \beta) - B(\beta)$ for $x \in \mathbb{R}$; thus, taking $x = z - \beta$, we obtain

$$\max_{z\in\mathbb{R}}\{B(z) - (z-\beta)^2\} = \max_{x\in\mathbb{R}}\{\bar{B}(x) - x^2\} + B(\beta).$$

This implies that

$$M(B, \beta) = M(\bar{B}, 0) + B(\beta) \quad \text{and} \quad Z_i(B, \beta) - \beta = Z_i(\bar{B}, 0),$$

which proves the lemma. (Notice that the arg max does not change by summing $B(\beta)$.)

Proof of Theorem 3. By Lemma 3,

$$m(a) = \mathbb{E}\max_{z\in\mathbb{R}}\{B(z) - z^2 + az\} = \mathbb{E}\max_{z\in\mathbb{R}}\left\{B(z) - \left(z - \frac{a}{2}\right)^2\right\} + \frac{a^2}{4} = m(0) + \frac{a^2}{4};$$

hence,

$$m'(0) = 0. \tag{17}$$

Since the arg max does not change by a vertical shifting of $a^2/4$, by Lemma 3 we have

$$\mathbb{E}Z_i^a = \mathbb{E}Z_i\left(\frac{a}{2}\right) = \frac{a}{2} + \mathbb{E}Z_i,$$

which shows that

$$\lim_{a\to 0}\mathbb{E}Z_i^a = \mathbb{E}Z_i. \tag{18}$$

On the other hand, by (10) we obtain

$$0 \le \frac{M^a - M}{a} - Z_i \le Z_i^a - Z_i \quad \text{for } a > 0,$$

and

$$0 \ge \frac{M^a - M}{a} - Z_i \ge Z_i^a - Z_i \quad \text{for } a < 0.$$

Together with (18), these inequalities yield

$$\mathbb{E}Z_i = m'(0);$$

hence, $\mathbb{E}Z_1 = \mathbb{E}Z_2$. Since $Z_1 \le Z_2$, we have $Z_1 = Z_2$ a.s., which proves (1). By (17), $\mathbb{E}Z = 0$. To compute the limiting value of $\mathbb{E}Z_+$, use (4).

To evaluate the second moment of Z, we add a quadratic perturbation to our original process and compute the derivative with respect to that. We follow the same notation as in [8] and set $X_\gamma(z) := B(z) - \gamma z^2$ for $z \in \mathbb{R}$, $M_\gamma := M(X_\gamma)$, and $V_\gamma := \arg\max(X_\gamma)$. (If the maximum is reached at a single value, then we refer to the point as the arg max.) Note that

$$M_{1-a} = \max_{z\in\mathbb{R}}\{B(z) - z^2 + az^2\}.$$

By scaling invariance of Brownian motion, we have

$$M_\gamma \stackrel{\mathrm{D}}{=} \gamma_1^{1/3}\gamma^{-1/3}M_{\gamma_1} \quad \text{and} \quad V_\gamma \stackrel{\mathrm{D}}{=} \gamma_1^{2/3}\gamma^{-2/3}V_{\gamma_1}. \tag{19}$$

Therefore,

$$n(a) := \mathbb{E}M_{1-a} = (1-a)^{-1/3} n(0);$$

thus,

$$n'(0) = \frac{n(0)}{3}. \tag{20}$$

On the other hand, as in the proof of (10),

$$0 \le (M_{1-a} - M_1) - aV_1^2 \le a(V_{1-a}^2 - V_1^2),$$

which implies that

$$0 \le \frac{M_{1-a} - M_1}{a} - V_1^2 \le V_{1-a}^2 - V_1^2 \quad \text{for } a > 0,$$

and that

$$0 \ge \frac{M_{1-a} - M_1}{a} - V_1^2 \ge V_{1-a}^2 - V_1^2 \quad \text{for } a < 0.$$

By taking expectations on both sides of the last inequalities, and then using (19), we have

$$\left| \frac{n(a) - n(0)}{a} - v(0) \right| \le |v(a) - v(0)| = (1-a)^{-4/3} |v(0)|,$$

where $v(a) = \mathbb{E}V_{1-a}^2$. Hence,

$$n'(0) = v(0).$$

Together with (20), this shows that

$$\mathbb{E}Z^2 = v(0) = n'(0) = \frac{n(0)}{3} = \frac{\mathbb{E}M}{3}.$$

We note that, by (19), this also shows that $\mathbb{E}V_\gamma^2 = (3\gamma)^{-1}\mathbb{E}M_\gamma$.

2.4. Proof of Theorem 4

Lemma 4. *Let A be a stationary process and, for $\beta \in \mathbb{R}$, let*

$$M(\beta) = M(A, \beta) := \max_{s \in \mathbb{R}} \{A(s) - (s - \beta)^2\}.$$

Let $Z_1(\beta) = Z_1(\beta)$ and $Z_2(\beta) = Z_2(\beta)$ denote the left-most and right-most locations of the maximum of $A(z) - (z - \beta)^2$, respectively. Then, for each fixed $\beta \in \mathbb{R}$,

$$M(\beta) \stackrel{\mathrm{D}}{=} M(0) \quad \textit{and} \quad Z_i(\beta) - \beta \stackrel{\mathrm{D}}{=} Z_i(0).$$

Proof. By stationarity, $\bar{A} \stackrel{\mathrm{D}}{=} A$, where $\bar{A}(x) := A(x + \beta)$. On the other hand,

$$M(A, \beta) = \max_{z \in \mathbb{R}} \{A(z) - (z - \beta)^2\} = M(\bar{A}, 0) \quad \text{and} \quad Z_i(A, \beta) - \beta = Z_i(\bar{A}, 0)$$

(take $x = z - a$), which proves the lemma.

Proof of Theorem 4. By Lemma 4,

$$m(a) = \mathbb{E}\max_{z\in\mathbb{R}}\{A(z) - z^2 + az\} = \mathbb{E}\max_{z\in\mathbb{R}}\left\{A(z) - \left(z - \frac{a}{2}\right)^2\right\} + \frac{a^2}{4} = m(0) + \frac{a^2}{4};$$

hence,

$$m'(0) = 0. \tag{21}$$

By (5), we have $\mathbb{E}|Z_i| < \infty$. Since the arg max does not change by a vertical shifting of $a^2/4$, by Lemma 4 we have

$$\mathbb{E}Z_i^a = \mathbb{E}Z_i\left(\frac{a}{2}\right) = \frac{a}{2} + \mathbb{E}Z_i,$$

which shows that

$$\lim_{a\to 0}\mathbb{E}Z_i^a = \mathbb{E}Z_i. \tag{22}$$

On the other hand, by (10) we obtain

$$0 \le \frac{M^a - M}{a} - Z_i \le Z_i^a - Z_i \quad \text{for } a > 0,$$

and

$$0 \ge \frac{M^a - M}{a} - Z_i \ge Z_i^a - Z_i \quad \text{for } a < 0.$$

Together with (22), these inequalities yield

$$\mathbb{E}Z_i = m'(0).$$

Thus, $\mathbb{E}Z_1 = \mathbb{E}Z_2$. Since $Z_1 \le Z_2$, we have $Z_1 = Z_2$ a.s., which proves (1). By (21), $\mathbb{E}Z = 0$.

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References

[1] Balázs, M., Cator, E. and Seppäläinen, T. (2006). Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Prob.* **11,** 1094–1132.

[2] Cator, E. and Groeneboom, P. (2006). Second class particles and cube root asymptotics for Hammersley's process. *Ann. Prob.* **34,** 1273–1295.

[3] Corwin, I. and Hammond, A. (2014). Brownian Gibbs property for Airy line ensembles. *Invent. Math.* **195,** 441–508.

[4] Daniels, H. E. and Skyrme, T. H. R. (1985). The maximum of a random walk whose mean path has a maximum. *Adv. Appl. Prob.* **17,** 85–99, 475.

[5] Ferrari, P. A. and Fontes, L. R. G. (1994). Current fluctuations for the asymmetric simple exclusion process. *Ann. Prob.* **22,** 820–832.

[6] Groeneboom, P. (1989). Brownian motion with a parabolic drift and Airy functions. *Prob. Theory Relat. Fields* **81,** 79–109.

[7] GROENEBOOM, P. (2011). Vertices of the least concave majorant of Brownian motion with parabolic drift. *Electron. J. Prob.* **16,** 2234–2258.

[8] JANSON, S. (2013). Moments of the location of the maximum of Brownian motion with parabolic drift. *Electron. Commun. Prob.* **18,** 8pp.

[9] JANSON, S., LOUCHARD, G. AND MARTIN-LÖF, A. (2010). The maximum of Brownian motion with parabolic drift. *Electron. J. Prob.* **15,** 1893–1929.

[10] JOHANSSON, K. (2003). Discrete polynuclear growth and determinantal processes. *Commun. Math. Phys.* **242,** 277–329.

[11] KIM, J. AND POLLARD, D. (1990). Cube root asymptotics. *Ann. Statist.* **18,** 191–219.

[12] MORENO FLORES, G., QUASTEL, J. AND REMENIK, D. (2013). Endpoint distribution of directed polymers in 1+1 dimensions. *Commun. Math. Phys.* **317,** 363–380.

[13] PRÄHOFER, M. AND SPOHN, H. (2002). Scale invariance of the PNG droplet and the Airy process. *J. Statist. Phys.* **108,** 1071–1106.

[14] SEPPÄLÄINEN, T. (2012). Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Prob.* **40,** 19–73.