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THE PROFILE NEAR QUENCHING TIME FOR THE SOLUTION OF A SINGULAR SEMILINEAR HEAT EQUATION*

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We study the profile near quenching time for the solutions of the first and second initial boundary value problems (IBVP) for a semilinear heat equation. Under certain conditions, one-point quenching occurs for both first and second IBVPs. Furthermore, we derive the asymptotic self-similar quenching rate for both problems.

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1. Introduction

In this paper we consider the following semilinear heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{b}{x}\frac{\partial u}{\partial x} + \frac{1}{(1-u)^\beta} \qquad \text{for } x \in (0,a),\ t > 0, \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x) \qquad \text{for } x \in (0,a), \tag{1.2}$$

where $a > 0$ is a constant. Throughout this paper, we assume that

$$b < 1, \quad \beta > 0, \quad \text{and } 0 \le u_0(x) \le 1 - c_0 \quad (0 \le x \le 1) \tag{1.3}$$

for some $c_0 > 0$.

Denote the operator $\mathcal{L}$ by

$$\mathcal{L}[u] = u_t - u_{xx} - \frac{b}{x}u_x.$$

Then the adjoint operator of $\mathcal{L}[u] = 0$ is given by

$$v_t = v_{xx} - \left(\frac{b}{x}v\right)_x.$$

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The above equation arises in many applications in stochastic processes. For example, the function v is the density function for a Markov process which is the limit of a sequence of random walks. Here b is some limit of the second conditional moment (cf. [17]). It is the Fokker–Planck equation of a singular diffusion in which b is related to the drift (cf. [7]). Also, $\mathcal{L}u = 0$ is equivalent to the following degenerate elliptic-parabolic equation

$$w_t = zw_{zz} + \frac{b+1}{2} w_z$$

introduced by Fichera (cf. [8]), through the transformation

$$z = \frac{x^2}{4}, \; w(z, t) = u(x, t).$$

For more references about the operator $\mathcal{L}$, we refer the reader to the paper of Alexiades [1].

The system (1.1)–(1.2) is supplemented with either the Dirichlet boundary conditions

$$u(0, t) = u(a, t) = 0 \qquad \text{for } t > 0, \tag{1.4}$$

or the Dirichlet and Neumann boundary conditions

$$u(0, t) = 0, \; \frac{\partial u}{\partial x}(a, t) = 0 \quad \text{for } t > 0. \tag{1.5}$$

Throughout this paper, we shall refer to the system (1.1)–(1.2), (1.4) as the First IBVP, and the system (1.1)–(1.2), (1.5) as the Second IBVP.

We shall assume that the initial datum $u_0(x)$ is smooth, say, $C^3[0, a]$, and satisfies

$$u_0''(x) + \frac{b}{x} u_0'(x) + \frac{1}{(1 - u_0(x))^\beta} \geq 0 \qquad \text{for } x \in (0, a). \tag{1.6}$$

For the First IBVP, we assume that there exists $c^* \in (0, a)$ such that

$$u_0(0) = 0, \quad u_0(a) = 0, \tag{1.7}$$

$$u_0'(x) \geq 0 \qquad \text{for } 0 \leq x < c^*, \quad u_0'(x) \leq 0 \qquad \text{for } c^* < x \leq a. \tag{1.8}$$

For the Second IBVP, we assume that

$$u_0(0) = 0, \quad u_0'(a) = 0, \tag{1.9}$$

$$u_0'(x) \geq 0 \qquad \text{for } 0 \leq x \leq a. \tag{1.10}$$

The solution to either First or Second IBVP is unique (see **[1]**). We say that a solution quenches if its maximum reaches 1 at some finite time. Let T be the quenching time. A point c is said to be a quenching point if there is a sequence $\{(x_n, t_n)\}$ such that $x_n \to c$, $t_n \to T$, and $u(x_n, t_n) \to 1$ as $n \to \infty$.

The quenching problem for parabolic equations has been studied by many authors, since the work of Kawarada **[15]** in 1975. We refer the reader to the survey papers Chan **[3, 4]** and Levine **[18, 19]** for more references.

The quenching may or may not occur, depending on the length a and the initial datum. Throughout this paper, we assume that the quenching occurs at $t = T$. Under the assumptions (1.6), (1.7) or (1.9), we have

$$u_t(x, t) > 0 \qquad \text{for } 0 < x < a, 0 < t < T, \tag{1.11}$$

for either First or Second IBVP. We state our results in terms of the following theorems.

Theorem 1.1 (First IBVP). *Let the assumptions* (1.6), (1.7) *and* (1.8) *be in force. Then we have the following:*

(i) *There is exactly one quenching point.*

(ii) *If the quenching point $x = c$ is away from the boundary $x = 0$ and $x = a$, then we have*

$$\lim_{t \to T} [1 - u(x, t)](T - t)^{-\gamma} = (\beta + 1)^{\gamma}, \qquad \gamma = \frac{1}{\beta + 1},$$

uniformly for $|x - c| \le C\sqrt{T - t}$ for any positive constant C.

(iii) *In particular, for $0 < \beta < 1$ the quenching point is away from the boundary $x = 0$ and $x = a$ and therefore* (ii) *holds in this case.*

Theorem 1.2 (Second IBVP). *Under the assumptions* (1.6), (1.9) *and* (1.10), *there is exactly one quenching point which occurs necessarily at $x = a$. Moreover,*

$$\lim_{t \to T} [1 - u(x, t)](T - t)^{-\gamma} = (\beta + 1)^{\gamma}, \qquad \gamma = \frac{1}{\beta + 1},$$

uniformly for $0 \le a - x \le C\sqrt{T - t}$ for any positive constant C.

The new feature of our system is the presence of the convection term involving the coefficient b/x. It is degenerate at $x = 0$, and the presence of this term invalidates any direct reflection argument used in **[10]** for zero-set analysis for u_x. We shall prove, under certain assumptions, that quenching will not occur at the boundary $x = 0$, and therefore the coefficient b/x is not degenerate at the quenching point. To prove the single-point-quenching for the First IBVP, we modify the reflection technique, and use a delicate argument in Lemmas 2.2 and 2.3. All of the modifications are necessary to take care of the *asymmetric* term introduced by reflecting the b/x term.

We shall give the proof of Theorem 1.1 in Sections 2 and 3. The proof of Theorem 1.2 will be given in Section 4.

2. The first IBVP

Let $Q_T = (0, a) \times (0, T)$. The following lemma is crucial for studying the quenching set. Since no boundary condition is needed in the lemma, it is valid for both the First and Second IBVPs.

Lemma A. *Suppose that* $u_x > 0$ (*or* $u_x < 0$) *in the set* $Q = (d, e) \times (t_0, T)$, *where* $0 \le d < e \le a$ *and* $0 < t_0 < T$. *Suppose also that* $u_t > 0$ *in the set* Q. *Then any point* c *in* $[d, e)$ (*or* $(d, e]$, *respectively*) *cannot be a quenching point.*

Proof. We only consider the case that $u_x > 0$ in Q. Suppose that there is a quenching point $c \in [d, e)$. Recall that $u_t > 0$ in Q_T. Then we have

$$\lim_{t \uparrow T} u(x, t) = 1 \tag{2.12}$$

uniformly for x in compact subset of (c, e). Take any l, m so that $c < l < m < e$. We consider the function

$$J(x, t) = u_x(x, t) - \eta h(x)$$

in $S = (l, m) \times (t_1, T)$, where the function $h(x)$ is given by

$$h(x) = \begin{cases} (x - l)^2, & l \le x \le l + \delta \\ k(x), & l + \delta \le x \le m - \delta \\ (m - x)^2, & m - \delta \le x \le m, \end{cases}$$

with $\eta > 0$, $\delta > 0$, and $t_1 \in (t_0, T)$ to be determined. Here the function $k(x)$ is chosen so that $h \in C^2(l, m)$. We compute that

$$J_t - J_{xx} - \frac{b}{x} J_x - \left[\beta(1 - u)^{-\beta-1} - \frac{b}{x^2}\right] J = \eta P, \tag{2.13}$$

where

$$P = \left(h'' + \frac{b}{x} h'\right) + \left[\beta(1 - u)^{-\beta-1} - \frac{b}{x^2}\right] h. \tag{2.14}$$

We claim that $P \ge 0$ in S.

For $l \le x \le l+\delta$, we have

$$P = 2\left[1 + b\,\frac{x-l}{x}\right] + \left[\beta(1-u)^{-\beta-1} - \frac{b}{x^2}\right](x-l)^2.$$

For $b > 0$, it follows from (2.12) that $P \ge 0$ in S if t_1 is chosen sufficiently close to T. For $b < 0$, since $(x-l)/x \le \delta/l$, it follows that $P \ge 0$ in S if δ is chosen sufficiently small.

Similarly, for $m-\delta \le x \le m$, we have

$$P = 2\left[1 - b\,\frac{m-x}{x}\right] + \left[\beta(1-u)^{-\beta-1} - \frac{b}{x^2}\right](m-x)^2.$$

It is trivial that $P \ge 0$ in S if $b < 0$. For $0 < b < 1$, we also have $P \ge 0$ in S if δ is chosen so that $\delta \le m - \delta$ and t_1 is chosen sufficiently close to T.

Finally, for $l+\delta \le x \le m-\delta$, we note that the quantities k'/k and k''/k are bounded. Hence

$$P = k\left[\beta(1-u)^{-\beta-1} - \frac{b}{x^2} + \frac{k''}{k} + \frac{b}{x}\frac{k'}{k}\right]$$

is non-negative if t_1 is chosen sufficiently close to T.

Fix t_1, we can choose η so small that $J \ge 0$ on the parabolic boundary of S. Hence it follows from the maximum principle that $J \ge 0$ in S, i.e.,

$$u_x(x,t) \ge \eta h(x) \text{ in } S.$$

Integrating this inequality from l to m, we reach a contradiction and the lemma follows. □

We begin the study of the First IBVP with a stronger assumption in place of (1.8), namely,

$$u_0'(x) > 0 \quad \text{for } 0 < x < c^*, \qquad u_0'(x) < 0 \quad \text{for } c^* < x \le a. \tag{2.15}$$

This assumption will be removed at the end of this section.

Lemma 2.1. *Let* (2.15) *be in force. Then for each* $t \in [0, T)$, *there is exactly one point* $x = s(t)$, *such that* $u_x(s(t), t) = 0$. *Furthermore,* $s \in C^\infty(0, T)$.

Proof. For each fixed $t \in [0, T)$, $u(\cdot, t)$ attains its maximum at an interior point $x = s(t)$, on which we must have $u_x(s(t), t) = 0$.

We now prove that there is at most one point on which $u_x = 0$. Notice that this is certainly true for $t = 0$. Take $x = c^*/2$ so that $u_x(c^*/2, 0) > 0$. By continuity, $u_x(c^*/2, t) > 0$

for $0 \le t \le \delta$. Notice that u_x satisfies the equation

$$(u_x)_t = (u_x)_{xx} + \frac{b}{x}(u_x)_x - \frac{b}{x^2}u_x + \frac{\beta u_x}{(1-u)^{\beta+1}}.$$

by the maximum principle,

$$u_x(x,t) > 0 \qquad \text{for } (x,t) \in (0, c^*/2) \times (0,\delta).$$

(Notice that the equation is singular at $x = 0$; however, we can always approximate this system with a nonsingular system and then apply the maximum principle.) Furthermore, the number of zeros of u_x will not increase in the interval $x \in [c^*/2, a]$ for $t \in [0,\delta)$ (see, for example, **[2, 6]**). Since the number of zero of u_x remains 1 for $t \in [0,\delta)$, it follows (see **[2, 6]**) that

$$u_{xx}(s(t),t) \neq 0 \qquad \text{for } 0 \le t < \delta,$$

(if it equals 0 for some $t \in (0,\delta)$, then the number of zeros of u_x must decrease by 1, which is not the case here) where $x = s(t)$ is the zero of u_x, which coincides with the maximum of $u(\cdot,t)$. Since the solution to the parabolic PDE is C^∞, we have proved that $s \in C^\infty(0,\delta)$. It is clear that this process can be continued beyond $t = \delta$. Now let $(0, T^*)$ be the maximal interval on which the lemma is valid, then $T^* \ge \delta$. It is clear that

$$u_x(x,t) > 0 \qquad \text{for } 0 < x < s(t), 0 < t < T^*$$

$$u_x(x,t) < 0 \qquad \text{for } s(t) < x < a, 0 < t < T^*.$$

We claim that $T^* = T$. Suppose that this is not true, then the previous procedure can be continued beyond $t = T^*$, provided

$$\liminf_{t \to T^*-0} s(t) > 0. \tag{2.16}$$

Thus, in order to establish the lemma for all $0 < t < T$, it suffices to establish a lower bound for $s(t)$.

For any small $\eta > 0$, we let $v(x) = Kx^{2\alpha}$, where $0 < \alpha < \min\left(\frac{1}{2}, \frac{1-b}{2}\right)$ and K is a constant. Since $u(x,t)$ is bounded for $0 \le t \le T - \eta$ (although the bound may depend on η), we can take $K = K_\eta$ to be large enough so that

$$K\eta^{2\alpha} \ge 1, \qquad u_0(x) \le v(x),$$

$$(1-u)^{-\beta} \le 2\alpha K \eta^{2\alpha-2}(1-b-2\alpha) \qquad \text{for } (x,t) \in (0,\eta) \times (0, T-\eta).$$

Then $v(x)$ is a supersolution in $(0,\eta) \times (0, T-\eta)$ and by the maximum principle

$$u(x,t) \le Kx^{2\alpha} \qquad \text{for } (x,t) \in (0,\eta) \times (0, T-\eta).$$

It follows that

$$K[s(t)]^{2\alpha} > u(s(t), t) = \max_{x \in [0,a]} u(x, t) \geq u\left(\frac{a}{2}, t\right) \geq u\left(\frac{a}{2}, \delta\right) \qquad \text{for } 0 < t < T - \eta.$$

as long as $x = s(t)$ is still well defined. This lower bound for $s(t)$, together with the continuation argument, implies that $T^* \geq T - \eta$, for any $\eta > 0$. Since η is arbitrary, the lemma holds. □

Recall that

$$u_x(x, t) > 0 \qquad \text{for } 0 < x < s(t), 0 < t < T,$$

$$u_x(x, t) < 0 \qquad \text{for } s(t) < x < a, 0 < t < T.$$

The main lemma of this section is the following.

Lemma 2.2. *Let* (2.15) *be in force. Then the limit of* $s(t)$ *as* $t \to T - 0$ *exists.*

Proof. If this is not true, then

$$0 \leq a_1 = \liminf_{t \to T-0} s(t) < a_2 = \limsup_{t \to T-0} s(t) \leq a. \tag{2.17}$$

We first prove that any $x^* \in [a_1, a_2]$ is a quenching point. In fact, for each $x^* \in (a_1, a_2)$, the curve $x = s(t)$ intersects with $x = x^*$ infinitely many times. On the intersection sequence, $u(x^*, t_j) = u(s(t_j), t_j)$ takes its maximum, and hence converges to 1 as $t_j \to T - 0$. Since $u_t > 0$, we conclude that

$$\lim_{t \to T-0} u(x^*, t) = 1 \qquad \text{for } x^* \in (a_1, a_2).$$

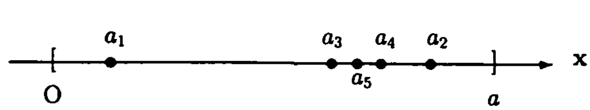
We now take x^* to be $a_1 + \varepsilon$ and $a_2 - \varepsilon$. It is clear that $u(x, t) \geq \min[u(a_1 + \varepsilon, t), u(a_2 - \varepsilon, t)]$. Hence,

$$\lim_{t \to T-0} u(x, t) = 1$$

uniformly for $x \in [a_1 + \varepsilon, a_2 - \varepsilon]$, for any $\varepsilon > 0$. Moreover, by Lemma A, any point in $[0, a] \backslash [a_1, a_2]$ is not a quenching point.

Now take $a_3 = \dfrac{a_1}{3} + \dfrac{2a_2}{3}$, $a_4 = (a_3 + a_2)/2$ and $a_5 = (a_3 + a_4)/2$, then

$$0 \leq a_1 < a_3 < a_5 < a_4 < a_2 \leq a.$$



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We take $t_j \to T-0$ so that $s(t_j) < a_1 + \varepsilon$ for a small $\varepsilon > 0$. Consider the function

$$w(x,t) = u(x,t) - u(2a(t)-x,t) \qquad \text{for } b(t) < x < a(t),\, t_j < t < T, \tag{2.18}$$

where $b(t) = \max(2a(t)-a, a_5-(a_2-a_1)/2)$, and $a(t) = a_5 + K(t-t_j)$. It is clear that

$$w_t - w_{xx} - \frac{b}{x}w_x + q(x,t)w = -u_x(2a(t)-x,t)\left[\frac{b}{x} + \frac{b}{2a(t)-x} + 2K\right]. \tag{2.19}$$

We take K to be large enough so that $\left[\frac{b}{x} + \frac{b}{2a(t)-x} + 2K\right] > 0$ in $\{b(t) < x < a(t),\, t_j < t < T\}$. We then take t_j to be sufficiently close to T so that

$$a_3 < a(t) < a_4 \qquad \text{for } t_j < t < T.$$

Clearly, $w = 0$ on $x = a(t)$.

Since $b(t_j) > a_1 + \varepsilon > s(t_j)$, we have $w_x(x,t_j) = u_x(x,t_j) + u_x(2a(t_j)-x,t_j) < 0$ for $b(t_j) < x < a(t_j)$ and hence $w(x,t_j) \geq w(a(t_j),t_j) = 0$ for $x \in (b(t_j), a(t_j))$.

Finally, on $\{x = b(t)\}$, if $b(t) = 2a(t) - a$, then clearly $w(b(t),t) > 0$. On the other hand if $b(t) = a_5 - (a_2-a_1)/2$, then $x = b(t)$ (for all $t_j < t < T$) are quenching points of $u(x,t)$ and therefore $u(b(t),t)$ is uniformly close to 1. On the other hand $1 - u(2a(t)-b(t),t)$ remains uniformly away from 0. Thus we conclude that $w(b(t),t) > 0$, for all $t_j < t < T$.

Since $a_3 < a(t) < a_4$, the curve $x = s(t)$ must intersect the line segment $x = a(t)$ infinitely many times. Let t^* be the first time of such an intersection, i.e.,

$$s(t) < a(t) \qquad \text{for } t_j \leq t < t^*, \qquad s(t^*) = a(t^*).$$

Then by definition of $s(t)$, we have $u_x(2a(t)-x,t) < 0$, for $b(t) < x < a(t)$, $t_j \leq t \leq t^*$. It follows by the maximum principle that $w > 0$ for $b(t) < x < a(t)$, $t_j \leq t \leq t^*$. The strong maximum principle implies that $w_x(a(t^*),t^*) < 0$, which is a contradiction. $\square$

We have proved that the limit

$$s^* \equiv \lim_{t \to T-0} s(t)$$

exists. By Lemma A, any point other than s^* is not a quenching point. Therefore, we have proved

Lemma 2.3. *There is a single point quenching which occurs at $x = s^*$.*

The first order term $\frac{b}{x}u_x$ represents convection. Therefore the following result is natural. We include this result here although it will not be used in the proof of Theorem 1.1.

Proposition 2.4. *If $b \geq 0$, then $x = a$ is not a quenching point. If $b \leq 0$, then $x = 0$ is not a quenching point.*

Proof. We consider two different cases.

Case 1. $b \geq 0$.
In this case, we claim that

$$s(t) \leq \frac{s(T/2)+a}{2} \qquad \text{for } \frac{T}{2} < t < T. \tag{2.20}$$

Thus Lemma 2.3 implies that $x = a$ is not a quenching point.

Let

$$d = \frac{s(T/2)+a}{2},$$

$$w(x,t) = u(x,t) - u(2d-x,t) \qquad \text{for } 2d-a < x < d, \quad \frac{T}{2} < t < T.$$

Since $d = (s(T/2)+a)/2$, we have $2d - a = s(T/2)$. As in the proof of Lemma 2.2, we have $w(x,t) \geq 0$ on the parabolic boundary of the domain $G \equiv \{2d-a < x < d, \ T/2 < t < T\}$.

Clearly,

$$w_t - w_{xx} - \frac{b}{x}w_x + q(x,t)w = -u_x(2d-x,t)\left[\frac{b}{x} + \frac{b}{2d-x}\right] \text{ in } G. \tag{2.21}$$

The right-hand side of the above inequality is non-negative if $2d - x \geq s(t)$, which is valid if $d \geq s(t)$. By our definition of d, we have $d > s(T/2)$. If the estimate $d > s(t)$ is not valid for all $T/2 < t < T$, then there must be a $t^* \in (T/2, T)$ such that

$$s(t) < d \qquad \text{for } \frac{T}{2} \leq t < t^*, \qquad s(t^*) = d.$$

It follows from the maximum principle that $w \geq 0$ in $G \cap \{t \leq t^*\}$. The strong maximum principle implies that $w_x(d, t^*) < 0$, which is a contradiction to $u_x(s(t^*), t^*) = 0$. Therefore $s(t) < d$ for $T/2 < t < T$ and (2.20) follows.

Case 2. $b \leq 0$. In this case a similar argument (the singularity at $x = 0$ does not cause any problem) shows that

$$s(t) \geq \frac{s(T/2)}{2} \qquad \text{for } \frac{T}{2} < t < T, \tag{2.22}$$

which implies that $x = 0$ is not a quenching point. □

Since $u_t \geq 0$,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{b}{x}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial t} - (t-u)^{-\beta} \\ &\geq -(1-u)^{-\beta} \qquad \text{for } t > 0, \quad 0 < x < a.\end{aligned}$$

Since $u_x < 0$ for $s(t) < x < a$, the above inequality implies

$$\begin{aligned}\frac{1}{2}u_x^2(x,t)+\int_{s(t)}^{x}\frac{b}{\xi}u_x^2(\xi,t)\,d\xi &= \int_{s(t)}^{x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{b}{x}\frac{\partial u}{\partial x}\right)u_x\,d\xi \\ &\le -\int_{s(t)}^{x}(1-u)^{-\beta}u_x\,d\xi \qquad (2.23)\\ &< \int_{u(x,t)}^{u(s(t),t)}(1-g)^{-\beta}\,dg \qquad \text{for } s(t)<x<a.\end{aligned}$$

It follows that (using Gronwall's inequality in the case $b < 0$),

$$u_x^2(x,t) \le 2\left[\frac{x}{s(t)}\right]^{\max(0,-2b)}\int_{u(x,t)}^{u(s(t),t)}(1-g)^{-\beta}\,dg \qquad \text{for } s(t)<x<a. \tag{2.24}$$

Similarly, since $u_x < 0$ for $0 < x < s(t)$, we have

$$\frac{1}{2}u_x^2(x,t)-\int_{x}^{s(t)}\frac{b}{\xi}u_x^2(\xi,t)\,d\xi \le \int_{u(x,t)}^{u(s(t),t)}(1-g)^{-\beta}\,dg \qquad \text{for } 0<x<s(t),$$

which leads to

$$u_x^2(x,t) \le 2\left[\frac{s(t)}{x}\right]^{\max(0,2b)}\int_{u(x,t)}^{u(s(t),t)}(1-g)^{-\beta}\,dg \qquad \text{for } 0<x<s(t). \tag{2.25}$$

Notice that the quantity $\int_{u(x,t)}^{u(s(t),t)}(1-g)^{-\beta}\,dg$ is bounded if $\beta < 1$. Thus we have

Lemma 2.5. *If $0 < \beta < 1$, then $x = 0$ and $x = a$ are not quenching points.*

Proof. If $x = a$ is a quenching point, then by Lemma 2.3, $\lim_{t\to T-0} s(t) = a$. Therefore (2.24) implies that

$$|u_x(x,t)| \le C \qquad \text{for } s(t)<x<a,\ \ T-t \ll 1.$$

Therefore

$$1 = \limsup_{t\to T-0}\,[u(s(t),t)-u(a,t)] \le \limsup_{t\to T-0} C(a-s(t)) = 0,$$

which is a contradiction.

Similarly, (2.25) implies that

$$x^{\max(0,b)}|u_x(x,t)| \le C \qquad \text{for } 0<x<s(t).$$

Since $b < 1$, we have

$$u(s(t), t) \leq \int_0^{s(t)} \frac{C}{x^{\max(0,b)}}\, dx,$$

which implies that $x = 0$ is not a quenching point. □

To end this section, we shall establish our theorems without assuming (2.15). We state this in terms of the following lemma.

Lemma 2.6. *(2.15) is valid for $u(x, \varepsilon)$ for some small ε. i.e., there exists $c^* \in (0, a)$ such that*

$$u_x(x, \varepsilon) > 0 \quad \textit{for } 0 < x < c^*, \qquad u_x(x, \varepsilon) < 0 \quad \textit{for } c^* < x \leq a. \tag{2.26}$$

Proof. By approximating the initial datum with the initial datum satisfying (2.15), we find that $u_x(x, \varepsilon)$ will change sign at most once. The solution $u(x, t)$ is analytic in x in the interior of the domain. Since $u_x(x, \varepsilon)$ is analytic in x and $u_x(x, \varepsilon) \not\equiv 0$, the lemma follows. □

3. Asymptotic behaviour for the first IBVP

Assume that quenching does not occur at the boundary. Let $c \in (0, a)$ be the quenching point. We define the similarity transformation as in **[11, 12]**,

$$y = \frac{x - c}{\sqrt{T - t}},\ s = -\log(T - t),\ w(y, s) = (1 - u(x, t))(T - t)^{-\gamma}, \tag{3.27}$$

where $\gamma = 1/(1 + \beta)$. Then w satisfies

$$w_s = w_{yy} - \frac{1}{2} y w_y + \gamma w - w^{-\beta} + \frac{b}{y + ce^{s/2}} w_y \text{ in } W_1, \tag{3.28}$$

$$w(y, s) = e^{\gamma s} \text{ for } y = -ce^{s/2}, (a - c)e^{s/2} \text{ and } s \geq s_0, \tag{3.29}$$

where $W_1 = \{(y, s) \mid 0 < c + ye^{-s/2} < a, s > s_0\}$ and $s_0 = -\log(T)$.

Lemma 3.1. *There is a positive constant A such that $w \geq A$ in W_1.*

Proof. Recall that $u_t > 0$ in Q_T. Since the quenching point is away from the boundary, there exist $\sigma > 0$ and $\delta > 0$ sufficiently small such that $u_t(x, t) \geq \sigma$ on the parabolic boundary of $Q \equiv (\delta, a - \delta) \times (\delta, T)$. Following **[10]**, we consider the function

$$J(x, t) = u_t(x, t) - \eta(1 - u)^{-\beta} \text{ in } Q$$

for some positive constant η. By applying the maximum principle, we conclude that $J(x, t) \geq 0$ in Q. Thus the lemma follows. □

Next, we estimate the first derivative of w as follows.

Lemma 3.2. *There are positive constants B_1, B_2, and B_3 such that*

$$|w_y(y, s)| \leq B_1 \text{ if } \beta > 1; \tag{3.30}$$

$$|w_y(y, s)| \leq B_2 w^{(1-\beta)/2}(y, s) \text{ if } \beta < 1; \tag{3.31}$$

$$|w_y(y, s)| \leq B_3[1 + \log w(y, s)] \text{ if } \beta = 1 \tag{3.32}$$

for all $(y, s) \in W_1$.

Proof. For $\beta > 1$, (2.24)–(2.25) imply that

$$\begin{aligned} u_x^2(x, t) &\leq C\{[1 - u(s(t), t)]^{1-\beta} - [1 - u(x, t)]^{1-\beta}\} \\ &\leq C[1 - u(s(t), t)]^{1-\beta} \qquad \text{for } \frac{a}{2} < x < a, 0 < t < T, \end{aligned}$$

for some positive constant C. Thus

$$w_y(y, s) = u_x(x, t)\,(T - t)^{1/2-\gamma} \leq C[\min_y w(y, s)]^{(1-\beta)/2} \leq C,$$

and (3.30) follows.

Similarly, if $\beta < 1$, then (2.24)–(2.25) imply that

$$u_x^2(x, t) \leq C[1 - u(x, t)]^{1-\beta} \qquad \text{for } |x - c| \leq \varepsilon, 0 < t < T,$$

and (3.31) follows immediately.

For $\beta = 1$, (2.24)–(2.25) imply that

$$u_x^2(x, t) \leq C \log \frac{1 - u(x, t)}{1 - u(s(t), t)} \qquad \text{for } |x - c| \leq \varepsilon, 0 < t < T.$$

Thus (notice that now $\gamma = 1/2$),

$$w_y(y, s) = u_x(x, t) \leq C \log \frac{w(y, s)}{\min_y w(y, s)}.$$

Since $w(y, s)$ is uniformly bounded below, (3.32) follows. □

The following lemma plays the same role as the nondegeneracy lemma for the corresponding blowup problem ([**13**, Lemma 3.7]). Here we shall use straightforward Hölder estimates.

Lemma 3.3. *We have*

$$w(0, s) \le C \qquad \textit{for } s_0 < s < \infty. \tag{3.33}$$

Proof. Take δ to be small so that $c \in [2\delta, a - 2\delta]$. By Lemma 3.1,

$$(1 - u(x, t))^{-\beta} \le C(T - t)^{-\gamma\beta} \qquad \text{for } \delta < x < a - \delta, 0 < t < T.$$

Since $\gamma\beta = \beta/(1 + \beta) < 1$ and u_x is estimated by (2.24)–(2.25), the function

$$f(x, t) = \frac{b}{x} u_x + [1 - u(x, t)]^{-\beta}$$

satisfies

$$|f(x, t)| \le C(T - t)^{-\gamma\beta}, \qquad \gamma\beta = \frac{\beta}{\beta + 1} < 1. \tag{3.34}$$

Let

$$v(x, t) = \int_0^t \int_\delta^{a-\delta} G(x - y, t - \tau) f(y, \tau)\, dy\, d\tau,$$

where

$$G(x, y) = \frac{1}{\sqrt{4\pi(t - \tau)}} \exp\left(-\frac{(x - y)^2}{4(t - \tau)}\right).$$

The estimate (3.34) implies that v is Hölder continuous (see [**16**, Chapter V, Theorem 1.1, p. 419]). For any $T/2 < t_1 < t_2 < T$,

$$v(c, t_1) - v(c, t_2) = J_1 + J_2 + J_3,$$

where

$$J_1 = \int_{t_1-(t_2-t_1)}^{t_2} \int_\delta^{a-\delta} G(c - y, t_2 - \tau) f(y, \tau)\, dy\, d\tau,$$

$$J_2 = \int_{t_1-(t_2-t_1)}^{t_1} \int_\delta^{a-\delta} G(c - y, t_1 - \tau) f(y, \tau)\, dy\, d\tau,$$

and

$$J_3 = \int_0^{t_1-(t_2-t_1)} \int_\delta^{a-\delta} [G(c - y, t_1 - \tau) - G(c - y, t_1 - \tau)] f(y, \tau)\, dy\, d\tau.$$

Using the bound from (3.34), it is not hard to show that

$$|J_1| + |J_2| \leq C\{[(T - t_1 + (t_2 - t_1))]^\gamma - (T - t_2)^\gamma\} \leq C(t_2 - t_1)^\gamma,$$

and by the mean value theorem,

$$\begin{aligned}
|J_3| &\leq C(t_2 - t_1) \int_0^{t_1-(t_2-t_1)} \int_\delta^{a-\delta} \frac{(T-\tau)^{-\gamma\beta}}{(\hat{t}(\tau) - \tau)^{3/2}} \exp\left(-\frac{(c-y)^2}{8(\hat{t}(\tau) - \tau)}\right) dy\, d\tau, \\
&\qquad (t_1 < \hat{t}(\tau) < t_2) \\
&\leq C(t_2 - t_1) \int_0^{t_1-(t_2-t_1)} \left[\frac{(T-\tau)^{-\gamma\beta}}{(\hat{t}(\tau) - \tau)} \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{8}\right) d\xi\right] d\tau \\
&\leq C(t_2 - t_1) \int_0^{t_1-(t_2-t_1)} \frac{(T-\tau)^{-\gamma\beta}}{t_1 - \tau}\, d\tau \\
&\leq C(t_2 - t_1) \int_0^{t_1-(t_2-t_1)} \frac{d\tau}{(t_1 - \tau)^{1+\gamma\beta}} \\
&\leq c(t_2 - t_1)^{1-\gamma\beta} = C(t_2 - t_1)^\gamma.
\end{aligned}$$

Combining the estimates for J_1, J_2 and J_3, we obtain

$$|v(c, t_1) - v(c, t_2)| \leq C|t_1 - t_2|^\gamma. \tag{3.35}$$

Clearly,

$$(u - v)_t - (u - v)_{xx} = 0 \qquad \text{for } (x, t) \in (\delta, a - \delta) \times (0, T).$$

Thus the standard parabolic estimates imply that

$$\max_{T/2 \leq t \leq T} |(u - v)_t| \leq C < \infty.$$

Combining this inequality with (3.35), we obtain

$$u(c, t_2) - u(c, t) \leq C(t_2 - t)^\gamma, \qquad \text{for } t < t_2 < T. \tag{3.36}$$

Notice that $u = (u - v) + v$ where both $(u - v)$ and v are Hölder continuous on $[2\delta, a - 2\delta] \times [T/2, T]$. The point $x = c$ is a quenching point, therefore by continuity $\lim_{t \to T-0} u(c, t) = 1$. Letting $t_2 \to T - 0$ in (3.36), we conclude that

$$1 - u(c, t) \leq C(T - t)^\gamma, \qquad \text{for } t < T.$$

Rewrite this inequality in terms of w, the lemma follows. □

For all three cases of β, Lemmas 3.2 and 3.3 imply that $w(y,s) \le C(1+y^2)$ and $w_y(y,s) \le C(1+|y|)$ for some $C>0$. Especially, $|w_s - w_{yy}| \le C(1+y^2)$, for some constant $C>0$. It follows from L^p interior boundary estimates and embedding theorem (apply the estimates in the domain $[y, y+1] \times [s, s+1]$) that the Hölder norm of w_y is bounded by $C(1+y^2)$ in $[0,y] \times [s^*, \infty)$, for s^* sufficiently large. Applying Schauder's interior boundary estimates, we obtain that w_{yy}, w_s and their Hölder norms are bounded by $C(1+|y|^3)$.

We now define the energy function (or Lyapunov function) as

$$E[w](s) = \int_{-s}^{s} \left[\frac{w_y^2(y,s)}{2} - F(w(y,s)) \right] e^{-y^2/4}\, dy, \tag{3.37}$$

where $F(w)$ is chosen so that $F'(w) = \gamma w - w^{-\beta}$. It is easy to compute that

$$\frac{d}{ds} E[w](s) = -\int_{-s}^{s} w_s^2(y,s) e^{-y^2/4}\, dy + J_1(s) + J_2(s) + J_3(s), \tag{3.38}$$

where

$$J_1(s) = -\int_{-s}^{s} \frac{b}{ae^{s/2} - y} w_y(y,s) w_s(y,s) e^{-y^2/4}\, dy$$

$$J_2(s) = (w_y(s,s) w_s(s,s) - w_y(-s,s) w_s(-s,s)) e^{-s^2/4}$$

$$J_3(s) = \left[\frac{w_y^2(s,s)}{2} - F(w(s,s)) + \frac{w_y^2(-s,s)}{2} - F(w(-s,s)) \right] e^{-s^2/4}.$$

We claim that $J_i(s), i = 1,2,3$, are integrable over (s_0, ∞). For J_1, both w_y and w_s are bounded by a polynomial in y, it follows that $J_1(s)$ is bounded by $Ce^{-s/4}$ and hence integrable over (s_0, ∞). Clearly, $J_2(s)$ and $J_3(s)$ are integrable over (s_0, ∞), owing to the polynomial growth estimates for w, w_y and w_s as $y \to \infty$. Therefore, it follows that

$$\frac{d}{ds} E[w](s) \le -\int_{-s}^{s} w_s^2(y,s) e^{-y^2/4}\, dy + J(s) \tag{3.39}$$

with $J(s)$ integrable over (s_0, ∞).

Now, by applying the energy method of Giga and Kohn [11] (for details see [13, Theorem 3.10]), we can show that $w(y,s)$ tends to a positive solution $w_\infty(y)$ of

$$w'' - \frac{1}{2} y w' + \gamma w - w^{-\beta} = 0, \qquad -\infty < y < \infty, \tag{3.40}$$

such that w_∞ has the same bound as w. Furthermore, $u_t > 0$ implies that w_∞ satisfies

$$\gamma w - \frac{1}{2} y w_y \ge 0, \qquad -\infty < y < \infty. \tag{3.41}$$

It is proved in ([**14**, Theorem 2.1]) that any non-constant solution of (3.40) bounded by a polynomial $P(y)$ must be a slow orbit. It is proved in ([**14**, Theorem 2.6]) that for any slow orbit, $g(y) \equiv \gamma w(y) - \frac{1}{2} y w_y(y)$ can not remain positive for all sufficiently large y. (In fact, $V(y)$ in the proof of ([**14**, Theorem 2.6]) will go to $+\infty$, if $g(y) \geq 0$ for all large y.) Thus $g(y)$ and $g'(y)$ can not vanish at the same time for all large y, according to the definition of $V(y)$. Thus (3.41) implies that $g(y) > 0$ for all large y, which leads to a contradiction as in the proof of ([**14**, Theorem 2.6]). Therefore, (3.41) eliminates any slow orbit. It follows that there is no non-constant solution of (3.40) bounded by a polynomial and satisfying (3.41). Thus $w(y, s)$ converges to the constant solution $(\beta + 1)^\gamma$.

We have established the following quenching rate estimate.

Theorem 3.4. *As $t \to T$, we have*

$$[1 - u(x, t)](T - t)^{-\gamma} \to (\beta + 1)^\gamma$$

uniformly for $|x - c| \leq C\sqrt{T - t}$ for any positive constant C.

4. The second IBVP

In this section we study the second IBVP. Recall that $u_x \geq 0$ in Q_T (cf. [**5**, Lemma 1]). Since u is a classical solution in Q_T, it follows from the strong maximum principle that $u_x > 0$ in Q_T. Then the following result is a direct consequence of Lemma A.

Theorem 4.1. *Single point quenching holds and this occurs necessarily at $x = a$.*

Since we have $u_t \geq 0$ and $u_x(a, t) = 0$, the estimate (2.25) is still valid for the second IBVP, namely,

$$u_x^2(x, t) \leq 2\left[\frac{a}{x}\right]^{\max(0,2b)} \int_{u(x,t)}^{u(a,t)} (1 - g)^{-\beta}\, dg \qquad \text{for } 0 < x < a. \tag{4.42}$$

We next study the asymptotic behaviour. Define the similarity transformation by

$$y = \frac{a - x}{\sqrt{T - t}},\ s = -\log(T - t),\ w(y, s) = (1 - u(x, t))(T - t)^{-\gamma}, \tag{4.43}$$

where $\gamma = 1/(1 + \beta)$. Then w satisfies

$$w_s = w_{yy} - \frac{1}{2} y w_y + \gamma w - w^{-\beta} + \frac{b}{y - ae^{s/2}} w_y \text{ in } W_2, \tag{4.44}$$

$$w_y(0, s) = 0,\ s \geq s_0; \quad w(y, s) = e^{\gamma s},\ y = ae^{s/2},\ s \geq s_0, \tag{4.45}$$

where $W_2 = \{(y, s) \mid 0 < y < ae^{s/2}, s > s_0\}$ and $s_0 = -\log(T)$.
First, we derive a uniform lower bound for w as follows.

Lemma 4.2. *There is a positive constant A such that $w \geq A$ in W_2.*

Proof. The proof is essentially the same as in Section 3 and is omitted here. □

Since $u_x(a, t) = 0$ and $u_{xx}(a, t) \leq 0$, we easily derive from the equation that $w(0, s) \leq (\beta + 1)^\gamma$. Furthermore, since $w_y(y, s) = u_x(x, t)(T - t)^{1/2-\gamma}$ and $u_x > 0$ in Q_T, we see that $w_y > 0$ for $y > 0$. Equation (4.42) will give the upper bounds for w_y, which we state in terms of the following lemma.

Lemma 4.3. *There are positive constants B_1, B_2, B_3, such that*

$$w_y(y, s) \leq B_1 \text{ if } \beta > 1; \tag{4.46}$$

$$w_y(y, s) \leq B_2 w^{(1-\beta)/2}(y, s) \text{ if } \beta < 1; \tag{4.47}$$

$$w_y(y, s) \leq B_3(1 + \log w(y, s)) \text{ if } \beta = 1; \tag{4.48}$$

for all $(y, s) \in W_0 = \{(y, s) \mid 0 < y < (a/2)e^{s/2}, s > s_0\}$.

Proof. For $\beta > 1$, (4.42) implies that

$$\begin{aligned} u_x^2(x, t) &\leq C\{[1 - u(a, t)]^{1-\beta} - [1 - u(x, t)]^{1-\beta}\} \\ &\leq C[1 - u(a, t)]^{1-\beta} \qquad \text{for } \frac{a}{2} < x < a, 0 < t < T, \end{aligned}$$

for some positive constant C. Thus

$$w_y(y, s) = u_x(x, t)(T - t)^{1/2-\gamma} \leq C[w(0, s)]^{(1-\beta)/2} \leq C,$$

and (4.46) follows.
Similarly, if $\beta < 1$, then (4.42) implies that

$$u_x^2(x, t) \leq C[1 - u(x, t)]^{1-\beta} \qquad \text{for } \frac{a}{2} < x < a, 0 < t < T,$$

and (4.47) follows immediately.
For $\beta = 1$, (4.42) implies that

$$u_x^2(x, t) \leq C \log \frac{1 - u(x, t)}{1 - u(a, t)} \qquad \text{for } \frac{a}{2} < x < a, 0 < t < T.$$

Thus (notice that now $\gamma = 1/2$),

$$w_y(y, s) = u_x(x, t) \le C \log \frac{w(y, s)}{w(0, s)}.$$

Since $w(0, s)$ is bounded above and below, (4.48) follows. $\square$

Remark 1. As an easy consequence of Lemma 4.3, for $\beta < 1$, we have

$$0 < 1 - u(x, T - 0) \le [C(a - x)]^{2\gamma}$$

for some positive constant C. This gives the spatial asymptotic estimate of u near the quenching point $x = a$ at time T. Comparing this estimate with that in [9], it is certainly not optimal. But the only factor which is missing from the estimate is the logarithmic factor, which decays to zero much slower than $[(a - x)]^{2\gamma}$.

Remark 2. The spatial asymptotics of u at $t = T$ obtained in [9] depends heavily on the asymptotic estimates at the quenching time $t = T$ (Theorems 1.1 and 1.2). So our Theorems 1.1 and 1.2 are the first step towards obtaining estimates for $u(x, T - 0)$. The spatial asymptotic behaviour for $u(x, T - 0)$ is the same as that in [9]. In order to obtain such an estimate, the asymptotic expansion for the next order will be needed, and an estimate in the variable $\xi = x/[(T - t)|\ln(T - t)|]^{1/2}$ is also needed. The extra term introduced from the b/x term decays to zero exponentially fast as $s = -\ln(T - t) \to \infty$ in the similarity variable equation and therefore is very unlikely to cause any problems in the parabolic estimates.

We now continue the discussion for the second IBVP. Recall that $w(0, s) \le (\beta + 1)^\gamma$. For all three cases of β, Lemma 4.3 implies that $w(y, s) \le C(1 + y^2)$ and $w_y(y, s) \le C(1 + |y|)$ for some $C > 0$. Moreover, $|w_s - w_{yy}| \le C(1 + y^2)$, for some constant $C > 0$. It follows from L^p interior boundary estimates and embedding theorem (apply the estimates in the domain $[y, y + 1] \times [s, s + 1]$) that the Hölder norm of w_y is bounded by $C(1 + y^2)$ in $[0, y] \times [s^*, \infty)$, for s^* sufficiently large. Applying Schauder's interior boundary estimates, we obtain that w_{yy}, w_s and their Hölder norms are bounded by $C(1 + |y|^3)$.

We now define the energy function (or Lyapunov function) as before

$$E[w](s) = \int_0^s \left[\frac{w_y^2(y, s)}{2} - F(w(y, s))\right] e^{-y^2/4}\, dy, \tag{4.49}$$

where $F(w)$ is chosen so that $F'(w) = \gamma w - w^{-\beta}$. As in the previous section (here we use the boundary condition $w_y(0, s) = 0$)

$$\frac{d}{ds} E[w](s) = -\int_0^s w_s^2(y, s) e^{-y^2/4}\, dy + J(s), \tag{4.50}$$

where $J(s)$ is integrable over (s_0, ∞). Therefore, it follows that

$$\frac{d}{ds}E[w](s) \leq -\int_0^s w_s^2(y,s)e^{-y^2/4}\,dy + J(s) \tag{4.51}$$

with $J(s)$ integrable over (s_0, ∞).

Now, the energy method of Giga and Kohn implies that $w(y,s)$ tends to a positive symmetric solution $w_\infty(y)$ of

$$w'' - \frac{1}{2}yw' + \gamma w - w^{-\beta} = 0 \tag{4.52}$$

such that w_∞ has the same bound as w with $\gamma w_\infty - \frac{1}{2}y(w_\infty)_y \geq 0$. Proceeding as in Section 3, we obtain the following quenching rate estimate.

Theorem 4.4. *As $t \to T$, we have*

$$[1 - u(x,t)](T-t)^{-\gamma} \to (\beta+1)^\gamma$$

uniformly for $a - x \leq C\sqrt{T-t}$ for any positive constant C.

We remark that as far as the quenching rate is concerned the constant b is irrelevant, as long as the quenching does not occur at $x = 0$.

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