SUBSEMIGROUPS OF FREE PRODUCTS OF SEMIGROUPS

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Given amalgams of semigroups $[U; T_i]$ and $[U; S_i]$ with $U \subseteq T_i \subseteq S_i$, it was proved by Howie in [6] that $\prod_U^* T_i$ need not be embedded in $\prod_U^* S_i$. We use the homological techniques developed by Renshaw in [7, 9] and study three new conditions each of which imply the embeddability of the above free products.

Howie [6] showed that, unlike the case for groups, if $[U; S_i]$ is an amalgam and if T_i are subsemigroups of S_i such that $[U; T_i]$ is an amalgam, then $\prod_U^* T_i$, the free product of the amalgam $[U; T_i]$, need not be embeddable in $\prod_U^* S_i$, the free product of the amalgam $[U; S_i]$. He proved, among other things, that if U and T_i are unitary in S_i, then the free products are embeddable. We extended these results in Renshaw [8] to show that the same conclusion holds if the inclusion maps $U \to T_i$ and $T_i \to S_i$ are pure. We provide here three more conditions for the free products to be embeddable, one related to Hall and Howie's *perfect* subsemigroups, one connected with *flatness* of U-sets and one concerning *right generalised inverse* semigroups.

1. The "homological" machinery

For definitions of U-sets, U-maps etc. see for example Renshaw [7]. The category of left U-sets will be denoted by U-ENS. For convenience, we shall mainly work from now on with monoids. It is an easy matter in any case to extend these results to semigroups (see for example Renshaw [8, 9]).

Suppose that $[U; S_i : i \in I]$ is an amalgam of monoids and that there exists a family of monoids $\{T_i : i \in I\}$ with the property that $U \subseteq T_i \subseteq S_i$ for all i in I. We shall call the amalgam $[U; T_i]$ an *amalgam of submonoids* of the amalgam $[U; S_i]$. Our main problem is to determine some conditions under which $\prod_U^* T_i$, the free product of the amalgam $[U; T_i]$, is embeddable in $\prod_U^* S_i$, the free product of the amalgam $[U; S_i]$. It was shown in Renshaw [8] that we need only consider amalgams over a finite index set.

Let $X \in \text{ENS-}U$. From now on, all tensor products are over U unless otherwise stated. We say that X is *(right) flat* if for all left U-monomorphisms $\lambda: A \to B$ the induced map $1 \otimes \lambda: X \otimes A \to X \otimes B$ is one to one. We shall say that X is *(right) quasi-flat* if for all left U-monomorphisms $\lambda: A \to B$ with B flat, the map $1 \otimes \lambda: X \otimes A \to X \otimes B$ is one to one

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(clearly flat implies quasi-flat). A right U-monomorphism $f:X \to Y$ is said to be (*right*) *pure* if for all left U-sets A, the induced map $f \otimes 1 : X \otimes A \to Y \otimes A$ is one to one. A right U-monomorphism $f:X \to Y$ is said to be *perfect* if f is right pure and Y is right flat. (This is a generalisation of Howie's *left perfect subsemigroups* in Howie [5]. See also Hall [2] and Renshaw [9].) Let $f:X \to Y$ be a right U-map and $\lambda : A \to B$ a left U-map and consider the commutative diagram

$$
\begin{array}{ccc}
X \otimes A & \xrightarrow{\quad f \otimes 1_A \quad} & Y \otimes A \\
{\scriptstyle 1_X \otimes \lambda} \downarrow & & \downarrow {\scriptstyle 1_Y \otimes \lambda} \\
X \otimes B & \xrightarrow{\quad f \otimes 1_B \quad} & Y \otimes B
\end{array}
\tag{1}
$$

We say that the pair (f, λ) is *stable* if $\operatorname{im}(f \otimes \lambda) = \operatorname{im}(1_Y \otimes \lambda) \cap \operatorname{im}(f \otimes 1_B)$. If $f:X \to Y$ is a right U-monomorphism and if (f, λ) is stable for all left U-monomorphisms $\lambda : A \to B$ then we say that f is *stable*. We shall say that a right U-monomorphism is *quasi-stable* if (f, λ) is stable for all left U-monomorphisms $\lambda : A \to B$ with B flat.

A monoid U is said to be *left (right) reversible* if any two principal right (left) ideals of U intersect. Let $X \in U$-ENS and consider the left U-congruence $\equiv$ on X defined by $x \equiv y$ if and only if either $x = y$ or there exists $x_1, \ldots, x_n \in X$, $u_1, \ldots, u_n$, $v_1, \ldots, v_n \in U$ such that

$$
x = u_1 x_1, \ v_1 x_1 = u_2 x_2, \ v_2 x_2 = u_3 x_3, \ldots, \ v_n x_n = y.
$$

There is clearly a dual definition for right U-sets. It is easy to demonstrate that if $x \otimes y = x' \otimes y'$ in $X \otimes Y$ then $x \equiv x'$ in X and $y \equiv y'$ in Y. Let $f:X \to Y$ be a U-monomorphism and consider the right U-congruence on Y defined by

$$
\rho_f = \operatorname{im} f \times \operatorname{im} f \cup 1_Y.
$$

We shall denote the quotient Y/ρ_f by Y/X and an element $y\rho_f$ by $\bar{y}$. The following results are easy to establish, see Renshaw [7].

Lemma 1.1. *Let $f:X \to Y$ be a right U-monomorphism and let $A \in U$-ENS. Then*

(i) $\ker(Y \otimes A \to Y/X \otimes A) = \{(f(x) \otimes a, f(x') \otimes a') : a \equiv a'\} \cup 1_{Y \otimes A}$,

(ii) *if Y/X is quasi-flat then U is right reversible,*

(iii) *if $y \otimes a = f(x) \otimes a'$ in $Y \otimes A$ then $\bar{y} \otimes a = \overline{f(x)} \otimes a$ in $Y/X \otimes A$,*

(iv) *if X and Y/X are quasi-flat then Y is quasi-flat,*

(v) *if Y is (quasi-)flat then Y/X is (quasi-)flat if and only if U is right reversible and f is (quasi-)stable.*

The following result will be required later.

Lemma 1.2. *If $f:X\to Y$ and $g:Y\to Z$ are right U-monomorphisms and $\lambda:A\to B$ is a left U-monomorphism then*

(i) *if $g\otimes 1:Y\otimes B\to Z\otimes B$ is one to one then (gf,λ) is stable if both (g,λ) and (f,λ) are stable,*

(ii) *if we let $\bar{g}:Y/X\to Z/X$ be given by $\bar{g}(\bar{y})=\overline{g(y)}$, then $(\bar{g},\lambda)$ is stable whenever (g,λ) is stable,*

(iii) *if $g\otimes 1:Y\otimes B\to Z\otimes B$ is one to one and (gf,λ) is stable then (f,λ) is stable,*

(iv) *if (gf,λ) and $(\bar{g},\lambda)$ are stable then (g,λ) is stable.*

In particular, g and f are quasi-stable if and only if gf and $\bar{g}$ are quasi-stable.

Proof. (i) Suppose $gf(x)\otimes b=z\otimes\lambda(a)$ in $Z\otimes B$. Then since (g,λ) is stable, there exists $y_1\otimes a_1$ in $Y\otimes A$ such that $gf(x)\otimes b=g(y_1)\otimes\lambda(a_1)$. So $f(x)\otimes b=y_1\otimes\lambda(a_1)$ in $Y\otimes B$ and since (f,λ) is stable, there exists $x_2\otimes a_2$ in $X\otimes A$ such that $f(x)\otimes b=f(x_2)\otimes\lambda(a_2)$. Hence, $gf(x)\otimes b=gf(x_2)\otimes\lambda(a_2)$ and (gf,λ) is stable.

(ii) Suppose that $\bar{g}(\bar{y})\otimes b=\bar{z}\otimes\lambda(a)$ in $Z/X\otimes B$. Then from Lemma 1.1(i), we see that either

(a) $g(y)\otimes b=z\otimes\lambda(a)$ in $Z\otimes B$, from which we deduce that there exists $y_1\otimes a_1$ in $Y\otimes A$ such that $g(y)\otimes b=g(y_1)\otimes\lambda(a_1)$ and consequently $\bar{g}(\bar{y})\otimes b=\bar{g}(\bar{y}_1)\otimes\lambda(a_1)$ and so $(\bar{g},\lambda)$ is stable; or

(b) $g(y)\otimes b=gf(x_1)\otimes b_1,\ z\otimes\lambda(a)=gf(x_2)\otimes b_2$ with $b_1\equiv b_2$. By Lemma 1.1(iii) $\bar{z}\otimes\lambda(a)=gf(x_2)\otimes\lambda(a)=\bar{g}(f(x_2))\otimes\lambda(a)$ and so $(\bar{g},\lambda)$ is stable.

(iii) If $f(x)\otimes b=y\otimes\lambda(a)$ in $Y\otimes B$ then $g(y)\otimes\lambda(a)=gf(x)\otimes b$ and so $g(y)\otimes\lambda(a)=gf(x')\otimes\lambda(a')$ for some x' in X, a' in A, since (gf,λ) is stable. Hence $y\otimes\lambda(a)=f(x')\otimes\lambda(a')$ as required.

(iv) Suppose $z\otimes\lambda(a)=g(y)\otimes b$ in $Z\otimes B$. Then $\bar{z}\otimes\lambda(a)=\overline{g(y)}\otimes b$ in $Z/X\otimes B$ and since $(\bar{g},\lambda)$ is stable, we deduce that $\bar{z}\otimes\lambda(a)=\overline{g(y')}\otimes\lambda(a')$ for some y' in Y, a' in A. From Lemma 1.1(i) we see that either

(a) $z\otimes\lambda(a)=g(y')\otimes\lambda(a')$ as required, or

(b) $z\otimes\lambda(a)=gf(x_1)\otimes b_1,\ g(y')\otimes\lambda(a')=gf(x_2)\otimes b_2$ with $b_1\equiv b_2$. Hence, since (gf,λ) is stable, we see that $z\otimes\lambda(a)=gf(x_3)\otimes\lambda(a_3)$ and so (g,λ) is stable.

The following lemma generalises Lemma 1.1(i) and (iii) and the proofs are similar.

Lemma 1.3. *Let $A\in E\text{-}NS\text{-}U$, $B\in U\text{-}ENS$, $X,Y\in U\text{-}ENS\text{-}U$ and $f:X\to Y$ a (U,U)-monomorphism. Then*

(i) $\ker(A\otimes Y\otimes B\to A\otimes(Y/X)\otimes B)=\{(a\otimes f(x)\otimes b,a'\otimes f(x')\otimes b'):a\equiv a',b\equiv b'\}\cup 1_{A\otimes Y\otimes B}.$

(ii) *if $a\otimes y\otimes b=a'\otimes f(x')\otimes b'$ in $A\otimes Y\otimes B$ then $a\otimes\bar{y}\otimes b=a'\otimes\overline{f(x')}\otimes b$ in $A\otimes(Y/X)\otimes B$.*

Lemma 1.4. *Let $f:A\to B$ be a right U-monomorphism, $g:C\to D$ be a (U,U)-monomorphism. Then*

(i) *if D, D/C are flat (both right and left flat) and A is (quasi-)flat then $A\otimes C\to A\otimes D$ is (quasi-)stable,*

(ii) *if D is flat and B/A is (quasi-)flat than $A\otimes D\to B\otimes D$ is (quasi-)stable,*

(iii) *if D, D/C are flat and A, B/A are quasi-flat then $A\otimes C\to B\otimes D$ is a quasi-stable monomorphism and hence $(B\otimes D)/(A\otimes C)$ is quasi-flat.*

Proof. (i) and (ii) are generalisations of Renshaw [7, Theorem 4.10] and the proofs are similar. In (iii), $A\otimes C\to A\otimes D$ is one to one since A is quasi-flat and $A\otimes D\to B\otimes D$ is one to one since D is flat. Now $A\otimes C\to A\otimes D$ is quasi-stable by (i) and $A\otimes D\to B\otimes D$ is quasi-stable by (ii) and so the composite $A\otimes C\to B\otimes D$ is quasi-stable by Lemma 1.2. Finally, $B\otimes D$ is quasi-flat since B is quasi-flat (Lemma 1.1(iv)) and D is flat. Hence by Lemma 1.1(v) we see that $(B\otimes D)/(A\otimes C)$ is quasi-flat.

Consider the following commutative diagram of U-monomorphisms
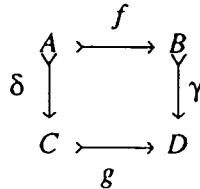
Define $\bar{\gamma}:B/A\to D/C$ by $\bar{\gamma}(\bar{b})=\overline{\gamma(b)}$. Then $\bar{\gamma}$ is well-defined and it is straightforward to show that $\bar{\gamma}$ is one to one if and only if the above diagram is a pullback.

Lemma 1.5. *Suppose that the above diagram is a pullback (so that $\bar{\gamma}$ and $\bar{g}$ are one to one). Then $(D/C)/(B/A)\simeq(D/B)/(C/A)$.*

Proof. Consider the following commutative diagram

(A)
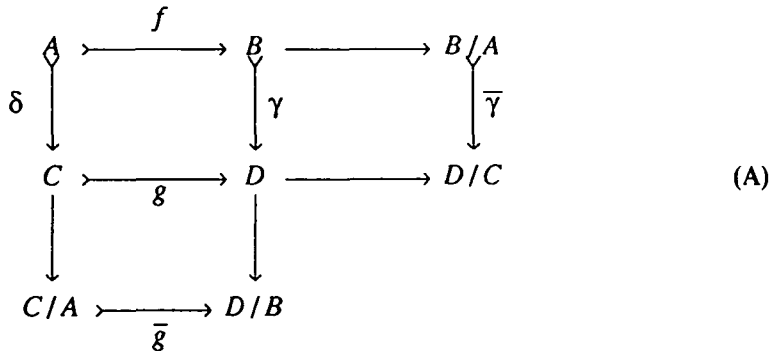
Let $\alpha:D/C\to(D/C)/(B/A)$ and $\beta:D/B\to(D/B)/(C/A)$ be the canonical maps and define $h:(D/C)/(B/A)\to(D/B)/(C/A)$ by $h(\alpha(\bar{d}))=\beta(\bar{d})$. Then h is well defined, since, if $\alpha(\bar{d})=\alpha(\bar{d}')$ then either

(i) $\bar{d}=\bar{d}'$ in D/C, in which case either

    (a) $d=d'$ giving $\beta(\bar{d})=\beta(\bar{d}')$, or

    (b) $d=g(c)$, $d'=g(c')$ for some c,c' in C. In this case $\bar{d}=\overline{g(c)}=\bar{g}(\bar{c})$ and $\bar{d}'=\bar{g}(\bar{c}')$. Hence $\beta(\bar{d})=\beta(\bar{d}')$; or

(ii) $\bar{d}=\bar{\gamma}(\bar{b})=\overline{\gamma(b)}$ and $\bar{d}'=\bar{\gamma}(\bar{b}')=\overline{\gamma(b')}$ in D/C. Now either

    (c) $d=\gamma(b)$ and so $\bar{d}=\overline{\gamma(b)}=\overline{\gamma f(a)}=\overline{g\delta(a)}$ in D/B (any $a\in A$), or

    (d) $d=g(c)$, $\gamma(b)=g(c')$ for some c,c' in C.

Either way, $\bar{d}=\bar{g}(\bar{c})$ in D/B for some c in C. Similarly, $\bar{d}'=\bar{g}(\bar{c}')$ in D/B. Hence $\beta(\bar{d})=\beta(\bar{d}')$.

A similar argument shows that h is one to one and it is easy to see that h is onto and is a U-map.

Lemma 1.6. *With the notation and conditions as in Lemma 1.5, suppose that $\lambda:X\to Y$ is a left U-monomorphism and that (γ,λ) is stable and $D/C\otimes X\to D/C\otimes Y$ is one to one. Then $(\bar{g},\lambda)$ is stable. In particular, if D, D/B and D/C are (quasi)-flat then so is $(D/B)/(C/ A)$ (and hence $(D/C)/(B/A)$ by Lemma 1.5).*

Proof. Suppose $\bar{g}(\bar{c})\otimes y=\bar{d}\otimes\lambda(x)$ in $D/B\otimes Y$. Then from Lemma 1.1(i), we deduce that either

    (i) $g(c)\otimes y=d\otimes\lambda(x)$ in $D\otimes Y$, or

    (ii) $g(c)\otimes y=\gamma(b_1)\otimes y_1$, $d\otimes\lambda(x)=\gamma(b_2)\otimes y_2$ with $y_1\equiv y_2$.

In case (i), $\bar{d}\otimes\lambda(x)=\overline{g(c)}\otimes\lambda(x)$ in $D/C\otimes Y$ by Lemma 1.1(iii) and so $\bar{d}\otimes x=\overline{g(c)}\otimes x$ in $D/C\otimes X$. Hence, by Lemma 1.1(i) we have either

    (a) $d\otimes x=g(c)\otimes x$ in $D\otimes X$, giving $\bar{d}\otimes\lambda(x)=\bar{g}(\bar{c})\otimes\lambda(x)$ as required, or

    (b) $d\otimes x=g(c_1)\otimes x_1$, $g(c)\otimes x=g(c_2)\otimes x_2$ which again means that $\bar{d}\otimes\lambda(x)=\bar{g}(c_1)\otimes\lambda(x_1)$ as required.

In case (ii), since (γ,λ) is stable than $d\otimes\lambda(x)=\gamma(b_3)\otimes\lambda(x_3)$ for some b_3 in B and x_3 in X. Hence, in $D/B\otimes Y$ we have, for any a in A, $\bar{d}\otimes\lambda(x)=\overline{\gamma(b_3)}\otimes\lambda(x_3)=\overline{\gamma f(a)}\otimes\lambda(x_3)=\overline{g\delta(a)}\otimes\lambda(x_3)=\bar{g}(\overline{\delta(a)})\otimes\lambda(x_3)$ as required.

Lemma 1.7. *Let $f:X\to Y$ be a right U-monomorphism and $\lambda:A\to B$ a left U-monomorphism and suppose that all the maps in diagram (1), above, are one to one. Then diagram (1) is a pullback if and only if (f,λ) is stable. In particular, if*

    (i) *f is perfect, or*

    (ii) *X and Y/X are quasi-flat and B is flat,*

then diagram (1) is a pullback.

Proof.   That (f, λ) is stable if and only if diagram (1) is a pullback follows from the note before Theorem 4.8 in Renshaw [7]. Also, if X and Y/X are quasi-flat then by Lemma 1.1(iv), Y is quasi-flat and so all the maps in (1) are one to one. Hence by Lemma 1.1(v) diagram (1) is a pullback. If f is (right) perfect then again all the maps in (1) are one to one. To show that (f, λ) is stable, we generalise (Renshaw [9, Lemma 4.15]). Consider the following pushout diagram (in U-ENS):

$$
\begin{array}{ccc}
A & \xrightarrow{\ \lambda\ } & B \\
{\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \alpha} \\
B & \xrightarrow[\ \beta\ ]{} & P
\end{array}
$$

Since tensor products preserve pushouts, then

$$
\begin{array}{ccc}
X \otimes A & \xrightarrow{\ 1\otimes\lambda\ } & X \otimes B \\
{\scriptstyle 1\otimes\lambda}\downarrow & & \downarrow{\scriptstyle 1\otimes\alpha} \\
X \otimes B & \xrightarrow[\ 1\otimes\beta\ ]{} & X \otimes P
\end{array}
$$

is also a pushout (and hence, since $1 \otimes \lambda$ is one to one, also a pullback). It is a pushout in ENS. Now if $y \otimes \lambda(a) = f(x) \otimes b$ in $Y \otimes B$ then

$$f(x) \otimes \alpha(b) = y \otimes \alpha\lambda(a) = y \otimes \beta\lambda(a) = f(x) \otimes \beta(b) \quad \text{in} \quad Y \otimes P.$$
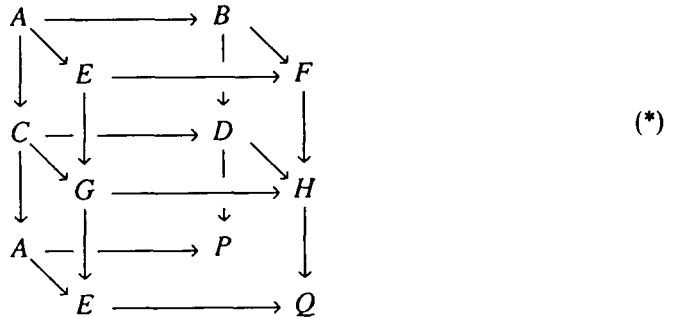
Since f is (right) pure then $x \otimes \alpha(b) = x \otimes \beta(b)$ in $X \otimes P$ and so (since pushouts of monomorphisms along monmorphisms are also pullbacks in ENS) there exists $x' \otimes a'$ in $X \otimes A$ with $x \otimes b = x' \otimes \lambda(a')$ in $X \otimes B$ which means $y \otimes \lambda(a) = f(x') \otimes \lambda(a')$ in $Y \otimes B$ as required.

The following rather technical result, which was proved in Renshaw [8, Lemmas 2.2 and 2.3], forms the backbone of the machinery we need.

Lemma 1.8.   *Consider the following commutative diagrams*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \alpha_1\ } & B \\
{\scriptstyle \beta_1}\downarrow & & \downarrow{\scriptstyle f_1} \\
C & \xrightarrow[\ g_1\ ]{} & D \\
{\scriptstyle \gamma_1}\downarrow & & \downarrow{\scriptstyle \sigma_1} \\
A & \xrightarrow[\ \delta_1\ ]{} & P
\end{array}
\qquad
\begin{array}{ccc}
E & \xrightarrow{\ \alpha_2\ } & F \\
{\scriptstyle \beta_2}\downarrow & & \downarrow{\scriptstyle f_2} \\
G & \xrightarrow[\ g_2\ ]{} & H \\
{\scriptstyle \gamma_2}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\
E & \xrightarrow[\ \delta_2\ ]{} & Q
\end{array}
$$

where the top squares are pullbacks and the bottom squares are pushouts and where $\gamma_1\beta_1 = 1_A$, $\gamma_2\beta_2 = 1_E$ and f_1, g_1, f_2 and g_s are all one to one. Suppose we have "connecting" monomorphisms $\varphi: A \to E$, $\vartheta: C \to G$, $\varepsilon: B \to F$ and $\psi: D \to H$ such that the completed diagram

$$\text{(*)}$$

commutes and such that

$$
\begin{array}{ccc}
C & \xrightarrow{\;g_1\;} & D \\
\vartheta \downarrow & & \downarrow \psi \\
G & \xrightarrow{\;g_2\;} & H
\end{array}
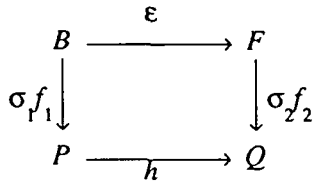\qquad
\begin{array}{ccc}
B & \xrightarrow{\;\varepsilon\;} & F \\
f_1 \downarrow & & \downarrow f_2 \\
D & \xrightarrow{\;\psi\;} & H
\end{array}
$$
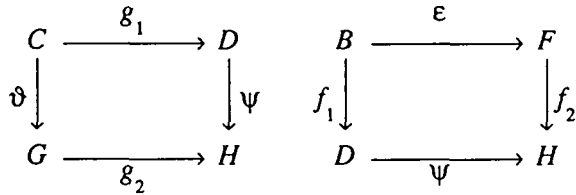
are pullbacks. Then there exists a unique monomorphism $h: P \to Q$ such that the bottom "cube" in (*) commutes and such that

$$
\begin{array}{ccc}
B & \xrightarrow{\;\varepsilon\;} & F \\
\sigma_1 f_1 \downarrow & & \downarrow \sigma_2 f_2 \\
P & \xrightarrow{\;h\;} & Q
\end{array}
$$

is also a pullback.

Recall that if U is a submonoid of a monoid S and if $X \in \text{ENS-}S$, $Y \in \text{ENS-}U$ and $f: X \to Y$ is a U-map then the *free S-extension of X and Y* consists of a right S-set, $F(S; X, Y)$, together with a right U-map $g: Y \to F(S; X, Y)$ such that the composite $g \circ f: X \to F(S; X, Y)$ is a right S-map and such that the pair $(F(S; X, Y), g)$ is universal with respect to this property. It was shown in Renshaw [9, Theorem 4.19] that $F(S; X, Y)$ is the pushout in the category ENS-S of the diagram

$$
\begin{array}{ccc}
X \otimes S & \longrightarrow & Y \otimes S \\
\downarrow & & \\
X & &
\end{array}
$$
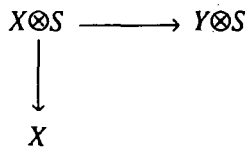
The idea behind the next two theorems is to take a diagram of the form

$$
\begin{array}{ccc}
A \longrightarrow & B \longrightarrow & F(T;A,B) \\
\downarrow & \downarrow & \downarrow \\
C \longrightarrow & D \longrightarrow & F(S;C,D)
\end{array}
$$

and show that for some monoids $U \subseteq T \subseteq S$, if the first square has a particular property, P, say, then so does the second square. This will form the basis for an inductive process in Section 2.

Theorem 1.9. *Let $U \subseteq T \subseteq S$ with $U \to T$ and $U \to S$ perfect as right U-monomorphisms. Then $T \to S$ is perfect if and only if the following condition holds*:

Whenever $A \in ENS\text{-}T$, $B, D \in ENS\text{-}U$ and $C \in ENS\text{-}S$ and whenever $\alpha_1 : A \to B$, $\alpha_2 : C \to D$, $\delta : A \to C$ and $\varepsilon : B \to D$ are perfect U-monomorphisms such that for all left U-sets X the commutative diagram

$$
\begin{array}{ccc}
A \otimes X & \xrightarrow{\alpha_1 \otimes 1_X} & B \otimes X \\
{\scriptstyle \delta \otimes 1_X} \downarrow & & \downarrow {\scriptstyle \varepsilon \otimes 1_X} \\
C \otimes X & \xrightarrow[\alpha_2 \otimes 1_X]{} & D \otimes X
\end{array}
$$

is a pullback, then the canonical maps $\beta_1 : B \to F(T; A, B)$ and $\beta_2 : D \to F(S; C, D)$ are perfect U-monomorphisms and there exists a unique perfect U-monomorphism $\psi : F(T; A, B) \to F(S; C, D)$ such that the commutative diagram

$$
\begin{array}{ccc}
B \otimes X & \xrightarrow{\beta_1 \otimes 1_X} & F(T;A,B) \otimes X \\
{\scriptstyle \varepsilon \otimes 1_X} \downarrow & & \downarrow {\scriptstyle \psi \otimes 1_X} \\
D \otimes X & \xrightarrow[\beta_2 \otimes 1_X]{} & F(S;C,D) \otimes X
\end{array}
$$

is also a pullback for all left U-sets X.

(In other words the property P mentioned above is: "all maps in the square are perfect and 'tensoring' on the right by any X gives a pullback".)

Proof. For the sake of brevity let $P = (T; A, B)$ and $Q = F(S; C, D)$. Suppose that the inclusion $T \to S$ is perfect. From Renshaw [**10**, Theorem V.1.3] (the proof is similar to Theorem 4.20 in Renshaw [**9**]) we can deduce immediately that the canonical maps $B \to P$ and $D \to Q$ are perfect monomorphisms. In particular, P and Q are right flat. We apply Lemma 1.8 to the diagram:

$$
\begin{array}{ccccccc}
A\otimes X & \longrightarrow & B\otimes X & & & & \\
\downarrow \searrow & & \downarrow \searrow & & & & \\
 & C\otimes X & \longrightarrow & D\otimes X & & & \\
A\otimes T\otimes X & \longrightarrow & B\otimes T\otimes X & & & & \\
\downarrow \searrow & & \downarrow \searrow & & \downarrow & & \\
 & C\otimes S\otimes X & \longrightarrow & D\otimes S\otimes X & & & \\
A\otimes X & \longrightarrow & P\otimes X & & & & \\
\searrow \downarrow & & \searrow & & \downarrow & & \\
 & C\otimes X & \longrightarrow & Q\otimes X & & &
\end{array}
$$
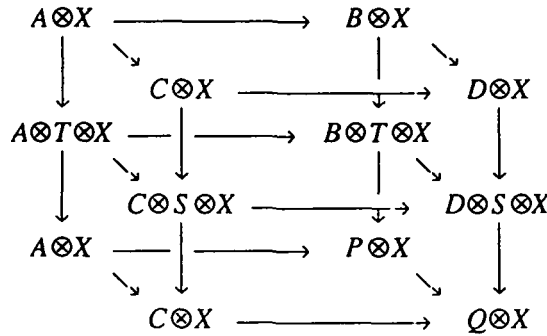
This is reasonably straightforward. The only part that requires special mention is that the commutative diagrams

(a)
$$
\begin{array}{ccc}
A\otimes X & \longrightarrow & B\otimes X \\
\downarrow & & \downarrow \\
A\otimes T\otimes X & \longrightarrow & B\otimes T\otimes X
\end{array}
$$

(b)
$$
\begin{array}{ccc}
C\otimes X & \longrightarrow & D\otimes X \\
\downarrow & & \downarrow \\
C\otimes S\otimes X & \longrightarrow & D\otimes S\otimes X
\end{array}
$$

(c)
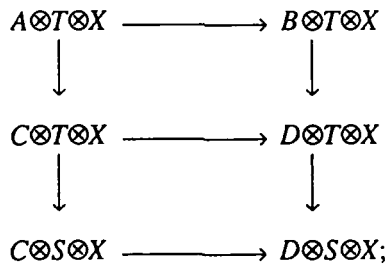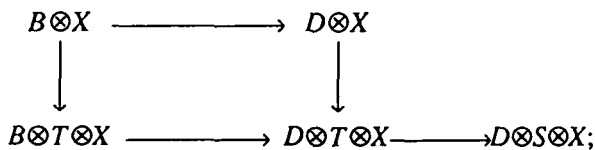$$
\begin{array}{ccc}
A\otimes T\otimes X & \longrightarrow & B\otimes T\otimes X \\
\downarrow & & \downarrow \\
C\otimes S\otimes X & \longrightarrow & D\otimes S\otimes X
\end{array}
$$

(d)
$$
\begin{array}{ccc}
B\otimes X & \longrightarrow & D\otimes X \\
\downarrow & & \downarrow \\
B\otimes T\otimes X & \longrightarrow & D\otimes S\otimes X
\end{array}
$$
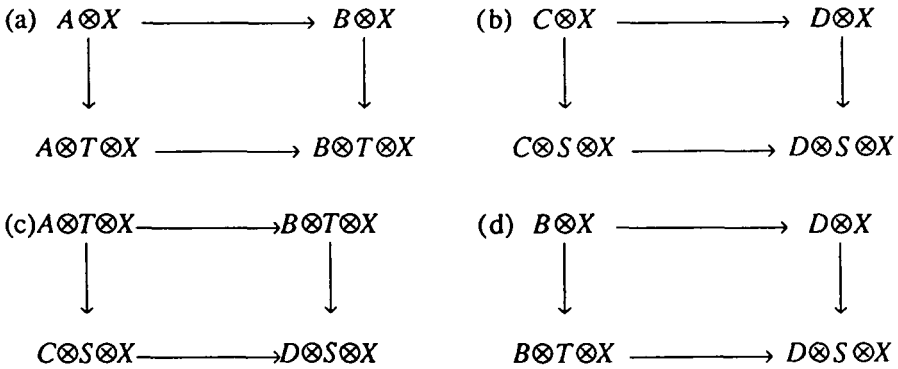
are all pullbacks. In fact diagrams (a) and (b) are pullbacks by Lemma 1.7; we can "redraw" diagram (c)

$$
\begin{array}{ccc}
A\otimes T\otimes X & \longrightarrow & B\otimes T\otimes X \\
\downarrow & & \downarrow \\
C\otimes T\otimes X & \longrightarrow & D\otimes T\otimes X \\
\downarrow & & \downarrow \\
C\otimes S\otimes X & \longrightarrow & D\otimes S\otimes X;
\end{array}
$$

the top square is a pullback, by assumption and the bottom square is a pullback by Lemma 1.7; diagram (d) can be "redrawn"

$$
\begin{array}{ccccc}
B\otimes X & \longrightarrow & D\otimes X & & \\
\downarrow & & \downarrow & & \\
B\otimes T\otimes X & \longrightarrow & D\otimes T\otimes X & \longrightarrow & D\otimes S\otimes X;
\end{array}
$$

the square is a pullback by Lemma 1.7 and the map $D \otimes T \otimes X \to D \otimes S \otimes X$ is one to one since $T \to S$ is right pure and D is right flat. Hence, by Lemma 1.8, we deduce that $P \otimes X \to Q \otimes X$ is one to one and so $P \to Q$ is perfect and the commutative diagram

$$
\begin{array}{ccc}
B \otimes X & \longrightarrow & P \otimes X \\
\downarrow & & \downarrow \\
D \otimes X & \longrightarrow & Q \otimes X
\end{array}
$$

is a pullback.

Conversely, let $A = T$, $B = T \cup U$, $C = T \otimes S$ and $D = T \otimes S \cup U$ in the statement of the theorem. Then it is a straightforward matter to check that the given conditions hold. Now it is easy to establish that $F(T; A, B) = T \cup T$ and $F(S; C, D) = T \otimes S \cup S$ and so we can deduce that $T \cup T \to T \otimes S \cup S$ is perfect and so in particular the inclusion $T \to S$ is perfect.

The next result is of a similar nature.

Theorem 1.10. *Let $U \subseteq T \subseteq S$ and suppose that S, T, S/U, T/U and S/T are all flat. Whenever $A \in ENS\text{-}T$, $B, D \in ENS\text{-}U$ and $C \in ENS\text{-}S$ and whenever $\alpha_1 : A \to B$, $\alpha_2 : C \to D$, $\delta : A \to C$ and $\varepsilon : B \to D$ are U-monomorphisms such that A, B/A, C, D/C, C/A and D/B are all quasi-flat and for all flat left U-sets X, the commutative diagram*

$$
\begin{array}{ccc}
A \otimes X & \longrightarrow & B \otimes X \\
\downarrow & & \downarrow \\
C \otimes X & \longrightarrow & D \otimes X
\end{array}
\tag{3}
$$

is a pullback, then there exists a unique U-monomorphism $\psi : F(T; A, B) \to F(S; C, D)$ such that B, $F(T; A, B)/B$, D, $F(S; C, D)/D$ and $F(S; C, D)/F(T; A, B)$ are all quasi-flat and the commutative diagram

$$
\begin{array}{ccc}
B \otimes X & \longrightarrow & F(T;A,B) \otimes X \\
\downarrow & & \downarrow \\
D \otimes X & \longrightarrow & F(S;C,D) \otimes X
\end{array}
\tag{4}
$$

is also a pullback for all flat left U-sets X.

Proof. Again we let $P = F(T; A, B)$ and $Q = F(S; C, D)$. It follows immediately from Renshaw [7, Theorem 4.15] that the canonical maps $B \to P$ and $D \to Q$ are one to one and that B, P/B, D, Q/D (and hence by Lemma 1.1(iv) P and Q) are all quasi-flat. If we apply Lemma 1.8 to diagram (2) as before we can deduce that there is a unique monomorphism $P \to Q$ such that (4) is a pullback. Finally, we need to show that Q/P is quasi-flat. To this end, notice that (D/B) is quasi-flat (by assumption) and so from

Lemma 1.1(iv) Q/P will be quasi-flat if we can show that $(Q/P)/(D/B)$ is quasi-flat. Now from Lemma 1.5 $(Q/P)/(D/B) \simeq (Q/D)/(P/B)$. Also, from Renshaw [7, Lemma 3.10] (see also Renshaw [7, the proof of Theorem 4.15]), $Q/D \simeq D \otimes S/L$ and $P/B \simeq B \otimes T/K$ where L and K are, respectively, the pushouts of the following diagrams

$$
\begin{array}{ccc}
C & \xrightarrow{\ \alpha_2\ } & D \\
\downarrow & & \downarrow{\vartheta_2} \\
C \otimes S & \xrightarrow{\ \tau_2\ } & L
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\ \alpha_1\ } & B \\
\downarrow & & \downarrow{\vartheta_1} \\
A \otimes T & \xrightarrow{\ \tau_1\ } & K
\end{array}
\qquad (5)
$$

With Lemma 1.6 in mind, consider the commutative diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ \varphi\ } & L \\
\gamma_1 \downarrow & & \downarrow \gamma_2 \\
B \otimes T & \xrightarrow{\ \varepsilon \otimes \iota\ } & D \otimes S
\end{array}
$$

It is straightforward to show that γ_1, γ_2 and φ are defined by

$$
\gamma_1(k) = \begin{cases} b \otimes 1 & ;\quad k = \vartheta_1(b) \\ \alpha_1(a) \otimes t; & k = \tau_1(a \otimes t) \end{cases}
\qquad
\gamma_2(l) = \begin{cases} d \otimes 1 & ;\quad l = \vartheta_2(d) \\ \alpha_2(c) \otimes s; & l = \tau_2(c \otimes s) \end{cases}
$$

$$
\varphi(k) = \begin{cases} \vartheta_2(\varepsilon(b)) & ;\quad k = \vartheta_1(b) \\ \tau_2(\delta(a) \otimes t); & k = \tau_1(a \otimes t) \end{cases}
$$

We need to show that this diagram is a pullback, that all the maps in the diagram are one to one and that $D \otimes S$, $(D \otimes S)/L$ and $(D \otimes S)/(B \otimes T)$ are all quasi-flat. That γ_1 and γ_2 are one to one follows from Renshaw [7, Lemma 3.9] and the fact that $(\alpha_1, U \to T)$ and $(\alpha_2, U \to S)$ are stable and $\varepsilon \otimes \iota$ is one to one since it is the composite of the one to one maps $B \otimes T \to D \otimes T$ and $D \otimes T \to D \otimes S$. To show that φ is one to one, suppose that $\varphi(k) = \varphi(k')$. There are essentially three cases to consider

(a) $k = \vartheta_1(b)$ and $k' = \vartheta_1(b')$;

(b) $k = \tau_1(a \otimes t)$ and $k' = \tau_1(a' \otimes t')$; and

(c) $k = \vartheta_1(b)$ and $k' = \tau_1(a' \otimes t')$.

In cases (a) and (b), we can easily deduce that $k = k'$ when we note that τ_2, ϑ_2, ε and $\delta \otimes 1$ are all one to one. In case (c) we have $\vartheta_2 \varepsilon(b) = \tau_2(\delta(a') \otimes t')$ and so, since the diagrams in (5) are also pullback diagrams, there exists c in C with $\varepsilon(b) = \alpha_2(c)$ in D and $\delta(a') \otimes t' = c \otimes 1$ in $C \otimes S$ (and so in $C \otimes T$). Since (3) is a pullback when $X = U$, we see

that $b = \alpha_1(a_1)$ and $c = \delta(a_1)$ for some a_1 in A. Hence $a' \otimes t' = a_1 \otimes 1$ in $A \otimes T$ and so $k = \vartheta_1(b) = \vartheta_1 \alpha_1(a_1) = \tau_1(a_1 \otimes 1) = \tau_1(a' \otimes t') = k'$ as required. Now $D \otimes S$ is quasi-flat since D is quasi-flat and S is flat, $(D \otimes S)/L$ is quasi-flat since it is isomorphic to Q/D and from Lemma 1.4(iii) we see that $(D \otimes S)/(B \otimes T)$ is quasi-flat since S and S/T are flat and B and D/B are quasi-flat. Finally, we show that the above diagram is a pullback. Suppose then that $\gamma_2(l) = \varepsilon(b) \otimes t$ in $D \otimes S$. Then either

  (a) $d \otimes 1 = \varepsilon(b) \otimes t$ if $l = \vartheta_2(d)$, or

  (b) $\alpha_2(c) \otimes s = \varepsilon(b) \otimes t$ if $l = \tau_2(c \otimes s)$.

In case (a), $d \otimes 1 = \varepsilon(b) \otimes t$ in $D \otimes T$ since D is quasi-flat and so since ε is quasi-stable (and so $(\varepsilon, U \to T)$ is stable) then $\varepsilon(b) \otimes t = \varepsilon(b') \otimes 1$ in $D \otimes T$ for some b' in B. Hence $b \otimes t = b' \otimes 1 = \gamma_1(\vartheta_1(b'))$ and $l = \vartheta_2(d) = \vartheta_2(\varepsilon(b')) = \varphi(\vartheta_1(b'))$ as required.

  In case (b), since α_2 is quasi-stable, then $\varepsilon(b) \otimes t = \alpha_2(c_1) \otimes t_1$ in $D \otimes S$ (and so in $D \otimes T$). Notice also that $c \otimes s = c_1 \otimes t_1$ in $C \otimes S$. Now, putting $X = T$ in diagram (3) we deduce that $b \otimes t = \alpha_1(a_2) \otimes t_2 = \gamma_1(\tau_1(a_2 \otimes t_2))$ and $c_1 \otimes t_1 = \delta(a_2) \otimes t_2$ for some a_2 in A, t_2 in T. Consequently $\varphi(\tau_1(a_2 \otimes t_2)) = \tau_2(\delta(a_2) \otimes t_2) = \tau_2(c_1 \otimes t_1) = \tau_2(c \otimes s) = l$.
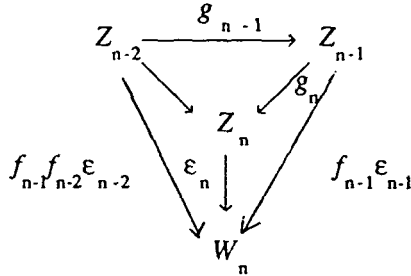
  Hence, by Lemma 1.6, $(D \otimes S/L)/(B \otimes T/K) \simeq (Q/D)/(P/B) \simeq (Q/P)/(D/B)$ is quasi-flat and the theorem is proved.

2. Subsemigroups of free products

  Let $[U; S_1, S_2]$ be an amalgam of monoids. Define $W_1 = S_1$, $W_2 = S_1 \otimes S_2$ and $f_1: W_1 \to W_2$ by $f_1(s_1) = s_1 \otimes 1$. Now define, inductively, $W_n = F(S_i; W_{n-2}, W_{n-1})$ ($i \equiv n \pmod 2$) and let $f_{n-1}: W_{n-1} \to W_n$ be the canonical map. It was shown in Renshaw [9] that there exist maps $\varphi_n: W_n \to S_1 *_U S_2$ for $n = 1, 2, \ldots$ such that $\varphi_n \circ f_{n-1} = \varphi_{n-1}$ ($n \geq 2$) and such that the system $(S_1 *_U S_2, \varphi_n)$ is the direct limit in the category of U-sets of (W_n, f_n). It is clear that if $[U; T_1, T_2]$ is an amalgam of submonoids then a similar construction, say (Z_n, g_n), can be made. So we can construct a commutative diagram
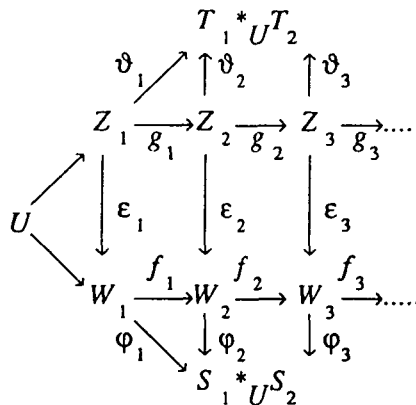
where $\varepsilon_1 : Z_1 \to W_1$ is the inclusion, $\varepsilon_2 : Z_2 \to W_2$ is given by $\varepsilon_2(t_1 \otimes t_2) = t_1 \otimes t_2$, and in general $\varepsilon_n : Z_n \to W_n$ is the unique T_i-map $(i \equiv n(\mathrm{mod}\, 2))$ which makes the diagram

$$
\begin{array}{ccc}
Z_{n-2} & \xrightarrow{\;\;g_{n-1}\;\;} & Z_{n-1} \\
& \searrow \quad Z_n \quad \swarrow g_n & \\
f_{n-1}f_{n-2}\varepsilon_{n-2} & \varepsilon_n \downarrow & f_{n-1}\varepsilon_{n-1} \\
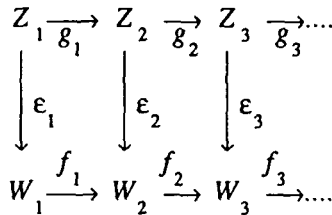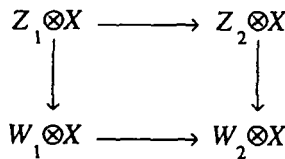& W_n & \\
\end{array}
$$

commute.

If we denote the canonical map $T_1 *_U T_2 \to S_1 *_U S_2$ by ψ, it is easy to show (see Renshaw [8]) that if each ε_i is one to one then so is ψ. Our aim therefore is to show that under certain conditions imposed on U, S_i and T_i, each ε_i is a U-monomorphism.

Theorem 2.1.   Let $[U; T_1, T_2]$ be an amalgam of submonoids of an amalgam $[U; S_1, S_2]$ such that the inclusions $U \to T_i$ and $T_i \to S_i$ are perfect right U-monomorphisms, $i = 1, 2$. Then the canonical map $T_1 *_U T_2 \to S_1 *_U S_2$ is also a perfect U-monomorphism.

Proof.   Consider the commutative diagram

$$
\begin{array}{ccccccc}
Z_1 & \xrightarrow{\;g_1\;} & Z_2 & \xrightarrow{\;g_2\;} & Z_3 & \xrightarrow{\;g_3\;} & \cdots \\
\downarrow{\scriptstyle \varepsilon_1} & & \downarrow{\scriptstyle \varepsilon_2} & & \downarrow{\scriptstyle \varepsilon_3} & & \\
W_1 & \xrightarrow{\;f_1\;} & W_2 & \xrightarrow{\;f_2\;} & W_3 & \xrightarrow{\;f_3\;} & \cdots \\
\end{array}
$$

We see that we need only establish the conditions of Theorem 1.9 for the first square and it will then follow by induction that each ε_i is perfect. It is easy to establish that f_1 and g_1 are perfect monomorphisms. For example, since S_1 is flat then f_1 is one to one; since S_1 and S_2 are both flat then $W_2 = S_1 \otimes S_2$ is flat and since the inclusion $U \to S_2$ is right pure and S_1 is right flat then f_1 is right pure. Now ε_1 is perfect by assumption. It is easy to check that the tensor product of two perfect monomorphisms is another perfect monomorphism and so ε_2 is perfect. Finally, if $X \in U$-ENS, then the commutative diagram

$$
\begin{array}{ccc}
Z_1 \otimes X & \longrightarrow & Z_2 \otimes X \\
\downarrow & & \downarrow \\
W_1 \otimes X & \longrightarrow & W_2 \otimes X \\
\end{array}
$$

is a pullback, since it can be "redrawn" as

$$
\begin{array}{ccc}
T_1 \otimes X & \longrightarrow & T_1 \otimes T_2 \otimes X \\
\downarrow & & \downarrow \\
S_1 \otimes X & \longrightarrow & S_1 \otimes T_2 \otimes X \longrightarrow S_1 \otimes S_2 \otimes X
\end{array}
$$

The square is a pullback, by Lemma 1.7 and since $T_2 \to S_2$ is right pure and S_1 is right flat then the map $S_1 \otimes T_2 \otimes X \to S_1 \otimes S_2 \otimes X$ is one to one. So the conditions of Theorem 1.9 hold and therefore each ε_i is a perfect U-monomorphism. It now follows that the map $T_1 *_U T_2 \to S_1 *_U S_2$ is pure and since it is known that $S_1 *_U S_2$ is also flat (Renshaw [10, V.1.4] and Bulman-Fleming and McDowell [1, Theorem 4]) the result then follows. (Note that direct limits of flat U-sets are flat.)

Using the well-known fact that amalgamated free products are associative, it is then an easy matter to extend this result to amalgams with a finite index set and then to an arbitrary index set using Renshaw [8, Theorem 1.2].

The next theorem can be proved in a similar way using Theorem 1.10 in place of Theorem 1.9. The details have been omitted.

Theorem 2.2. *Let $[U; T_1, T_2]$ be an amalgam of submonoids of an amalgam $[U; S_1, S_2]$ such that T_i, T_i/U, S_i, S_i/U and S_i/T_i are all flat for $i = 1, 2$. Then the canonical map $T_1 *_U T_2 \to S_1 *_U S_2$ is a U-monomorphism.*

Say that a monoid U is a *perfect amalgamation base* if

(1) U is an amalgamation base, and

(2) whenever $[U; T_i]$ is an amalgam of submonoids of $[U; S_i]$ then $\Pi_U^* T_i \to \Pi_U^* S_i$ is one to one.

It is now clear that if U is absolutely flat (in other words, every U-set is flat) then U is a perfect amalgamation base. Since it is known that inverse semigroups are absolutely flat we can deduce:

Corollary 2.3. *Inverse semigroups are perfect amalgamation bases.*

It is easy to show that U is absolutely flat if and only if U is absolutely perfect (i.e. the inclusion $U \to S$ is perfect as a right and left U-map, for every containing monoid, S) (see also, for example, Renshaw [9 or 10]). Hence absolutely perfect monoids are perfect amalgamation bases. The converse of this result was established in Renshaw [8]. Note, however, that we need the inclusions to be perfect *both* as left U-maps *and* right U-maps. For example the two element right zero semigroup, U, is such that $U \to S$ is always perfect as a left U-map but not always as a right U-map (see Hall [2]). Since perfect amalgamation bases need have both these properties, we deduce that the two element right zero semigroup is not a perfect amalgamation base. Other examples of

amalgamation bases that are not perfect amalgamation bases can be found among the cyclic semigroups. Hall [2] has shown that every finite cyclic semigroup is an amalgamation base and that the only cyclic semigroups that are absolutely perfect are the groups. Hence the only cyclic, perfect amalgamation bases are the cyclic groups.

Since inverse semigroups are perfect amalgamation bases, it is natural to ask: if $[U; T_i]$ is an amalgam of subsemigroups of the amalgam $[U; S_i]$ and if U, T_i and S_i are all inverse, is it true that $\mathcal{T}*_U T_i$ the free inverse product of the amalgam $[U; T_i]$, is embeddable in $\mathcal{T}*_U S_i$ the free inverse product of the amalgam $[U; S_i]$? The answer is in the affirmative. To see this we note that in any class of semigroups with amalgamated free products, it is straightforward to check the following,

  (i) $(A*_U B)*_S C \cong A*_U(B*_S C)$,

  (ii) $U*_U D \cong D$,

whenever U, S, A, B, C and D are semigroups such that $U \subseteq A, B, D$ and $S \subseteq B, C$. Now suppose that $\mathscr{C}$ is a class of semigroups with amalgamated free products (e.g. inverse semigroups) and suppose that $[U; T_1, T_2]$ is an amalgam of subsemigroups of $[U; S_1, S_2]$ where all the semigroups belong to $\mathscr{C}$. Then using the above isomorphisms we can deduce that $S_1*_U S_2 \cong ((T_1*_U T_2)*_{T_2}S_2)*_{T_1}S_1$ and so if, in $\mathscr{C}$, U, T_1 and T_2 are all amalgamation bases (e.g. if they are all inverse) it follows that $T_1*_U T_2$ can be embedded in $S_1*_U S_2$.

3. Right generalised inverse semigroups

Lemma 3.1 (Bulman-Fleming and McDowell [1, Lemma 1.2]). *$x \otimes y = x' \otimes y'$ in $X \otimes Y$ if and only if there exists $x_1, \ldots, x_n$ in X, $y_2, \ldots, y_n$ in Y, $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ in U such that*

$$
\begin{aligned}
x = x_1 u_1 \qquad\qquad u_1 y = v_1 y_2 \\
x_1 v_1 = x_2 u_2 \qquad\qquad u_2 y_2 = v_2 y_3 \\
\cdots\cdots \\
x_{n-1} v_{n-1} = x_n u_n \qquad\qquad u_n y_n = y' \\
x_n v_n = x'
\end{aligned}
$$

A set of equations such as those above is called a *U-scheme over X and Y joining* (x, y) *to* (x', y').

In what follows we will denote by ${}^1U$ the monoid obtained from the semigroup U by adjoining an identity, 1, regardless of whether U already has an identity. See Renshaw [9] for results concerning amalgams of this type of monoid. A semigroup, S, is called *right generalised inverse* if it is regular and $efg = feg$ for all idempotents e, f and g. Hall [3, Theorem 4] proved that if U is a regular subsemigroup of a semigroup S such that $efs = fes$ for all idempotents $e, f \in S$ and for all $s \in S$, then ${}^1U \to {}^1S$ is perfect as a left ${}^1U$-map. In particular, this will be true if U is an inverse semigroup or if both U and S are right generalised inverse semigroups. We extend this result as follows:

Theorem 3.2. *Let U be a regular subsemigroup of a semigroup T and T a subsemigroup of a semigroup S and suppose that $efs = fes$ for all $s \in S$, and all $e, f \in E(T)$. Then the inclusion $^1T \to {}^1S$ is a left pure ^1U-monomorphism.*

Proof.   Let $Z \in \mathrm{ENS}\text{-}^1U$ and suppose that $z \otimes t = z' \otimes t'$ in $Z \otimes_{1_U} {}^1S$. Then we can construct a ^1U-scheme of the form

$$
\begin{array}{ll}
z = z_1 u_1 & u_1 t = v_1 s_2, \\[4pt]
z_1 v_1 = z_2 u_2 & u_2 s_2 = v_2 s_3, \\[4pt]
\multicolumn{2}{c}{\cdots\cdots} \\[4pt]
z_{n-1} v_{n-1} = z_n u_n & u_n s_n = v_n t' \\[4pt]
z_n v_n = z'
\end{array}
$$

where $z_i \in Z$, $u_i, v_i \in {}^1U$, $s_i \in {}^1S$ and $t, t' \in {}^1T$. We can assume, without loss of generality, that each $u_i, v_i \in U$ (otherwise a smaller ^1U-scheme could be formed). We therefore need to consider three cases:

Case I, $t, t' \in T$ and each $s_i \in S$.
Let us define

$$
x_0 = 1,\ x_1 = v_1^{-1},\ x_i = v_i^{-1} u_i x_{i-1} \qquad 2 \leqq i \leqq n
$$

$$
y_0 = 1,\ y_1 = u_n^{-1},\ y_i = u_{n-i+1}^{-1} v_{n-i+1} y_{i-1} \qquad 2 \leqq i \leqq n,
$$

where u_i^{-1} and v_i^{-1} are fixed inverses (in U) of u_i and v_i. So in other words,

$$
x_i = v_i^{-1} u_i v_{i-1}^{-1} u_{i-1} \ldots v_2^{-1} u_2 v_1^{-1},
$$

$$
y_i = u_{n-i+1}^{-1} v_{n-i+1} u_{n-i+2}^{-1} \ldots u_{n-1}^{-1} v_{n-1} u_n^{-1}.
$$

Define

$$
x_0^{-1} = 1,\quad x_1^{-1} = v_1,\quad x_i^{-1} = v_1 u_2^{-1} \ldots u_i^{-1} v_i,
$$

$$
y_0^{-1} = 1,\quad y_1^{-1} = u_n,\quad y_i^{-1} = u_n v_{n-1}^{-1} \ldots v_{n-i+1}^{-1} u_{n-i+1}.
$$

We shall prove:

Lemma.

  (i) $u_1 t = v_1 x_1 u_1 t,$

  (ii) $u_i x_{i-1} u_1 t = u_i x_{i-1} x_i^{-1} s_{i+1}, \qquad 2 \leqq i \leqq n \quad (\text{where } s_{n+1} = t'),$

  (iii) $u_i x_{i-1} u_1 t = v_i x_i u_1 t, \qquad\qquad 2 \leqq i \leqq n,$

(iv) $v_n t' = u_n y_1 v_n t'$,

(v) $v_i y_{n-i} v_n t' = v_i y_{n-i} y_{n-i+1}^{-1} s_i$, $1 \leqq i \leqq n-1$ (where $s_1 = t$),

(vi) $v_i y_{n-i} v_n t' = u_i y_{n-i+1} v_n t'$, $1 \leqq i \leqq n-1$,

(vii) $v_n x_n u_1 t = u_n x_{n-1} x_n^{-1} t'$,

(viii) $v_i x_i x_n^{-1} t' = u_i x_{i-1} x_n^{-1} t'$, $2 \leqq i \leqq n-1$,

(ix) $v_1 x_1 x_n^{-1} t' = v_1 y_{n-1} v_n t'$.

Proof of lemma.

(i) $u_1 t = v_1 s_2 = v_1 v_1^{-1} v_1 s_2 = v_1 v_1^{-1} u_1 t = v_1 x_1 u_1 t$.

(ii) We use induction. When $i = 2$, we have

$$u_2 x_1 u_1 t = u_2 v_1^{-1} u_1 t = u_2 v_1^{-1} v_1 s_2 = u_2 u_2^{-1} u_2 v_1^{-1} v_1 s_2$$

$$= u_2 v_1^{-1} v_1 u_2^{-1} u_2 s_2 = u_2 x_1 v_1 u_2^{-1} v_2 s_3 = u_2 x_1 x_2^{-1} s_3.$$

Now assume the result is true for $i = k$, say and consider

$$u_{k+1} x_k u_1 t = u_{k+1} v_k^{-1} u_k x_{k-1} u_1 t = u_{k+1} v_k^{-1} u_k x_{k-1} x_k^{-1} s_{k+1}, \text{ by induction,}$$

$$= u_{k+1} u_{k+1}^{-1} u_{k+1} v_k^{-1} u_k x_{k-1} x_k^{-1} s_{k+1} = u_{k+1} u_{k+1}^{-1} u_{k+1} x_k x_k^{-1} s_{k+1}$$

$$= u_{k+1} x_k x_k^{-1} u_{k+1}^{-1} u_{k+1} s_{k+1} = u_{k+1} x_k x_k^{-1} u_{k+1}^{-1} v_{k+1} s_{k+2}$$

$$= u_{k+1} x_k x_{k+1}^{-1} s_{k+2}.$$

The result follows by induction.

(iii) $u_i x_{i-1} u_1 t = u_i x_{i-1} x_i^{-1} s_{i+1} = u_i x_{i-1} x_{i-1}^{-1} u_i^{-1} v_i s_{i+1}$,

$$= u_i x_{i-1} x_{i-1}^{-1} u_i^{-1} v_i v_i^{-1} v_i s_{i+1} = v_i v_i^{-1} u_i x_{i-1} x_{i-1}^{-1} u_i^{-1} v_i s_{i+1}$$

$$= v_i v_i^{-1} u_i x_{i-1} u_1 t = v_i x_i u_1 t.$$

(iv) $v_n t' = u_n s_n = u_n u_n^{-1} u_n s_n = u_n u_n^{-1} v_n t' = u_n y_1 v_n t'$.

(v) When $i = n-1$, we have

$$v_{n-1} y_1 v_n t' = v_{n-1} u_n^{-1} v_n t' = v_{n-1} u_n^{-1} u_n s_n = v_{n-1} v_{n-1}^{-1} v_{n-1} u_n^{-1} u_n s_n$$

$$= v_{n-1} u_n^{-1} u_n v_{n-1}^{-1} v_{n-1} s_n = v_{n-1} y_1 u_n v_{n-1}^{-1} u_{n-1} s_{n-1},$$

$$= v_{n-1} y_1 y_2^{-1} s_{n-1}.$$

Assume the result is true for $i = k$ and consider

$$v_{k-1}y_{n-k+1}v_nt' = v_{k-1}u_k^{-1}v_ky_{n-k}v_nt',$$

$$= v_{k-1}u_k^{-1}v_ky_{n-k}y_{n-k+1}^{-1}s_k, \quad \text{by induction}$$

$$= v_{k-1}v_{k-1}^{-1}v_{k-1}y_{n-k+1}y_{n-k+1}^{-1}s_k,$$

$$= v_{k-1}y_{n-k+1}y_{n-k+1}^{-1}v_{k-1}^{-1}v_{k-1}s_k,$$

$$= v_{k-1}y_{n-k+1}y_{n-k+1}^{-1}v_{k-1}^{-1}u_{k-1}s_{k-1},$$

$$= v_{k-1}y_{n-k+1}y_{n-k+2}^{-1}s_{k-1}.$$

The result follows by induction.

(vi)
$$v_iy_{n-i}v_nt' = v_iy_{n-i}y_{n-i+1}^{-1}s_i = v_iy_{n-i}y_{n-i}^{-1}v_i^{-1}u_is_i,$$

$$= v_iy_{n-i}y_{n-i}^{-1}v_i^{-1}u_iu_i^{-1}u_is_i = u_iu_i^{-1}v_iy_{n-i}y_{n-i}^{-1}v_i^{-1}u_is_i,$$

$$= u_iu_i^{-1}v_iy_{n-i}v_nt' = u_iy_{n-i+1}v_nt'.$$

(vii)
$$v_nx_nu_1t = u_nx_{n-1}x_n^{-1}t' \qquad \text{by (ii) and (iii).}$$

(viii) First, notice that $x_n^{-1} = x_i^{-1}y_{n-i}v_n$, $1 \le i \le n-1$.

$$v_ix_ix_n^{-1}t' = v_iv_i^{-1}u_ix_{i-1}x_i^{-1}y_{n-i}v_nt' = v_iv_i^{-1}u_ix_{i-1}x_{i-1}^{-1}u_i^{-1}v_iy_{n-i}v_nt',$$

$$= u_ix_{i-1}x_{i-1}^{-1}u_i^{-1}v_iv_i^{-1}v_iy_{n-i}v_nt' = u_ix_{i-1}x_{i-1}^{-1}u_i^{-1}v_iy_{n-i}v_nt'$$

$$= u_ix_{i-1}x_i^{-1}y_{n-i}v_nt' = u_ix_{i-1}x_n^{-1}t'.$$

(ix)
$$v_1x_1x_n^{-1}t' = v_1v_1^{-1}x_1^{-1}y_{n-1}v_nt', \quad \text{since } x_n^{-1} = x_1^{-1}y_{n-1}v_n,$$

$$= v_1v_1^{-1}v_1y_{n-1}v_nt' = v_1y_{n-1}v_nt'.$$

If we now return to the proof of the theorem and put $a_i = x_iu_1t$, $b_i = x_ix_n^{-1}t'$ and $c_i = y_{n-i+1}v_nt'$, then it is easy to check that the following is a ^1U-scheme over Z and ^1T joining (z, t) and (z', t') and so $z \otimes t = z' \otimes t'$ in $Z \otimes_{1U} {}^1T$ as required.

$$z = z_1u_1 \qquad\qquad u_1t = v_1a_1$$

$$z_1v_1 = z_2u_2 \qquad\qquad u_2a_1 = v_2a_2$$

$$\ldots\ldots$$

$$z_i v_i = z_{i+1} u_{i+1} \qquad\qquad u_{i+1} a_i = v_{i+1} a_{i+1}$$

$$\ldots\ldots$$

$$z_{n-2} v_{n-2} = z_{n-1} u_{n-1} \qquad\qquad u_{n-1} a_{n-2} = v_{n-1} a_{n-1}$$

$$z_{n-1} v_{n-1} = z_n u_n \qquad\qquad u_n a_{n-1} = u_n b_{n-1}$$

$$z_n u_n = z_{n-1} v_{n-1} \qquad\qquad v_{n-1} b_{n-1} = u_{n-1} b_{n-2}$$

$$\ldots\ldots$$

$$z_i u_i = z_{i-1} v_{i-1} \qquad\qquad v_{i-1} b_{i-1} = u_{i-1} b_{i-2}$$

$$\ldots\ldots$$

$$z_3 u_3 = z_2 v_2 \qquad\qquad v_2 b_2 = u_2 b_1$$

$$z_2 u_2 = z_1 v_1 \qquad\qquad v_1 b_1 = v_1 c_2$$

$$z_1 v_1 = z_2 u_2 \qquad\qquad u_2 c_2 = v_2 c_3$$

$$\ldots\ldots$$

$$z_i v_i = z_{i+1} u_{i+1} \qquad\qquad u_{i+1} c_{i+1} = v_{i+1} c_{i+2}$$

$$\ldots\ldots$$

$$z_{n-1} v_{n-1} = z_n u_n \qquad\qquad u_n c_n = v_n t'$$

$$z_n v_n = z'$$

Case II   Some of the $s_i = 1$.

Suppose that i is the smallest index with $s_i = 1$ and that j is the largest index with $s_j = 1$. Put $x = v_{i-1} x_{i-1} x_{i-1}^{-1}$ and $y = u_j y_{n-j+1} y_{n-j+1}^{-1}$ and notice that $x, y \in U$. Then it is straightforward to check that the following is a $^1 U$-scheme over Z and $^1 T$:

$$z = z_1 u_1 \qquad\qquad u_1 t = v_1 a_1$$

$$z_1 v_1 = z_2 u_2 \qquad\qquad u_2 a_1 = v_2 a_2$$

$$\ldots\ldots$$

$$z_{i-2} v_{i-2} = z_{i-1} u_{i-1} \qquad u_{i-1} a_{i-2} = x \cdot 1$$

$$z_{i-1} x = (z_{i-1} x)$$

and so we have $z \otimes t = z_{i-1} x \otimes 1$ in $Z \otimes {}^1 T$. In a similar way it is again easy to check the following is a $^1 U$-scheme over Z and $^1 T$:

$$z' = z_n v_n \qquad\qquad v_n t' = u_n c_n$$

$$z_n u_n = z_{n-1} v_{n-1} \qquad\qquad v_{n-1} c_n = u_{n-1} c_{n-1}$$

$$\ldots\ldots$$

$$z_{j+1} u_{j+1} = z_j v_j \qquad\qquad v_j c_{j+1} = y \cdot 1$$

$$z_j y = (z_j y)$$

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and so $z' \otimes t' = z_j y \otimes 1$ in $Z \otimes^1 T$. But $z \otimes t = z' \otimes t'$ in $Z \otimes^1 S$ and so $z_{i-1} x \otimes 1 = z_j y \otimes 1$ in $Z \otimes^1 S$. By Hall [3, Theorem 2] we deduce that $z_{i-1} x = z_j y$ and so, in $Z \otimes^1 T$ we have $z \otimes t = z_{i-1} x \otimes 1 = z_j y \otimes 1 = z' \otimes t'$ in $Z \otimes^1 T$, as required.

Case III $t - 1$ *or* $t' = 1$.

Suppose, without loss of generality that $t' = 1$. If $t = 1$, then by Hall [3, Theorem 2] it follows that $z = z'$ and so $z \otimes 1 = z' \otimes 1$ in $Z \otimes^1 T$. If $t \neq 1$, and $s_i = 1$ for some i, then by the above argument, there exists $z'' \in Z$ with $z \otimes t = z'' \otimes 1$ in $Z \otimes^1 T$. Hence $z' \otimes 1 = z'' \otimes 1$ in $Z \otimes^1 S$ and so by Hall [3, Theorem 2] again, $z' = z''$ giving $z \otimes t = z' \otimes 1$ in $Z \otimes^1 T$ as required. Suppose then that $t \neq 1$ and that each $s_i \neq 1$. In this case we see that the original scheme can be rewritten as:

$$z = z_1 u_1 \qquad\qquad u_1 t = v_1 s_2,$$
$$z_1 v_1 = z_2 u_2 \qquad\qquad u_2 s_2 = v_2 s_3,$$
$$\cdots\cdots$$
$$z_{n-1} v_{n-1} = z_n u_n \qquad\qquad u_n s_n = v_n (v_n^{-1} v_n),$$
$$z_n v_n = z'$$

and so using the procedure in Case I, we can deduce that $z \otimes t = z' \otimes v_n^{-1} v_n$ in $Z \otimes^1 T$. But $z' \otimes v_n^{-1} v_n = z' v_n^{-1} v_n \otimes 1 = z_n v_n v_n^{-1} v_n \otimes 1 = z_n v_n \otimes 1 = z' \otimes 1$ in $Z \otimes^1 T$. The proof of Theorem 3.2 is complete.

Using Hall [3, Theorem 4] and the dual of Theorem 2.1, we can now deduce immediately:

Theorem 3.3. *Let U be a regular semigroup and let $[U; T_i]$ be an amalgam of subsemigroups of an amalgam $[U; S_i]$ such that $e_i f_i s_i = f_i e_i s_i$ for all i and for all $s_i \in S_i$ and all idempotents $e_i, f_i \in T_i$. Then $\Pi_U^* T_i \to \Pi_U^* S_i$ is one to one. In particular, the conclusion holds if U is inverse or if U, T_i and S_i are all right generalised inverse.*

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