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ON THE GLOBAL EXISTENCE OF SOLUTIONS TO QUASILINEAR PARABOLIC EQUATIONS*

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Abstract. We prove, under quite general assumptions, the global existence of classical solutions for quasilinear parabolic equations in bounded domains with homogeneous Dirichlet boundary conditions. The same results for weakly coupled reaction-diffusion systems are also given.

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1. Introduction. Consider the following quasilinear parabolic problems

$$\begin{cases} u_t - \operatorname{div}(a(t,x,u)\nabla u) = f(t,x,u,\nabla u), & t>0,\ x\in\Omega, \\ u(t,x)=0, & t>0,\ x\in\partial\Omega, \\ u(0,x)=\varphi(x), & x\in\overline{\Omega}, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary.

Many authors have studied problem (1) (see [**1**, **3**, **4**] and citations therein) by discussing the existence and uniqueness of local solutions, the global existence of solutions, blow-up behavior of solutions, and so on. Due to the difficulty caused by the nonlinearities f and a, these problems for (1) still need to be investigated.

Our aim is to prove the global existence of classical solutions for problem (1). Our method is not based on maximum principles and the comparison method, but relies on the recent result for the time evolution of the extrema of a function [**1**]. This approach permits us to obtain a new result for (1) under quite general assumptions on the nonlinearities. This approach is also applicable to prove the global existence of classical solutions to weakly reaction-diffusion systems. The obtained results improve the recent results in [**1**, **6**].

Throughout this paper, we assume that

$$a \in C^2(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R}, \mathbb{R}^{n\times n});\ f \in C^1(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}),$$

and

$$\langle a(t,x,w)\eta,\ \eta\rangle \geq c|\eta|^2, \quad (t,x,w) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R},\ \eta \in \mathbb{R}^n,$$

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where $\mathbb{R}_+ = [0, \infty)$ and $c > 0$. Fix $p > n$. It is known from the classical parabolic theory [**3**, **4**] that, given any initial data $\varphi \in W_0^{s,p}(\Omega)$ with $s \in [1, \min\{1 + \frac{1}{p},\ 2 - \frac{n}{p}\})$, there exists a maximal $T = T(\varphi) > 0$ and a unique solution

$$u \in C^1((0, T); C(\overline{\Omega})) \cap C((0, T); C^2(\overline{\Omega})) \cap C\big([0, T); W_0^{s,p}(\Omega)\big).$$

Moreover, if $T < \infty$, one has

$$\limsup_{t \uparrow T} \|u(t, \cdot)\|_{L^\infty(\Omega)} = \infty$$

if the nonlinearity f satisfies the growth condition

$$|f(t, x, w, \eta)| \leq h(t, w)(1 + |\eta|^2), \tag{2}$$

where $(t, x, w, \eta) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ and some $h \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$.

2. Proof of the main result. We prove that under certain conditions on the nonlinearity f the solutions to (1) are global in time for any initial data $\varphi \in W_0^{s,p}(\Omega)$.

We will need the following lemma in the sequel.

LEMMA. [**1**] *Let $T > 0$ and $u \in W^{1,1}((0, T); C(\overline{\Omega}))$, where Ω is a bounded domain in $\mathbb{R}^n$. Then for every $t \in (0, T)$ there exists at least one pair of points $\xi(t),\ \zeta(t) \in \overline{\Omega}$ with*

$$m(t) := \min_{x \in \overline{\Omega}}[u(t, x)] = u(t, \xi(t)), \qquad M(t) := \max_{x \in \overline{\Omega}}[u(t, x)] = u(t, \zeta(t)),$$

and the functions $m(t),\ M(t)$ are almost everywhere differentiable on $(0, T)$ with

$$\frac{dm}{dt}(t) = u_t(t, \xi(t)) \quad \text{and} \quad \frac{dM}{dt}(t) = u_t(t, \zeta(t)), \qquad \text{a.e. on } (0, T).$$

We now present the main result of this paper.

THEOREM. *Assume that f satisfies (2) and the growth condition*

$$h(t, w) \leq \theta(t)\mu(w),$$

with some $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\mu \in C(\mathbb{R}, (0, \infty))$. Furthermore, let $\varphi \in W_0^{s,p}(\Omega)$. If for all $T > 0$

$$\int_0^T \theta(s)\, ds < \min\left\{\int_{-\infty}^{m(0)} \frac{ds}{\mu(s)},\ \int_{M(0)}^{\infty} \frac{ds}{\mu(s)}\right\},$$

where $m(0) = \min_{x \in \overline{\Omega}}\{\varphi(x)\}$ and $M(0) = \max_{x \in \overline{\Omega}}\{\varphi(x)\}$, then the corresponding unique classical solution to (1) is defined globally in time.

Proof. Let u be the classical solution of (1) with the initial data φ and let $T > 0$ be the maximal existence time of u. Set

$$m(t) := \min_{x \in \overline{\Omega}}[u(t, x)], \qquad M(t) := \max_{x \in \overline{\Omega}}[u(t, x)],$$

Define the open set $I = \{t \in (0, T) :\ M(t) > 0\}$. If $I \neq \emptyset$, then for $t \in I$, $M(t)$ is attained at some interior point $\zeta(t) \in \Omega$ and $\zeta(t) \notin \partial\Omega$. Therefore, it follows that (see [**2**]) $\nabla u(t, \zeta(t)) = 0$ and

$$\sum_{i,j=1}^{n} a_{ij}(t, \zeta(t), u(t, \zeta(t))) \frac{\partial^2 u(t, \zeta(t))}{\partial x_i \partial x_j} \le 0$$

In view of the growth conditions of f, applying the above Lemma and using the equation (1), we get

$$\begin{aligned} \frac{dM(t)}{dt} &= u_t(t, \zeta(t)) \\ &= \sum_{i,j=1}^{n} a_{ij}(t, \zeta(t), u(t, \zeta(t))) \frac{\partial^2 u(t, \zeta(t))}{\partial x_i \partial x_j} + f(t, \zeta(t), u(t, \zeta(t)), 0) \\ &\le \theta(t)\mu(M(t)) \end{aligned}$$

If $t \notin I$, then it follows that $t \in J = (0, T) \setminus I = \{t \in (0, T) : M(t) = 0\}$. Following the proof of Theorem 3.1 in [**1**], we get that $M'(t) = 0$ a.e. on J. Therefore, we have

$$\frac{M'(t)}{\mu(M(t))} \le \theta(t) \quad \text{a.e. on } (0, T). \tag{3}$$

In the case $I = \emptyset$, since $M(t) \equiv 0$, it follows that $M'(t) = 0$ a.e. on $(0, T)$ so that the above inequality is also true.

Similarly, we can prove that

$$-\frac{m'(t)}{\mu(m(t))} \le \theta(t) \quad \text{a.e. on } (0, T). \tag{4}$$

Provided $T < \infty$, it follows that $\limsup_{t \to T} \|u(t, \cdot)\|_{C(\overline{\Omega})} = \infty$. Thus, we obtain a sequence $\{t_k\} \subset (0, T)$ converging to T such that

$$\lim_{k \to \infty} M(t_k) = \infty \quad \text{or} \quad \lim_{k \to \infty} m(t_k) = -\infty$$

Suppose first that $\lim_{k \to \infty} M(t_k) = \infty$.

Since $\int_{M(0)}^{\infty} \frac{ds}{\mu(s)} > \int_0^T \theta(s)\, ds$, there is a $n \in N$ such that

$$\int_{M(0)}^{M(t_n)} \frac{ds}{\mu(s)} > l = \int_0^T \theta(s)\, ds.$$

Then (3) leads to the contradiction

$$l < \int_{M(0)}^{M(t_n)} \frac{ds}{\mu(s)} = \int_0^{t_n} \frac{M'(t)}{\mu(M(t))}\, dt \le \int_0^T \theta(s)\, ds = l.$$

Similarly, if $\lim_{k \to \infty} m(t_k) = -\infty$, using (4), we also obtain a contradiction. Therefore, $T = \infty$ and the solution u exists globally in time. This completes the proof of the theorem. □

EXAMPLE 1. Consider the problem (1) with

$$|f(t, x, w, \nabla w)| \leq \frac{1}{3(1+t^2)}(1+w^2)(1+|\nabla w|^2). \tag{5}$$

Take $\theta(t) = \frac{1}{3(1+t^2)}$ and $\mu(s) = 1 + s^2$, a straightforward computation shows that

$$\int_0^\infty \theta(t)\,dt = \frac{\pi}{6} < \frac{\pi}{4} = \min\left\{\int_1^\infty \frac{ds}{\mu(s)}, \int_{-\infty}^{-1} \frac{ds}{\mu(s)}\right\}.$$

Applying our theorem, we get that the solutions of (1) are global for any initial data satisfying $|\varphi(x)| \leq 1$ on $\overline{\Omega}$ if (5) holds. Observe that the recent results in [**1**] are not applicable.

REMARK. Note that Theorem 3.1 in [**1**] is a special case of our theorem (with $\theta(t) = 1$).

3. Weakly coupled systems. We shall prove that the method developed in the above Theorem can also be applicable to the following weakly coupled reaction-diffusion systems

$$\begin{cases} u_t - \operatorname{div}(\beta(t, x, u, v)\nabla u) = F(t, x, u, v), & t > 0,\ x \in \Omega, \\ v_t - \operatorname{div}(\beta(t, x, u, v)\nabla v) = G(t, x, u, v), & t > 0,\ x \in \Omega, \\ u(t, x) = v(t, x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0, x) = \varphi(x), & x \in \overline{\Omega}, \\ v(0, x) = \psi(x), & x \in \overline{\Omega}, \end{cases} \tag{6}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary. We assume that

$$\beta \in C^2(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^{n\times n});\ F, G \in C^1(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),$$

and

$$\langle \beta(t, x, w, z)\eta,\ \eta\rangle \geq c|\eta|^2, \quad (t, x, w, z) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R},\ \eta \in \mathbb{R}^n,$$

where $\mathbb{R}_+ = [0, \infty)$ and $c > 0$. Fix $p > n$. It is known from the classical parabolic theory [**3**, **4**, **5**] that, given any initial data $\varphi,\ \psi \in W_0^{s,p}(\Omega)$ with $s \in [1, \min\{1 + \frac{1}{p},\ 2 - \frac{n}{p}\})$, there exists a maximal $T = T(\varphi,\ \psi) > 0$ and a unique solution

$$u, v \in C^1((0, T); C(\overline{\Omega})) \cap C((0, T); C^2(\overline{\Omega})) \cap C\big([0, T); W_0^{s,p}(\Omega)\big).$$

Moreover, if $T < \infty$, one has

$$\limsup_{t\uparrow T} \|u(t, \cdot)\|_{L^\infty(\Omega)} = \infty \quad \text{or} \quad \limsup_{t\uparrow T} \|v(t, \cdot)\|_{L^\infty(\Omega)} = \infty,$$

As a corollary of the above Theorem, we have the following.

COROLLARY. *Assume that F and G satisfy the growth condition*

$$|F(t, x, w, z)| + |G(t, x, w, z)| \leq \theta(t)\mu(w+z), \quad (t, x, w, z) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R},$$

for some $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and* $\mu \in C(\mathbb{R}, (0, \infty))$. *Furthermore, let* $\varphi,\ \psi \in W_0^{s,p}(\Omega)$. *If for all* $T > 0$

$$\int_0^T \theta(s)\, ds < \min \left\{ \int_{-\infty}^{m(0)} \frac{ds}{\mu(s)},\ \int_{M(0)}^{\infty} \frac{ds}{\mu(s)} \right\},$$

where $m(0) = \min_{x\in\overline{\Omega}}\{\varphi(x) + \psi(x)\}$ *and* $M(0) = \max_{x\in\overline{\Omega}}\{\varphi(x) + \psi(x)\}$, *then the corresponding unique classical solution to* (6) *is defined globally in time.*

Proof. Let (u, v) be the classical solution of (6) with the initial data (φ, ψ) and let $T > 0$ be the maximal existence time of (u, v). Set

$$m(t) := \min_{x\in\overline{\Omega}}[u(t, x) + v(t, x)], \quad M(t) := \max_{x\in\overline{\Omega}}[u(t, x) + v(t, x)],$$

Summing up the two equations of (6), an analogous argumentation as in the proof of the above theorem yields

$$\frac{M'(t)}{\mu(M(t))} \le \theta(t) \quad \text{and} \quad -\frac{m'(t)}{\mu(m(t))} \le \theta(t) \quad \text{a.e. on } (0, T).$$

Provided $T < \infty$, it follows that $\limsup_{t\to T} \|(u+v)(t,\cdot)\|_{C(\overline{\Omega})} = \infty$. Thus, we obtain a sequence $\{t_k\} \subset (0, T)$ converging to T such that

$$\lim_{k\to\infty} M(t_k) = \infty \quad \text{or} \quad \lim_{k\to\infty} m(t_k) = -\infty$$

This leads to a contradiction which is the same as that of the above theorem. Therefore, $T = \infty$ and the solution (u, v) exists globally in time. □

EXAMPLE 2. Consider the weakly coupled reaction-diffusion system (6) with

$$|F(t, x, w, z)| + |G(t, x, w, z)| \le (1 + t)^{-2}(1 + |w + z|)^{\frac{3}{2}}. \tag{7}$$

Take $\theta(t) = (1 + t)^{-2}$ and $\mu(s) = (1+ \mid s \mid)^{\frac{3}{2}}$. A straightforward computation shows that

$$1 = \int_0^\infty \theta(t)\, dt = \int_3^\infty \frac{ds}{\mu(s)} = \int_{-\infty}^{-3} \frac{ds}{\mu(s)}.$$

Applying our Corollary, we get that the solution to (6) with condition (7) is global if $|\varphi(x) + \psi(x)| \le 3$ throughout $\overline{\Omega}$. Observe that the recent results in [**1**, **6**] are powerless.

REMARK. Note that Theorem 5.1 in [**1**] and Theorem 2 in [**6**] are special cases of our corollary (with $\theta(t) = 1$).

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