AN APPLICATION OF THE ADDITION THEOREM FOR DETERMINANTS

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THE integral evaluated in this note was suggested by the famous one connected with the Poincaré polynomials of the classical groups (see (**1**)).

Let X be an $n \times n$ matrix whose elements depend on k parameters. Denote by $\mathscr{X}$ a manifold in Euclidean space of dimension n^2, with the property that if $X \in \mathscr{X}$, then so does XI_{-i} for $1 \leqq i \leqq n$, where I_{-i} is the unit matrix I altered by a minus sign in the (i, i)th place. Suppose further that there exists on $\mathscr{X}$ a measure which is invariant under the transformation $X \rightarrow XI_{-i}$. Such manifolds and measures exist. For example (see (**2**), § 5), the set of all proper and improper $n \times n$ orthogonal matrices H is such a manifold, the H depending on $\frac{1}{2}n(n-1)$ parameters because of the orthogonality and normality of the columns of H. Since the set of all H is a compact topological group, an invariant measure exists.

Theorem. *If dX is an invariant measure on $\mathscr{X}$, such that $V = \int_{\mathscr{X}} dX$ exists and is finite, and if A, B, C are constant $n \times n$ matrices, and $| M |$ is the determinant of M, then*

$$\int_{\mathscr{X}} | A+BXC | \, dX = V \, | A |. \qquad (1)$$

Proof. Suppose $| C | \neq 0$, and let $D = AC^{-1}$, then

$$\int | A+BXC | \, dX = | C | \int | D+BX | \, dX.$$

Since the measure dX is invariant under the transformation $X \rightarrow XI_{-i}$,

$$J = \int | D+BX | \, dX = \int | D+BXI_{-i} | \, dX$$

$$= \tfrac{1}{2} \int \{ | D+BX | + | D+BXI_{-i} | \} dX.$$

Now $| D+BX |$ and $| D+BXI_{-1} |$ differ only in their first columns, so their sum is a determinant $| 2d_1, \ (D+BX)_{n-1} |$, whose first column is twice the first column, d_1, of $| D |$ and whose remaining columns $(D+BX)_{n-1}$, are the last $n-1$ columns of $| D+BX |$. So

$$J = \int | d_1, \quad (D+BX)_{n-1} | \, dX.$$

Now carry out the transformation $X \to XI_{-2}$ and let d_2 be the second column of $| D |$ and

$$J = \int | d_1, \quad d_2, \quad (D+BX)_{n-2} \, | \, dX.$$

Continuing,

$$J = \int | d_1, \quad d_2, \quad \ldots, \quad d_n \, | \, dX = V \, | \, D \, |.$$

This proves the theorem when $| C | \neq 0$. But when $| C | \neq 0$, (1) is an identity between two polynomials in the elements of C, and so by continuity, it still holds when $| C | = 0$.

Corollary. *Let $E_r(X)$ be the elementary symmetric functions of the latent roots of X, then*

$$\int_{\mathfrak{X}} E_r(X) dX = 0.$$

Proof. Let $A = zI$, $B = C = I$. Then since

$$| zI | = z^n \text{ and } | zI + X | = z^n + \sum_{r=1}^{n} z^{n-r} E_r(X),$$

$$\sum_{r=1}^{n} z^{n-r} \int E_r(X) dX = 0 \text{ for any number } z.$$

REFERENCES

(1) D. E. Littlewood, On the Poincaré polynomials of the classical groups, *Journ. London Math. Soc.* **28** (1953), 494-500.

(2) H. Jack and A. M. Macbeath, The volume of a certain set of matrices, *Proc. Camb. Phil. Soc.* **55** (1959), 213-223.

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