ON THE ATTACHED PRIME IDEALS OF CERTAIN ARTINIAN LOCAL COHOMOLOGY MODULES

by R. Y. SHARP

(Received 23rd May 1979)

1. Introduction

The study of the cohomological dimensions of algebraic varieties has produced some interesting results and problems in local algebra: the general local problem is that posed by Hartshorne and Speiser in (**4**, p. 57). We consider a (commutative, Noetherian) local ring A (with identity), a proper ideal $\mathfrak{a}$ of A, and ask the following question.

When are the local cohomology modules $H_{\mathfrak{a}}^i(A)$ *cofinite (that is Artinian), or zero, for large values of* i?

Let n denote the dimension of A. It is well known that $H_{\mathfrak{a}}^i(M)=0$ for all A-modules M whenever $i>n$: see (**2**, 1.12). Also, if c is an integer such that $H_{\mathfrak{a}}^i(A)=0$ whenever $i>c$, then it is a consequence of the fact that the formation of local cohomology modules commutes with the formation of direct limits that $H_{\mathfrak{a}}^i(M)=0$ for all A-modules M whenever $i>c$. For these reasons, a considerable amount of attention has been focussed on the module $H_{\mathfrak{a}}^n(A)$. Sufficient conditions for the vanishing of this module are provided by the following theorem, which was first proved by Hartshorne.

The Local Lichtenbaum–Hartshorne Theorem. (See (**3**, 3.1), and also (**11**, III, 3.1).) *Let* $\mathfrak{a}$ *be a proper ideal of the local ring* A; *denote* $\dim A$ *by* n. *Assume that, for every minimal prime ideal* $\mathfrak{q}$ *of* $\hat{A}$ *(the completion of* A*) of dimension* n, *it is the case that* $\dim(\hat{A}/(\mathfrak{a}\hat{A}+\mathfrak{q}))\geqq 1$. *Then* $H_{\mathfrak{a}}^n(A)=0$.

The converse of the local Lichtenbaum–Hartshorne Theorem is rather easier to establish (and we shall provide a proof of the converse in Section 3). We may restate the local Lichtenbaum–Hartshorne Theorem and its converse as follows: $H_{\mathfrak{a}}^n(A)=0$ if and only if the set

$$\{\mathfrak{q}\in \operatorname{Spec}(\hat{A}): \dim \hat{A}/\mathfrak{q}=n \text{ and } \dim(\hat{A}/(\mathfrak{a}\hat{A}+\mathfrak{q}))=0\}$$

is empty.

However, there is another finite set of prime ideals related to $H_{\mathfrak{a}}^n(A)$ with the property that its being empty is equivalent to the vanishing of $H_{\mathfrak{a}}^n(A)$. We shall show in Section 3 that $H_{\mathfrak{a}}^n(A)$ is always an Artinian A-module: there is a theory of "secondary representation" for Artinian A-modules which is in several respects dual to the well-known theory of primary decomposition available for Noetherian A-modules. In particular, this secondary representation theory associates with an Artinian A-module M a finite collection of prime ideals, called the attached prime ideals of M: the set

https://doi.org/10.1017/S0013091500003941 Published online by Cambridge University Press

which they form is denoted by Att (M) (or Att_A (M) if confusion is possible), and this set has the property that Att $(M)=\varnothing$ if and only if $M=0$.

As it is Artinian, $H^n_{\mathfrak{a}}(A)$ has a natural structure as an $\hat{A}$-module, and, in fact, is Artinian as such. We may therefore form the finite subset $\text{Att}_{\hat{A}}$ $(H^n_{\mathfrak{a}}(A))$ of Spec $(\hat{A})$, and this has the property that its being empty is equivalent to the vanishing of $H^n_{\mathfrak{a}}(A)$. The purpose of this paper is to prove that $\text{Att}_{\hat{A}}$ $(H^n_{\mathfrak{a}}(A))$ is precisely the set

$$\{\mathfrak{q}\in \text{Spec}\,(\hat{A}): \dim \hat{A}/\mathfrak{q}=n \text{ and } \dim(\hat{A}/(\mathfrak{a}\hat{A}+\mathfrak{q}))=0\}$$

mentioned earlier in connection with the local Lichtenbaum–Hartshorne Theorem, and in so doing to establish a link between that theorem and the theory of attached prime ideals for secondary representation of Artinian modules.

2. Notation, terminology, and some preliminary results

All rings considered in this paper will be commutative and Noetherian, and will have non-zero identities. R will always denote such a ring; A will always denote a local ring, and $\mathfrak{m}$ will always denote the maximal ideal of A. If M is an R-module, the injective envelope of M will be denoted by $E(M)$ (or $E_R(M)$).

We shall use the notation of (**12**) concerning local cohomology, except that, if $\mathfrak{b}$ is an ideal of R, then, for $i\geqq 0$, we shall use $H^i_{\mathfrak{b}}$ to denote the ith right derived functor of the local cohomology functor $L_{\mathfrak{b}}$ (**12**, §2); if M is an R-module, the result $H^i_{\mathfrak{b}}(M)$ of applying $H^i_{\mathfrak{b}}$ to M will be called the ith *local cohomology module of* M *with respect to* $\mathfrak{b}$.

Many of the local cohomology modules we shall investigate will be Artinian. Now there is available a theory which applies to Artinian R-modules and which is dual to the well-known theory of primary decomposition of proper submodules of Noetherian R-modules. Accounts of this dual theory are available in (**7**), (**5**), and (**9**); here we shall follow Macdonald's terminology from (**7**) and refer to this dual theory as "secondary representation theory". A brief review of the main facts from this theory is given in (**14**, pp. 143–4): we shall use the terminology employed therein. The reader should note in particular that, if N is a representable R-module, then $N=0$ if and only if Att $(N)=\varnothing$.

We shall use $\hat{A}$ to denote the completion of the local ring A, and $\hat{\mathfrak{m}}$ to denote the maximal ideal of $\hat{A}$. Our first lemma is useful in the study of Artinian A-modules.

Lemma 2.1. *Let M be an Artinian A-module. Then M has a natural structure as an (Artinian) $\hat{A}$-module, and there is an $\hat{A}$-isomorphism $\psi: M\otimes_A \hat{A}\to M$ for which*

$$\psi\left(\sum_{i=1}^{u} x_i\otimes \hat{a}_i\right)=\sum_{i=1}^{u}\hat{a}_i x_i$$

(for $x_1,\ldots,x_u\in M$ and $\hat{a}_1,\ldots,\hat{a}_u\in\hat{A}$). Also

$$\text{Att}_A\,(M)=\{\mathfrak{q}\cap A:\mathfrak{q}\in \text{Att}_{\hat{A}}\,(M)\}.$$

Proof. By (**15**, 3.21) and (**8**, 3.4(1)), each element of M is annihilated by some power of $\mathfrak{m}$. Let $x\in M$ and $\hat{a}\in\hat{A}$. Let $(a_n)_{n\geqq 1}$ be a Cauchy sequence of elements of A

which converges to $\hat{a}$ in $\hat{A}$. Then the values of the sequence $(a_n x)_{n\geq 1}$ of elements of M are ultimately constant. It is straightforward to check that M may be given the structure of an $\hat{A}$-module in such a way that $\hat{a}x$ is equal to the ultimate constant value of the sequence $(a_n x)_{n\geq 1}$. It follows that a subset of M is an A-submodule if and only if it is an $\hat{A}$-submodule.

The proofs of all the remaining statements in the lemma are now straightforward, and so will be left to the reader.

We shall also need to employ the following standard results about local cohomology modules: 2.1 is of assistance in the proof of the second of these.

Proposition 2.2. (Hartshorne (**3**, 2.2) and Ogus (**10**, 2.1).) *Let $\mathfrak{a}$ be an ideal of the local ring A, and let i be an integer $\geqq 0$. Then*

(i) *there is an isomorphism $H^i_{\mathfrak{a}}(A)\otimes_A \hat{A}\cong H^i_{\mathfrak{a}\hat{A}}(\hat{A})$ of $\hat{A}$-modules; and*

(ii) *$H^i_{\mathfrak{a}}(A)$ is an Artinian A-module if and only if $H^i_{\mathfrak{a}\hat{A}}(\hat{A})$ is an Artinian $\hat{A}$-module.*

The following result is also well known.

Proposition 2.3. *Let $\mathfrak{a}$ be an ideal of the local ring A, and let n denote* dim A. *Then, for all finitely generated A-modules M,*

$$H^n_{\mathfrak{a}}(M)\cong H^n_{\mathfrak{a}}(A)\otimes_A M.$$

Proof. By (**12**, 6.1), $H^{n+1}_{\mathfrak{a}}(X)=0$ for all A-modules X. Hence the functor $H^n_{\mathfrak{a}}$ is right exact, and so the result follows from a standard fact about certain right exact covariant functors: see, for example, (**13**, (3.2)).

3. The nth local cohomology modules in a local ring of dimension n

The local Lichtenbaum–Hartshorne Theorem was stated in the Introduction: we begin by proving its converse.

Theorem 3.1. *Let $\mathfrak{a}$ be a proper ideal of the local ring A; denote* dim A *by n. Assume that $H^n_{\mathfrak{a}}(A)=0$. Then*

$$\dim(\hat{A}/(\mathfrak{a}\hat{A}+\mathfrak{q}))\geqq 1$$

for every minimal prime ideal $\mathfrak{q}$ of $\hat{A}$ of dimension n.

Proof. Let $\mathfrak{q}$ be a minimal prime ideal of $\hat{A}$ of dimension n. Suppose

$$\dim(\hat{A}/(\mathfrak{a}\hat{A}+\mathfrak{q}))=0.$$

We use $\hat{\mathfrak{m}}$ to denote the maximal ideal of $\hat{A}$. By (**6**, 2.2), $H^n_{\hat{\mathfrak{m}}/\mathfrak{q}}(\hat{A}/\mathfrak{q})\neq 0$. However, the supposition that $\dim(\hat{A}/(\mathfrak{a}\hat{A}+\mathfrak{q}))=0$ implies that $(\mathfrak{a}\hat{A}+\mathfrak{q})/\mathfrak{q}$ is an $\hat{\mathfrak{m}}/\mathfrak{q}$-primary ideal of the local ring $\hat{A}/\mathfrak{q}$; hence the local cohomology functors $L_{\hat{\mathfrak{m}}/\mathfrak{q}}$ and $L_{(\mathfrak{a}\hat{A}+\mathfrak{q})/\mathfrak{q}}$ (on the category of all $\hat{A}/\mathfrak{q}$-modules and all $\hat{A}/\mathfrak{q}$-homomorphisms) coincide; it therefore follows that $H^n_{(\mathfrak{a}\hat{A}+\mathfrak{q})/\mathfrak{q}}(\hat{A}/\mathfrak{q})\neq 0$.

Use of (**12**, 4.3) therefore shows that $H^n_{\mathfrak{a}\hat{A}}(\hat{A}/\mathfrak{q}) \neq 0$; hence $H^n_{\mathfrak{a}\hat{A}}(\hat{A}) \neq 0$ by 2.3, and so $H^n_{\mathfrak{a}}(A) \neq 0$ by 2.2(i). With this contradiction the proof is complete.

As was mentioned in the Introduction, we may restate the local Lichtenbaum–Hartshorne Theorem and its converse in the following way.

Corollary 3.2. *Let the situation be as in* 3.1. *Then* $H^n_{\mathfrak{a}}(A) = 0$ *if and only if the set*

$$\{\mathfrak{q} \in \operatorname{Spec}(\hat{A}) : \dim \hat{A}/\mathfrak{q} = n \text{ and } \dim(\hat{A}/(\mathfrak{a}\hat{A} + \mathfrak{q})) = 0\}$$

is empty.

Our aim is to show how the finite subset of $\operatorname{Spec}(\hat{A})$ described in 3.2 arises in connection with the theory of attached prime ideals for secondary representation of Artinian modules. The next step is to show that $H^n_{\mathfrak{a}}(A)$ is an Artinian A-module.

Theorem 3.3. *Let* $\mathfrak{a}$ *be a proper ideal of the local ring* A; *denote* $\dim A$ *by* n. *Then* $H^n_{\mathfrak{a}}(A)$ *is an Artinian* A-*module.*

Proof. By virtue of 2.2(ii), we may assume that A is complete. Then we may use the Cohen structure theorems for complete local rings in the manner in which Peskine and Szpiro used them in (**11**, III, 3.1) to see that there exists a complete Gorenstein local ring B of dimension n and a surjective ring homomorphism $\phi : B \to A$. Set $\phi^{-1}(\mathfrak{a}) = \mathfrak{b}$. We may regard A as a B-module by means of ϕ and form the B-module $H^n_{\mathfrak{b}}(A)$. On the other hand, we may use ϕ to regard $H^n_{\mathfrak{a}}(A)$ as a B-module. By (**12**, 4.3), there is an isomorphism

$$H^n_{\mathfrak{a}}(A) \cong H^n_{\mathfrak{b}}(A)$$

of B-modules, from which we see that it is sufficient for us to show that $H^n_{\mathfrak{b}}(A)$ is an Artinian B-module.

Set $\mathfrak{c} = \ker \phi$, so that A and $B/\mathfrak{c}$ are isomorphic B-modules. It thus follows from 2.3 that

$$H^n_{\mathfrak{b}}(A) \cong H^n_{\mathfrak{b}}(B/\mathfrak{c}) \cong H^n_{\mathfrak{b}}(B) \otimes_B B/\mathfrak{c},$$

so that

$$H^n_{\mathfrak{b}}(A) \cong H^n_{\mathfrak{b}}(B)/\mathfrak{c}H^n_{\mathfrak{b}}(B).$$

It is thus sufficient for us to show that $H^n_{\mathfrak{b}}(B)$ is an Artinian B-module.

To see this, one should calculate $H^n_{\mathfrak{b}}(B)$ by applying the functor $L_{\mathfrak{b}}$ to a minimal injective resolution for B: since B is a Gorenstein local ring of dimension n, it follows from Bass's work in (**1**, §1) that, if we let $\mathfrak{r}$ denote the maximal ideal of B, then the nth term in a minimal injective resolution for B is isomorphic to $E_B(B/\mathfrak{r})$. Since $E_B(B/\mathfrak{r})$ is Artinian, by (**8**, 4.2), it follows that $H^n_{\mathfrak{b}}(B)$, which is isomorphic to a homomorphic image of a submodule of $E_B(B/\mathfrak{r})$, is also Artinian. This completes the proof.

Theorem 3.4. *Let* $\mathfrak{a}$ *be a proper ideal of the local ring* A; *denote* $\dim A$ *by* n. *Since, by* 3.3, $H^n_{\mathfrak{a}}(A)$ *is an Artinian* A-*module, it has, by* 2.1, *a natural structure as an* (*Artinian*) $\hat{A}$-*module. Then*

$$\operatorname{Att}_{\hat{A}}(H^n_{\mathfrak{a}}(A)) = \{\mathfrak{q} \in \operatorname{Spec}(\hat{A}) : \dim \hat{A}/\mathfrak{q} = n \text{ and } \dim(\hat{A}/(\mathfrak{a}\hat{A} + \mathfrak{q})) = 0\},$$

that is precisely the subset of Spec $(\hat{A})$ *which arose in* 3.2 *in connection with the local Lichtenbaum–Hartshorne Theorem.*

Proof. In view of 2.1 and 2.2(i), we may assume that A is complete. We shall use the characterisation of $\mathrm{Att}_A\,(H^n_{\mathfrak{a}}(A))$ afforded by (**6**, 1.4).

First of all, suppose that $\mathfrak{q} \in \mathrm{Att}_A\,(H^n_{\mathfrak{a}}(A))$. Then, by (**6**, 1.4), $H^n_{\mathfrak{a}}(A) \neq \mathfrak{q}H^n_{\mathfrak{a}}(A)$, whence $H^n_{\mathfrak{a}}(A) \otimes_A A/\mathfrak{q} \neq 0$. It thus follows from 2.3 that $H^n_{\mathfrak{a}}(A/\mathfrak{q}) \neq 0$. We may now use (**12**, 4.3) to see that, over the complete local integral domain $A/\mathfrak{q}$, the local cohomology module $H^n_{(\mathfrak{a}+\mathfrak{q})/\mathfrak{q}}(A/\mathfrak{q}) \neq 0$. It thus follows from (**12**, 6.1) that $\dim A/\mathfrak{q} = n$, while the local Lichtenbaum–Hartshorne Theorem shows that $\dim\,(A/(\mathfrak{a}+\mathfrak{q})) = 0$.

Conversely, suppose that $\mathfrak{q}$ is a prime ideal of A of dimension n for which

$$\dim\,(A/(\mathfrak{a}+\mathfrak{q})) = 0.$$

Thus, in the n-dimensional complete local integral domain $A/\mathfrak{q}$, the ideal $(\mathfrak{a}+\mathfrak{q})/\mathfrak{q}$ is $\mathfrak{m}/\mathfrak{q}$-primary. Thus $H^n_{(\mathfrak{a}+\mathfrak{q})/\mathfrak{q}}(A/\mathfrak{q}) = H^n_{\mathfrak{m}/\mathfrak{q}}(A/\mathfrak{q})$. Thus, by (**6**, 2.1 and 2.2), $H^n_{(\mathfrak{a}+\mathfrak{q})/\mathfrak{q}}(A/\mathfrak{q})$ is an Artinian $A/\mathfrak{q}$-module which is 0-secondary (that is $\mathfrak{q}/\mathfrak{q}$-secondary). Use of (**12**, 4.3) now shows that the A-module $H^n_{\mathfrak{a}}(A/\mathfrak{q})$ has annihilator equal to $\mathfrak{q}$. But, by 2.3, $H^n_{\mathfrak{a}}(A/\mathfrak{q})$ is a homomorphic image of $H^n_{\mathfrak{a}}(A)$, and so it follows from (**6**, 1.4) that $\mathfrak{q} \in \mathrm{Att}_A\,(H^n_{\mathfrak{a}}(A))$. This completes the proof.

Corollary 3.5. *Let the situation be as in* 3.4. *Then*

$$\{\mathfrak{q} \cap A : \mathfrak{q} \in \mathrm{Spec}\,(\hat{A}),\ \dim \hat{A}/\mathfrak{q} = n, \text{ and } \dim\,(\hat{A}/(\mathfrak{a}\hat{A} + \mathfrak{q})) = 0\}$$

is the set of attached prime ideals for secondary representation of the Artinian A-module $H^n_{\mathfrak{a}}(A)$.

Proof. This is immediate from 3.4 and 2.1.

In particular, it follows from 3.5 that (with the same notation)

$$\mathrm{Att}_A\,(H^n_{\mathfrak{m}}(A)) = \{\mathfrak{p} \in \mathrm{Spec}\,(A) : \dim A/\mathfrak{p} = n\};$$

this particular fact was proved in (**6**, 2.2).

REFERENCES

(**1**) H. Bass, On the ubiquity of Gorenstein rings, *Math. Z.* **82** (1963), 8–28.

(**2**) A. Grothendieck, *Local cohomology* (Lecture notes in mathematics No. 41, Berlin–Heidelberg–New York, Springer, 1967).

(**3**) R. Hartshorne, Cohomological dimension of algebraic varieties, *Annals of Math.* **88** (1968), 403–450.

(**4**) R. Hartshorne and R. Speiser, Local cohomological dimension in characteristic p, *Annals of Math.* **105** (1977), 45–79.

(**5**) D. Kirby, Coprimary decomposition of Artinian modules, *J. London Math. Soc.* **6** (1973), 571–576.

(**6**) I. G. Macdonald and R. Y. Sharp, An elementary proof of the non-vanishing of certain local cohomology modules, *Quart. J. Math. Oxford* (2) **23** (1972), 197–204.

(7) I. G. Macdonald, Secondary representation of modules over a commutative ring, *Symposia Mathematica* **11** (1973), 23–43.

(8) E. Matlis, Injective modules over Noetherian rings, *Pacific J. Math.* **8** (1958), 511–528.

(9) D. G. Northcott, Generalized Koszul complexes and Artinian modules, *Quart. J. Math. Oxford* (2) **23** (1972), 289–297.

(10) A. Ogus, Local cohomological dimension of algebraic varieties, *Annals of Math.* **98** (1973), 327–365.

(11) C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, *Inst. Haut. Étud. Sci., Publ. Math.* **42** (1973), 323–395.

(12) R. Y. Sharp, Local cohomology theory in commutative algebra, *Quart. J. Math. Oxford* (2) **21** (1970), 425–434.

(13) R. Y. Sharp, Some results on the vanishing of local cohomology modules, *Proc. London Math. Soc.* (3) **30** (1975), 177–195.

(14) R. Y. Sharp, Secondary representations for injective modules over commutative Noetherian rings, *Proc. Edinburgh Math. Soc.* (2) **20** (1976), 143–151.

(15) D. W. Sharpe and P. Vámos, *Injective modules* (Cambridge tracts in mathematics and mathematical physics No. 62, Cambridge University Press, 1972).

Department of Pure Mathematics
University of Sheffield
Sheffield S3 7RH