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WEAK-STAR CONTINUOUS ANALYTIC FUNCTIONS

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ABSTRACT. Let $\mathcal{X}$ be a complex Banach space, with open unit ball B. We consider the algebra of analytic functions on B that are weakly continuous and that are uniformly continuous with respect to the norm. We show these are precisely the analytic functions on B that extend to be weak-star continuous on the closed unit ball of $\mathcal{X}^{**}$. If $\mathcal{X}^*$ has the approximation property, then any such function is approximable uniformly on B by finite polynomials in elements of $\mathcal{X}^*$. On the other hand, there exist Banach spaces for which these finite-type polynomials fail to approximate. We consider also the approximation of entire functions by finite-type polynomials. Assuming $\mathcal{X}^*$ has the approximation property, we show that entire functions are approximable uniformly on bounded sets if and only if the spectrum of the algebra of entire functions coincides (as a point set) with $\mathcal{X}^{**}$.

1. **Introduction.** It will be convenient to work in the context of a dual Banach space $\mathcal{Z}$, with open unit ball U. When endowed with the weak-star topology, the closed unit ball $\bar{U}$ of $\mathcal{Z}$ is weak-star compact. We consider the algebra $A(U)$ of analytic functions on U that extend weak-star continuously to $\bar{U}$. The algebra $A(U)$ is a natural generalization of the disk algebra $A(\Delta)$ of analytic functions in the open unit disk Δ in the complex plane that extend continuously to the closure. The weak-star continuous linear functionals on $\mathcal{Z}$ form a closed linear subspace of the algebra $A(U)$, which already separates the points of $\bar{U}$. Thus $A(U)$ is a uniform algebra on $\bar{U}$, that is, a closed point-separating unital subalgebra of $C(\bar{U})$.

A *weak-star continuous finite-type polynomial* on $\mathcal{Z}$ is a finite linear combination of products of weak-star continuous linear functionals on $\mathcal{Z}$. We denote by $P(U)$ the algebra of uniform limits on $\bar{U}$ of the weak-star continuous finite-type polynomials. In other words, $P(U)$ is the uniform subalgebra of $A(U)$ generated by the weak-star continuous linear functionals.

In the one-dimensional case, the polynomials in the coordinate function are dense in the disk algebra, so that $P(\Delta) = A(\Delta)$. This result persists for finite-dimensional Banach spaces. In the general context, it seems that the algebra approximation problem of when $P(U) = A(U)$ is related to the linear approximation property of the underlying Banach space and its predual. Starting with a Banach space without the approximation property, we construct in Section 4 a dual Banach space for which $P(U) \neq A(U)$. In the positive

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direction, we observe in Section 5 that $P(U) = A(U)$ whenever the predual of $\mathcal{Z}$ has the approximation property.

The paper is organized as follows. In Section 2 we reduce the approximation problem to that of approximation of m-homogeneous entire functions, and in Section 3 we represent these by $(m-1)$-linear operators from $\mathcal{Z}$ to the predual $\mathcal{Y}$. Sections 4 and 5 are devoted to the results on polynomial approximation mentioned above. In Section 6 we specialize to the case $\mathcal{Z} = \mathcal{X}^{**}$, so that $U = B^{**}$ is the open unit ball of the bidual. Using a fundamental result from [3], we show that the algebra $A(B^{**})$ consists of precisely the extensions to B^{**} of the weakly continuous analytic functions on B that are uniformly continuous with respect to the norm. This overlaps with results obtained by L. Moraes in [9].

In Section 7 we consider approximation of entire functions. Let $H_b(\mathcal{X})$ denote the algebra of entire functions on $\mathcal{X}$ that are bounded on bounded sets, with the topology of uniform convergence on bounded sets. We show that if $\mathcal{X}^*$ has the approximation property, then the finite-type polynomials are dense in $H_b(\mathcal{X})$ if and only if the spectrum of the Frechet algebra $H_b(\mathcal{X})$ coincides with $\mathcal{X}^{**}$ (as a point set).

2. **Reduction to approximation of Taylor coefficients.** Each function f analytic on a neighborhood of 0 in $\mathcal{Z}$ has a Taylor series expansion $f = \sum f_m$, where the Taylor coefficient function f_m is an m-homogeneous analytic function on $\mathcal{Z}$. The Taylor series is convergent on any ball centered at 0 on which f is analytic and bounded. Furthermore, if f is analytic, bounded, and uniformly continuous (with respect to the norm) on a ball rU, then the Cesàro means of the partial sums of the series converge uniformly to f on the ball.

Recall that the *bounded-weak-star topology* for a dual Banach space $\mathcal{Z}$ is the largest topology that agrees with the weak-star topology on bounded sets. A function g on $\mathcal{Z}$ is *bounded-weak-star continuous* if its restriction to each bounded set is weak-star continuous. Thus an m-homogeneous analytic function g on $\mathcal{Z}$ is bounded-weak-star continuous if and only if the restriction of g to the closed unit ball $\bar{U}$ is weak-star continuous, that is, g belongs to $A(U)$. The bounded-weak-star continuous linear functionals on $\mathcal{Z}$ are all weak-star continuous (Theorem V.5.6 of [6]).

LEMMA 2.1. *Let $f \in H^\infty(U)$ have Taylor series $\sum f_m$. Then $f \in A(U)$ if and only if f is uniformly continuous on U (norm metric) and each Taylor coefficient f_m belongs to $A(U)$.*

PROOF. Suppose first that $f \in A(U)$. Suppose that $\{z_j\}$ and $\{w_j\}$ are sequences in U such that $\|z_j - w_j\| \longrightarrow 0$. Let $\{z_{j(\alpha)}\}$ be a subnet that converges weak-star to $z \in \bar{U}$. Then also $\{w_{j(\alpha)}\}$ converges weak-star to z. Hence $f(z_{j(\alpha)}) \longrightarrow f(z)$ and $f(w_{j(\alpha)}) \longrightarrow f(z)$, so that $|f(z_{j(\alpha)}) - f(w_{j(\alpha)})| \longrightarrow 0$. Since this holds for any weak-star convergent subnet of $\{z_j\}$, in fact $|f(z_j) - f(w_j)| \longrightarrow 0$. It follows that f is uniformly continuous on U.

To show that $f_m \in A(U)$, it suffices by the homogeneity of f_m to show that f_m is weak-star continuous on some ball rU. So fix $0 < r < 1$, and let $\{z_\alpha\}$ be a net in rU converging weak-star to z. Then λz_α converges weak-star to λz for all complex λ. By the weak-star continuity of f, $g_\alpha(\lambda) = f(\lambda z_\alpha)$ converges pointwise to $g(\lambda) = f(\lambda z)$ for $|\lambda| < 1/r$. Furthermore, the g_α's are uniformly bounded by $\|f\|_U$ for $|\lambda| \leq 1/r$. Hence for $|\lambda| \leq 1$,

$$g_\alpha(\lambda) = f(\lambda z_\alpha) = \sum f_m(\lambda z_\alpha) = \sum \lambda^m f_m(z_\alpha)$$

converges uniformly to

$$g(\lambda) = \sum \lambda^m f_m(z).$$

It follows that for each fixed $m \geq 0$, $f_m(z_\alpha)$ converges to $f_m(z)$, and f_m is weak-star continuous on rU.

For the converse, suppose that $f \in H^\infty(U)$ is uniformly continuous on U. Let $\{g_k\}$ be the sequence of Cesàro means of the partial sums of the Taylor series of f. The uniform continuity of f and standard estimates show that g_k converges uniformly to f. If the f_m's are weak-star continuous on $\bar{U}$, then so are the g_k's, and consequently so is f. ∎

LEMMA 2.2. *Let $f \in A(U)$ have Taylor series $\sum f_m$. Then $f \in P(U)$ if and only if each f_m is in $P(U)$.*

PROOF. If $g = \sum g_m$ is a weak-star continuous finite-type polynomial approximating f uniformly on U, the estimate $\|f_m - g_m\|_U \leq \|f - g\|_U$ shows g_m approximates f_m uniformly, and consequently $f_m \in P(U)$. Conversely if the f_m's are in $P(U)$, then since the Cesàro means of the partial sums of $\sum f_m$ approximate f uniformly, $f \in P(U)$. ∎

We consider in passing the problem of whether the pointwise bounded approximability of $f \in A(U)$ implies the uniform approximability. We do not know if this holds in general, but we can establish it under a hypothesis of separability.

THEOREM 2.3. *Assume that $\mathcal{Z}$ is separable. Let $f \in A(U)$. Suppose there is a net $\{g_\alpha\}$ of finite-type polynomials on $\mathcal{Z}$ that are uniformly bounded on U, such that g_α converges pointwise to f on U. Then $f \in P(U)$.*

PROOF. By Theorem 4.1 of [4], we can approximate the g_α's pointwise by weak-star continuous finite-type polynomials of the same norm. Thus we can assume that the g_α's are weak-star continuous. Let $\{z_j\}_{j=1}^\infty$ be a dense sequence in U. We can select a sequence $\{h_k\}$ from the g_α's that converges pointwise to f on the z_j's. Since the h_k's are uniformly bounded on U, they are equi-uniformly continuous on each ball rU, $0 < r < 1$, and consequently they converge pointwise to f on the entire ball U. In particular, they converge to f in the weak topology of the space $C(r\bar{U})$ of continuous functions on each $r\bar{U}$ (weak-star topology on $r\bar{U}$). It follows that f is uniformly approximable on $r\bar{U}$ by weak-star continuous finite-type polynomials. Thus the dilate f_r, defined by $f_r(z) = f(rz)$, $z \in U$, belongs to $P(U)$. Letting r increase to 1, we obtain $f \in P(U)$. ∎

3. **Representation of m-homogeneous functions in $A(U)$.** Each m-homogeneous analytic function f on $\mathcal{Z}$ is the restriction to the diagonal of a unique symmetric m-form F on $\mathcal{Z}$, and F can be expressed in terms of f via the polarization formula:

$$F(z_1,\ldots,z_m) = \frac{1}{m!\,2^m} \sum_{\substack{\varepsilon_j=\pm 1\\ 1\leq j\leq m}} \varepsilon_1\cdots\varepsilon_m f(\varepsilon_1 x_1 + \cdots + \varepsilon_m x_m).$$

For details see [10] or [11].

We say that an m-form F on $\mathcal{Z}$ is *bounded-weak-star continuous* if the restriction of F to $U\times\cdots\times U$ (m times) is continuous with respect to the product of the weak-star topologies. The polarization lemma shows that if the m-homogeneous analytic function f on $\mathcal{Z}$ is bounded-weak-star continuous, then the corresponding symmetric m-form F on $\mathcal{Z}$ is also bounded-weak-star continuous. Furthermore, F is separately weak-star continuous, since bounded-weak-star continuous linear functionals are weak-star continuous.

It would be useful to have conditions that guarantee that a separately weak-star continuous multilinear functional is bounded-weak-star continuous. For instance, it is easy to check that a separately weak-star continuous bilinear form that is bounded-weak-star continuous at the origin is actually bounded-weak-star continuous. However, this statement fails already for trilinear forms. The form $F(x,y,z) = x_1 \sum y_j z_j$ on ℓ^2 is separately weakly continuous, but F is not bounded-weak continuous at any (x,y,z) for which $x_1 \neq 0$.

Any continuous m-linear form F on an arbitrary Banach space $\mathcal{Z}$ can be represented in terms of a continuous $(m-1)$-linear operator T from $\mathcal{Z}^{m-1}$ to $\mathcal{Z}^*$ by

(1) $$F(z_1,\ldots,z_m) = \langle T(z_1,\ldots,z_{m-1}), z_m\rangle, \quad z_1,\ldots,z_m \in \mathcal{Z}.$$

Conversely, every such $(m-1)$-linear operator T determines an m-linear F by (1), and the correspondence is an isometry: $\|F\| = \|T\|$.

THEOREM 3.1. *Fix $m \geq 2$, and let F be an m-linear functional on the dual Banach space $\mathcal{Z}$. Then F is bounded-weak-star continuous if and only if F has the representation of the form (1), where T is an $(m-1)$-linear operator from $\mathcal{Z}^{m-1}$ to the predual $\mathcal{Y}$ of $\mathcal{Z}$ that is continuous from the bounded-weak-star topology of $\mathcal{Z}^{m-1}$ to the norm topology of $\mathcal{Y}$. In this case, T maps U^{m-1} to a subset of $\mathcal{Y}$ that is norm precompact.*

PROOF. Suppose that F is bounded-weak-star continuous. For fixed $z_1,\ldots,z_{m-1}$ in $\mathcal{Z}^{m-1}$, the functional $z_m \longrightarrow F(z_1,\ldots,z_m)$ is weak-star continuous, so there is $T(z_1,\ldots,z_{m-1})$ in the predual $\mathcal{Y}$ for which the representation (1) is valid. Evidently T is $(m-1)$-linear, and $\|T(z_1,\ldots,z_{m-1})\| \leq \|F\|\,\|z_1\|\cdots\|z_{m-1}\|$, so that T is norm-to-norm continuous.

Suppose that T is not continuous with respect to the bounded-weak-star topology on $\mathcal{Z}^{m-1}$. Then there are $\varepsilon > 0$ and weak-star convergent nets $z_j^{(\alpha)} \longrightarrow z_j$, $1 \leq j \leq m-1$, such that $\|z_j^{(\alpha)}\| \leq 1$ and $\|T(z_1^{(\alpha)},\ldots,z_{m-1}^{(\alpha)}) - T(z_1,\ldots,z_{m-1})\| \geq \varepsilon$. In view of the representation (1) and the Hahn-Banach theorem, there are $z_m^{(\alpha)} \in \mathcal{Z}$ such that $\|z_m^{(\alpha)}\| \leq 1$

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and $|F(z_1^{(\alpha)}, \ldots, z_m^{(\alpha)}) - F(z_1, \ldots, z_{m-1}, z_m^{(\alpha)})| \geq \varepsilon$. Passing to a subnet, we can assume that $z_m^{(\alpha)}$ converges weak-star to $z_m \in \mathcal{Z}$. However, in the limit this contradicts the continuity of F, and the continuity of T from bounded-weak-star topology to norm topology is established.

Conversely, suppose that F is represented by T with the properties above. Suppose $\{z_j^{(\alpha)}\}$, $1 \leq j \leq m$, are nets in U that converge respectively to z_j in the weak-star topology. Since $\bar{U}^{m-1}$ is weak-star compact, its image under T is norm compact. Passing to a subnet, we can assume that $T(z_1^{(\alpha)}, \ldots, z_{m-1}^{(\alpha)})$ converges to $T(z_1, \ldots, z_{m-1})$ in norm. Express $F(z_1^{(\alpha)}, \ldots, z_m^{(\alpha)})$ in the form

$$\langle T(z_1^{(\alpha)}, \ldots, z_{m-1}^{(\alpha)}) - T(z_1, \ldots, z_{m-1}), z_m^{(\alpha)} \rangle + \langle T(z_1, \ldots, z_{m-1}), z_m^{(\alpha)} \rangle.$$

The first summand here tends to 0, by the continuity of T, and the second summand tends to $F(z_1, \ldots, z_m)$. Hence F is bounded-weak-star continuous. ∎

4. **Bilinear forms.** For the moment, let $\mathcal{X}$ be an arbitrary Banach space. We say that a linear operator T from $\mathcal{X}$ to its dual $\mathcal{X}^*$ is *symmetric* if the corresponding bilinear form is symmetric, that is, if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{X}$. It is easy to check that this occurs if and only if the restriction of the adjoint T^* to $\mathcal{X}$ (regarded as a subspace of $\mathcal{X}^{**}$) coincides with T.

The notion of symmetry can also be defined for an operator from a dual Banach space to its predual. We define an operator T from $\mathcal{X}^*$ to $\mathcal{X}$ to be *symmetric* if $(Tz, w) = (z, Tw)$ for all $z, w \in \mathcal{X}^*$. It is easy to check that this occurs if and only if T coincides with T^* (the range space $\mathcal{X}$ of T being regarded as a subspace of $\mathcal{X}^{**}$).

The symmetry of an operator is related to the symmetry of its adjoint by the following two lemmas, which taken together show that the second adjoint of a symmetric operator is symmetric if and only if the operator is weakly compact.

LEMMA 4.1. *An operator $T: \mathcal{X}^* \longrightarrow \mathcal{X}$ is symmetric if and only if its dual $T^*: \mathcal{X}^* \longrightarrow \mathcal{X}^{**}$ is symmetric.*

PROOF. This is a straightforward consequence of the defining relation for symmetry. ∎

LEMMA 4.2. *Suppose $T: \mathcal{X} \longrightarrow \mathcal{X}^*$ is symmetric. Then $T^*: \mathcal{X}^{**} \longrightarrow \mathcal{X}^*$ is symmetric if and only if T is weakly compact.*

PROOF. If T^* is symmetric, then by Lemma 4.1, T^{**} is symmetric, and $T^{**} = T^*$. Thus the range of T^{**} is contained in $\mathcal{X}^*$, and by Theorem VI.4.2 of [6], T is weakly compact. Conversely, suppose that T is weakly compact, so that the range of T^{**} is contained in $\mathcal{X}^*$. Let $\phi, \psi \in \mathcal{X}^{**}$. Let $\{x_\alpha\}$ be a bounded net in $\mathcal{X}$ that converges weak-star to ϕ, and regard x_α also as an element of $\mathcal{X}^{**}$. Then $T^* x_\alpha = Tx_\alpha$, by symmetry of T. Using the fact that $T^{**}\phi \in \mathcal{X}^*$, we then obtain $\langle \phi, T^*\psi \rangle = \lim \langle x_\alpha, T^*\psi \rangle = \lim \langle Tx_\alpha, \psi \rangle = \lim \langle T^* x_\alpha, \psi \rangle = \lim \langle x_\alpha, T^{**}\psi \rangle = \langle \phi, T^{**}\psi \rangle = \langle T^*\phi, \psi \rangle$. Thus T^* is symmetric. ∎

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LEMMA 4.3. *If $T: \mathcal{X}^* \longrightarrow \mathcal{X}$ is compact and symmetric, then T is continuous from the bounded-weak-star topology of $\mathcal{X}^*$ to the norm topology of $\mathcal{X}$.*

PROOF. Suppose $\{z_\alpha\}$ is a bounded net in $\mathcal{X}^*$ that converges weak-star to z, such that $Tz_\alpha \longrightarrow x$ in the norm of $\mathcal{X}$. It suffices to show that $Tz = x$. For this, note that if $w \in \mathcal{X}^*$, then $\langle x, w\rangle = \lim\langle Tz_\alpha, w\rangle = \lim\langle z_\alpha, Tw\rangle = \langle z, Tw\rangle = \langle Tz, w\rangle$, so $Tz = x$. ∎

Now we return to our dual Banach space $\mathcal{Z}$, with predual $\mathcal{Y}$. In view of the representation theorem in the preceding section and Lemma 4.3, we have natural isomorphisms between the following spaces: (i) the space of bounded-weak-star continuous 2-homogeneous analytic functions f on $\mathcal{Z}$; (ii) the space of bounded-weak-star continuous symmetric bilinear forms F on $\mathcal{Z}$; and (iii) the space of compact symmetric operators T from $\mathcal{Z}$ to $\mathcal{Y}$. The correspondence is given by

$$f(z) = F(z, z) = \langle Tz, z\rangle, \quad z \in \mathcal{Z},$$

and F is recaptured from its values on the diagonal by the polarization formula. Moreover, $\|F\| = \|T\|$, while the polarization formula yields $\|f\| \leq \|F\| \leq 2\|f\|$. Under this correspondence, the 2-homogeneous functions in $A(U)$ that are of finite-type correspond precisely to the finite-dimensional symmetric operators from $\mathcal{Z}$ to $\mathcal{Y}$. This leads to the following approximation theorem.

THEOREM 4.4. *A 2-homogeneous analytic function f on a dual Banach space $\mathcal{Z}$ belongs to $P(U)$ if and only if the corresponding operator $T: \mathcal{Z} \longrightarrow \mathcal{Z}^*$ has range in the predual $\mathcal{Y}$ of $\mathcal{Z}$ and is approximable in operator norm by symmetric finite-dimensional operators from $\mathcal{Z}$ to $\mathcal{Y}$. This occurs if and only if T is approximable in operator norm by finite-dimensional operators from $\mathcal{Z}$ to $\mathcal{Y}$ that are continuous with respect to the weak-star topology of $\mathcal{Z}$.*

PROOF. If the operator T corresponding to f is approximable in operator norm by a sequence $\{T_n\}$ of finite-dimensional (not necessarily symmetric) operators from $\mathcal{Z}$ to $\mathcal{Y}$ that are continuous with respect to the weak-star topology of $\mathcal{Z}$, then $f(z) = \langle Tz, z\rangle$ is a uniform limit on U of the 2-homogeneous analytic functions $f_n(z) = \langle T_n z, z\rangle$, which are finite-type polynomials in $P(U)$. Conversely, if the finite-type polynomials $f_n \in P(U)$ converge uniformly to f on U, then the corresponding finite-dimensional symmetric operators from $\mathcal{Z}$ to $\mathcal{Y}$ converge to T in operator norm. This establishes the second assertion. The first assertion follows from the second and the remark that finite-dimensional symmetric operators from $\mathcal{Z}$ to $\mathcal{Y}$ are continuous with respect to the weak-star topology of $\mathcal{Z}$. ∎

THEOREM 4.5. *There is a reflexive Banach space $\mathcal{Z}$ such that $P(U) \neq A(U)$, and in fact the 2-homogeneous bounded-weak-star continuous polynomials of finite-type are not dense in the space of all 2-homogeneous bounded-weak-star continuous polynomials.*

PROOF. The proof is based on a version of Enflo's theorem (see [8]), that there is a compact linear operator V on some reflexive Banach space $\mathcal{X}$ that is not approximable by finite-dimensional operators. Let $\mathcal{Z} = \mathcal{X} \oplus \mathcal{X}^*$, and define an operator T from $\mathcal{Z}$ to $\mathcal{Z}^*$ by the matrix representation

$$T = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}.$$

Then T is compact and symmetric, and T is not approximable by finite-dimensional operators. By Theorem 4.4, $f(z) = \langle Tz, z \rangle$ is not approximable by finite-type polynomials. ∎

5. **An approximation theorem.** Recall that the Banach space $\mathcal{Y}$ has the *approximation property* if for every $\varepsilon > 0$ and compact subset E of $\mathcal{Y}$, there is a finite-dimensional operator Q on $\mathcal{Y}$ such that $\|Qy - y\| < \varepsilon$ for all $y \in E$. The following lemma is the analog for the weak-star topology of Corollary 2.11 of [3].

LEMMA 5.1. *Suppose that the predual $\mathcal{Y}$ of the dual Banach space $\mathcal{Z}$ has the approximation property. Let F be an m-linear functional on $\mathcal{Z}$ that is bounded-weak-star continuous. Then for any $\varepsilon > 0$, there exist $y_{jr} \in \mathcal{Y}$, $1 \le j \le N$, $1 \le r \le m$, such that $|F(z_1, \ldots, z_m) - \sum_{j=1}^{N} \langle z_1, y_{j1} \rangle \cdots \langle z_m, y_{jm} \rangle| < \varepsilon$ for all $z_1, \ldots, z_m \in \mathcal{Z}$ satisfying $\|z_j\| \le 1$, $1 \le j \le m$.*

PROOF. The lemma is true if $m = 1$, and we make the induction hypothesis that it holds for $(m-1)$-forms that are bounded-weak-star continuous. Let T be the $(m-1)$-linear operator from $\mathcal{Z}^{m-1}$ to $\mathcal{Y}$ that represents F as in Theorem 3.1. Since the image under T of U^{m-1} is precompact in $\mathcal{Y}$, there is a finite-dimensional operator Q on $\mathcal{Y}$ such that $\|QT(z_1, \ldots, z_{m-1}) - T(z_1, \ldots, z_{m-1})\| \le \varepsilon$ whenever $z_j \in U$, $1 \le j \le m-1$. Define $G(z_1, \ldots, z_m) = \langle QT(z_1, \ldots, z_{m-1}), z_m \rangle$. Since $|G(z_1, \ldots, z_m) - F(z_1, \ldots, z_m)| \le \varepsilon \|z_m\| < \varepsilon$ for $z_1, \ldots, z_m \in U$, it suffices to approximate G. Choose v_j's in $\mathcal{Z}$ and u_j's in $\mathcal{Y}$ such that $Qy = \sum \langle y, v_j \rangle u_j$ for $y \in \mathcal{Y}$. For each fixed v_j, the $(m-1)$-form $(z_1, \ldots, z_{m-1}) \longrightarrow \langle T(z_1, \ldots, z_{m-1}), v_j \rangle$ is bounded-weak-star continuous, because T is continuous from the bounded-weak-star to the norm topologies. By the induction hypothesis, each $\langle T(z_1, \ldots, z_{m-1}), v_j \rangle$ has a uniform approximant on U^{m-1} as in the statement of Lemma 5.1. It follows that $G(z_1, \ldots, z_m) = \sum \langle T(z_1, \ldots, z_{m-1}), v_j \rangle \langle u_j, z_m \rangle$ has a uniform approximant on U^m of the required form. ∎

THEOREM 5.2. *Suppose that the predual $\mathcal{Y}$ of $\mathcal{Z}$ has the approximation property. Then $P(U) = A(U)$. Moreover, any bounded-weak-star continuous entire function on $\mathcal{Z}$ can be approximated uniformly on bounded sets by weak-star continuous finite-type polynomials.*

PROOF. By restricting to the diagonal, we see from Lemma 5.1 that any m-homogeneous bounded-weak-star continuous analytic function on $\mathcal{Z}$ can be approximated uniformly on U by weak-star continuous finite-type polynomials. In view of Lemmas 2.1 and 2.2, this yields the first statement. Since entire functions are approximated uniformly on bounded sets by polynomials, the second statement can be proved along the same lines. ∎

In connection with Theorems 4.5 and 5.2, the following problem arises: to find necessary and sufficient conditions, in terms of the linear approximation property of the underlying Banach spaces, for analytic functions to be approximable by finite-type polynomials.

6. **Extension of bounded-weak continuous analytic functions.** Now we specialize to the case in which $\mathcal{Z} = \mathcal{X}^{**}$ is the double dual of a Banach space $\mathcal{X}$. We are interested in determining which analytic functions on the open unit ball B of $\mathcal{X}$ extend to be weak-star continuous on the closed unit ball $\bar{U}$ of $\mathcal{X}^{**}$. Any such extension is evidently unique.

A function on $\mathcal{X}$ is said to be *bounded-weak continuous* if its restriction to any bounded set is weakly continuous. If a function on $\mathcal{X}^{**}$ is bounded-weak-star continuous, then its restriction to $\mathcal{X}$ is bounded-weak continuous on $\mathcal{X}$. Our goal is to extend bounded-weak continuous analytic functions on $\mathcal{X}$ to bounded-weak-star continuous analytic functions on $\mathcal{X}^{**}$. We begin by extending m-homogeneous analytic functions.

An m-form F on $\mathcal{X}$ is *bounded-weak continuous* if the restriction of F to B^m is continuous with respect to the product of the weak topologies. From the polarization formula it follows that an m-homogeneous analytic function is bounded-weak continuous if and only if the associated symmetric m-form is bounded-weak continuous.

To a continuous m-form F on $\mathcal{X}$ we can associate the representation

$$F(x_1,\dots,x_m) = \langle T(x_1,\dots,x_{m-1}), x_m\rangle, \quad x_1,\dots,x_m \in \mathcal{X},$$

where T is an $(m-1)$-linear operator from $\mathcal{X}$ to $\mathcal{X}^*$. We also define

$$S_j(x)(x_1,\dots,x_{m-1}) = F(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_{m-1}), \quad x,x_1,\dots,x_{m-1} \in \mathcal{X},$$

so that S_j is a continuous linear operator from $\mathcal{X}$ to the Banach space of continuous $(m-1)$-linear functionals on $\mathcal{X}$.

LEMMA 6.1. *The following are equivalent, for a continuous m-linear form F on* $\mathcal{X}$:

(i) F is bounded-weak continuous.

(ii) F has an extension to $\mathcal{X}^{**}$ *that is bounded-weak-star continuous.*

(iii) S_j *is a compact operator, for* $1 \le j \le m$.

(iv) T is continuous, from bounded-weak to norm topologies.

(v) T extends to an $(m-1)$*-linear form from* $\mathcal{X}^{**}$ *to* $\mathcal{X}^*$ *that is continuous from bounded-weak-star to norm topologies.*

PROOF. The equivalence of (i) and (iii) is essentially proved in [3]; one need check only that with the hypothesis (iii) the symmetry assumption used in [3] can be dropped. It is also proved in [3] that these imply that F is uniformly continuous with respect to the weak topology on $\bar{B}^m$. The uniform continuity with respect to the weak topology implies that F extends continuously to the completion, which is the m-fold product of the unit ball in the double dual with the weak-star topology. Consequently (i) and (iii) imply (ii).

That (ii) implies (v) follows from Theorem 3.1, while (v) trivially implies (iv), and (iv) easily implies (i). ∎

As in Section 2, the problem of approximating an analytic function f on X can be reduced to the problem of approximating its Taylor coefficients. The proof of the following lemma is almost identical to that of Lemma 2.1.

LEMMA 6.2. *Let f be analytic on the open unit ball B in X, with Taylor series $\sum f_m$. If f is weakly continuous on B, then each f_m is bounded-weak continuous. Conversely, if each f_m is bounded-weak continuous, and if f is uniformly continuous on B with respect to the norm, then f extends to be weakly continuous on the closed unit ball $\bar{B}$.*

THEOREM 6.3. *If f is a bounded analytic function on the open unit ball B of X, then f extends to be weak-star continuous on the closed unit ball $\bar{U}$ of X^{**} if and only if f is weakly continuous on B and f is uniformly continuous with respect to the norm.*

PROOF. Suppose $f = \sum f_m$ is weakly continuous on B. By Lemma 6.2 each f_m is bounded-weak continuous, as is the corresponding symmetric m-form F_m. By Lemma 6.1, F_m extends to be bounded-weak-star continuous on X^{**}. Restricting to the diagonal, we obtain a weak-star continuous extension of f_m to $\bar{U}$. If furthermore f is uniformly continuous on B, the Cesàro means of the partial sums of $\sum f_m$ converge uniformly to f on B. Since B is weak-star dense in $\bar{U}$, the Cesàro means of the partial sums of the extensions converge uniformly on $\bar{U}$, to an extension of f that is weak-star continuous on $\bar{U}$. The reverse implication is trivial. ∎

Thus $A(U)$ is isometrically isomorphic to the algebra of uniformly continuous analytic functions on B that are weakly continuous. Combining Theorems 5.2 and 6.3 we obtain immediately the following, which is essentially Corollary 2.11 of [3].

COROLLARY 6.4. *Suppose that X^* has the approximation property. Then the uniform limits of finite-type polynomials on the closed unit ball of X consists of precisely the weakly continuous functions on the closed unit ball that are analytic on the open ball and uniformly continuous with respect to the norm.*

7. Entire functions.

Recall that $H_b(X)$ is the algebra of entire functions on X that are bounded on bounded sets, with the topology of uniform convergence on bounded sets. According to [1], each function $f \in H_b(X)$ has a canonical extension $\hat{f}$ to X^{**}, so that the extension operator $f \longrightarrow \hat{f}$ is an algebra isomorphism that embeds $H_b(X)$ as a closed subalgebra of $H_b(X^{**})$. For more on the algebra $H_b(X)$ and the canonical extension operator, see also [2]. The following theorem overlaps work in [9], in which the equivalence of (i) and (iii) is obtained in quite general circumstances.

THEOREM 7.1. *The following are equivalent, for $f \in H_b(X)$:*

(i) f is bounded-weak continuous.

(ii) $\lim f(x_\alpha)$ exists for any bounded net $\{x_\alpha\}$ in X converging weak-star in X^{**}.

(iii) f has an extension to X^{**} that is bounded-weak-star continuous.

(iv) The canonical extension $\hat{f}$ of f to X^{**} is bounded-weak-star continuous.

Further, if $\mathcal{X}^*$ has the approximation property, these are equivalent to:

(v) f is approximable uniformly on bounded subsets of $\mathcal{X}$ by finite-type polynomials.

PROOF. Since f is bounded on bounded sets, the Schwarz lemma shows that on any bounded set f is uniformly continuous with respect to the norm. Theorem 6.3, applied to dilates of the unit ball of $\mathcal{X}$, then shows that (i) implies (iii). Since (iii) implies (ii) and (ii) implies (i), these three statements are equivalent. That these statements imply (iv) can be seen from the construction of the extension operator given in [1]. It also follows from the theorem proved in [5] that for each $z \in \mathcal{X}^{**}$ there is a bounded net $\{x_\alpha\}$ in $\mathcal{X}$ such that $f(x_\alpha) \longrightarrow \hat{f}(z)$ for all $f \in H_b(\mathcal{X})$. Thus any bounded-weak-star continuous extension of f to $\mathcal{X}^{**}$ must coincide with $\hat{f}$. In any event, (i) through (iv) are equivalent. The remaining equivalence follows from Corollary 6.4. ■

The extension procedure of [1] is shown in [5] to determine an isometric algebra isomorphism $f \longrightarrow \hat{f}$ of $H^\infty(B)$ and a closed subalgebra of $H^\infty(U)$. The theorem cited in the proof above shows that this canonical extension $\hat{f}$ coincides with the weak-star continuous extension given in Theorem 6.3, whenever the latter exists.

Now denote by $M_b = M_b(\mathcal{X})$ the spectrum of $H_b(\mathcal{X})$. It consists of the nonzero continuous complex-valued homomorphisms of $H_b(\mathcal{X})$, endowed with the $H_b(\mathcal{X})$-topology. For a discussion of M_b, see [2].

Each $z \in \mathcal{X}^{**}$ determines a homomorphism $\varphi_z \in M_b$, obtained by evaluating the canonical extension of $f \in H_b(\mathcal{X})$ at z:

$$\varphi_z(f) = \hat{f}(z), \quad f \in H_b(\mathcal{X}),\ z \in \mathcal{X}^{**}.$$

Thus there is a natural embedding of $\mathcal{X}^{**}$ into $M_b(\mathcal{X})$, which is continuous with respect to the norm topology of $\mathcal{X}^{**}$ (but not the weak-star topology).

When the finite-type polynomials are dense in $H_b(\mathcal{X})$, each $\varphi \in M_b$ is determined completely by the linear functional $z = \varphi|_{\mathcal{X}^*}$ in $\mathcal{X}^{**}$, and φ coincides with the evaluation homomorphism φ_z. Thus M_b coincides with $\mathcal{X}^{**}$, at least as a point set. For instance, in the case $\mathcal{X}$ is the Banach space c_0 of null sequences, the Littlewood-Bogdanowicz-Pelczynski theorem (see [12], [2]) asserts that the finite-type polynomials are dense in $H_b(c_0)$, and $M_b(c_0)$ coincides with ℓ^∞ as a point set.

When $\mathcal{X}^*$ has the approximation property, a converse result is valid.

THEOREM 7.2. *Suppose that $\mathcal{X}^*$ has the approximation property. Then $M_b(\mathcal{X})$ coincides with $\mathcal{X}^{**}$ (as a point set) if and only if the finite-type polynomials are dense in $H_b(\mathcal{X})$.*

PROOF. As observed above, the backward implication is true even without the approximation property. For the forward implication, suppose $M_b = \mathcal{X}^{**}$. The main point of the proof is that the closed unit ball $\bar{U}$ of $\mathcal{X}^{**}$ is then compact with respect to the $H_b(\mathcal{X})$-topology. This is because the closed ball coincides with the subset of the spectrum of homomorphisms that are continuous with respect to the supremum norm over the unit ball of $\mathcal{X}$. A compactness argument then shows that the $H_b(\mathcal{X})$-topology coincides with the weak-star topology on $\bar{U}$. It follows that functions in $H_b(\mathcal{X})$ have extensions to $\mathcal{X}^{**}$ that are bounded-weak-star continuous. By part (v) of Theorem 7.1 they are uniformly approximable on bounded sets by finite-type polynomials. ∎

Thus, assuming $\mathcal{X}^*$ has the approximation property, we see that $M_b(\mathcal{X}) = \mathcal{X}$ if and only if $\mathcal{X}$ is reflexive and the finite-type polynomials are dense in $H_b(\mathcal{X})$. An example of a space with this property is the Tsirelson space (*cf.* [13]).

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