Canad. Math. Bull. Vol. 18 (5), 1975

# AN APPLICATION OF A GENERALIZATION OF TERQUEM'S PROBLEM

BY
STEPHEN M. TANNY

Moser and Abramson [4] proved: given $m \geq 2$ and $0 \leq k_1, k_2, \ldots, k_p < m$, the number of $p$-combinations

$$1 \leq x_1 < x_2 < \cdots < x_p \leq n$$

satisfying

$$x_1 \equiv 1+k_1 \pmod m, \qquad x_j \equiv x_{j-1}+1+k_j \pmod m, \qquad j = 2, 3, \ldots, p,$$

is

$$\text{(1)} \qquad f(n, p; m \mid k_1, k_2, \ldots, k_p) = \binom{\left[\frac{n+(m-1)p-(k_1+k_2+\cdots+k_p)}{m}\right]}{p}$$

($[x]$ denotes the greatest integer $\leq x$).

The case $m=2$, $k_j=0$, $j=1, 2, \ldots, p$ is the well-known Terquem's problem [8], while $k_j=0$, $j=1, 2, \ldots, p$ is Skolem's generalization [4] of Terquem's problem.

In another direction Terquem's problem can be generalized to: find the number $F(n, p; \alpha, \beta)$ of $p$-combinations in which the first $\alpha$ integers are of the same parity, the next $\beta$ are of opposite parity to that of the previous $\alpha$, the next $\alpha$ are of opposite parity to the previous $\beta$, and so on (the final group may have fewer than $\alpha$ or $\beta$ elements). The numbers $F(n, p; 1, 1)$ and $F(n, p; \alpha, 1)$ have been determined ([3], [6], [7]). The purpose of this note is to observe that $F(n, p; \alpha, \beta)$ can be obtained from (1) by letting $m=2$ and adding the two counts which result by taking $k_1=0$ or 1 (corresponding to the cases $x_1$ is odd or even) and

$$k_i = \begin{cases} 0 & \text{if} \quad i = \alpha+1, \quad \alpha+\beta+1, \quad 2\alpha+\beta+1, \quad 2\alpha+2\beta+1, \ldots, \\ 1 & \text{otherwise.} \end{cases}$$

It remains only to determine $k_1+k_2+\cdots+k_p$ i.e., how many of the $k_i$'s, with $i>1$, are equal to 0.

First write $p$ in the form

$$p = p_1\alpha+p_2\beta+r, \qquad 0 \leq p_1-p_2 \leq 1,$$
$$0 \leq r \leq \begin{cases} \alpha-1 & \text{if} \quad p_1 = p_2, \\ \beta-1 & \text{if} \quad p_1 = p_2+1. \end{cases}$$

* The author wishes to express his gratitude to the referee for improving the readability of this note.



https://doi.org/10.4153/CMB-1975-135-8 Published online by Cambridge University Press

Now it is easy to see that

$$k_1+k_2+\cdots+k_p = k_1+p-p_1-p_2-1-\delta_{r,0}$$

($\delta_{a,b}=1$ if $a=b$, and $=0$ if $a\neq b$). Adding the two counts corresponding to $k_1=0$ or 1, we obtain

$$F(n, p; \alpha, \beta) = \binom{\left[\frac{n+p_1+p_2+1-\delta_{r,0}}{2}\right]}{p}+\binom{\left[\frac{n+p_1+p_2-\delta_{r,0}}{2}\right]}{p}$$

In particular, when $\alpha=\beta=1$ we have: the number of alternating (in parity) $p$-combinations is

$$F(n, p; 1, 1) = \binom{\left[\frac{n+p}{2}\right]}{p}+\binom{\left[\frac{n+p-1}{2}\right]}{p}$$

([1], [2], [3], [6], [7]).

## References


1. C. A. Church and H. W. Gould, *Problem of Terquem and an extension of Fibonacci numbers*, Fibonacci Quarterly **5**, (1967), pp. 59–68.
2. F. Hering, *A problem on inequalities* Am. Math. Monthly, vol. **78**, no. 3, March (1971), pp. 275–276.
3. F. Hering, *A generalization of the arithmetic-geometric mean inequality and an application to finite sequences of zeros and ones*, Israel J. of Math., Vol. II, no. 1 (1972), pp. 14–30.
4. W. O. J. Moser and Morton Abramson, *Generalizations of Terquem's Problem*, J. of Comb. Theory, vol. **7**, no. 2, Sept. (1969) pp. 171–180.
5. J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.
6. S. M. Tanny, *On alternating subsets of integers*, Fibonacci Quarterly (to appear).
7. S. M. Tanny, *Generating functions and generalized alternating subsets*, Discrete Math. (to appear).
8. O. Terquem, *Sur un symbole combinatoire d'Euler et son utilité dans l'analyse*, J. Math. Pures Appl. **4** (1839), pp. 177–184.



Department of Mathematics,
University of Toronto.


https://doi.org/10.4153/CMB-1975-135-8 Published online by Cambridge University Press