GENERATION OF THE LOWER CENTRAL SERIES II

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1. Introduction. In this article, we obtain results on commutators in Sylow subgroups of the lower central series, extending the work of Dark and Newell [**2**], Rodney [**12**, **13**] and Aschbacher and the author [**1**, **6**, **7**].

Some notation is required for the statement of the main results. Let r be a positive integer and define

$$[x_1]=x_1, \qquad [x_1, x_2]=x_1^{-1}x_2^{-1}x_1x_2,$$

and

$$[x_1, \ldots, x_r]=[[x_1, \ldots, x_{r-1}], x_r] \quad \text{for} \quad r \geq 3$$

where $x_1, \ldots, x_r$ are elements in a group G. Let $\Gamma_rG=\{[x_1, \ldots, x_r] \mid x_i \in G\}$ be the set of r-fold commutators in G. Then $L_rG=\langle \Gamma_rG\rangle$ is the rth term in the lower central series of G. Set $L_\infty G = \cap L_rG$.

THEOREM A. *Suppose L_rG is finite and $P \in \mathrm{Syl}_p(L_rG)$ is abelian of rank at most* 2. *If any of the following conditions hold then $P \subset \Gamma_rG$.*

(i) $p \geq 5$.
(ii) *P is cyclic.*
(iii) *P has exponent p.*
(iv) $P \cap L_\infty G \neq 1$.
(v) $P \cap L_{r+1}G=1$.
(vi) $r \leq 2$.

This result is known for $r=2$. It was first proved by Rodney [**13**] for $P \in \mathrm{Syl}_p(G)$ of exponent p. The complete proof of (vi) is [**7**, Theorem A]. The main idea of the proof is to reduce to the case where $P=L_rG$. With this hypothesis, (iii) and (iv) are given in [**6**], while (ii) and (v) are proved in [**2**]. However, (i) is still a new result even in this more restricted situation. By examples in [**1**], [**2**], and [**6**], rank 2 cannot be replaced by rank 3. Moreover, (i) fails for $p=2$ (and possibly for $p=3$).

The proof that when $p \geq 5$ and $P=L_rG$ is an abelian rank 2 p-group then $P=\Gamma_rG$ splits essentially into two cases. The first is when $P=L_\infty G$ and is handled by [**1**, Theorem C]. The more difficult case is when G is a p-group. In fact, we consider the more general problem of when $P \subset (\Gamma_rG)^k$. This also breaks up into the two cases described above (Theorem 2.1). An example (Section 4) is given to show that the p-group situation is the relevant obstruction to determining k in terms of the rank of P.

Combining these techniques with a result of Gallagher, we obtain the following results.

THEOREM B. *Suppose G' is a p-group of order p^k with $k<n(n+1)$.*

(a) *If $L_\infty G$ is abelian, then $G'=(\Gamma_2G)^n$.*

(b) $G'=(\Gamma_2G)^{2n}$.

THEOREM C. *If $G'\leqslant Z(G)$ and G' is a p-group of rank less than $n(n+1)$, then $G'=(\Gamma_2G)^n$.*

By examples in [**5**], the bounds in Theorems B and C are of the right order of magnitude. We define a function $f=f(p,r,d)$ by the following: if $P=L_rG$ is an abelian p-group of rank d, then $P=(\Gamma_rG)^f$ and f is the least positive integer satisfying this.

THEOREM D. (i) $\sqrt{d}-1\leqslant f\leqslant 2d$, *for* $r=2$ *and* 3. (ii) $d^{(r-2)/(r-1)}/6r<f\leqslant 2d$, *for* $r>3$.

Finally, we note that:

THEOREM E. *If $|G|<96$ or $|L_rG|<8$, then $L_rG=\Gamma_rG$.*

Moreover, these bounds are best possible for $r>2$ (for $r=2$, replace 8 by 16). See [**6**] and [**7**] for examples.

The paper is organized as follows. In Section 2, the proof of Theorem A is reduced to the case $L_rG=P$. This case is handled in Section 3. Some examples pertaining to lower bounds for f are given in Section 4. Finally, Theorems B – E are proved in Section 5. We shall use notation as in [**4**]. I wish to thank the referee for his very careful reading of the article and many valuable comments.

2. Reduction of Theorem A. The first result describes how the condition that L_rG has an abelian Sylow p-subgroup splits a Sylow p-subgroup into two nicer pieces.

THEOREM 2.1. *Let G be a finite group with $S\in\mathrm{Syl}_p(G)$. Suppose $n\geqslant r$ and set $P=S\cap L_rG$ and $T=S\cap L_nG$.*

(a) $P=(L_rS)T$.

(b) *If T is abelian and $N=N_G(T)$, then $P=(L_rS)[T,H]\leqslant L_rN$, where H is a Hall p'-subgroup of N.*

(c) *If $S\lhd G$ and $V=S\cap L_\infty G$ is abelian, then $S=CV$ and $C\cap V=1$, where $C=C_S(H)$ and H is a Hall p'-subgroup of G. Moreover, if G is solvable and m is a positive integer such that $L_rC=(\Gamma_rC)^m$ and* $\operatorname{rank}V<2^{m+1}-1$, *then $P\subset(\Gamma_rG)^m$.*

Proof. Since G/L_nG is nilpotent, $(L_rS)L_nG/L_nG$ is a Sylow p-subgroup of L_rG/L_nG. Thus $P\geqslant(L_rS)T\in\mathrm{Syl}_p(L_rG)$ and (a) follows. If T is abelian, then by [**1**, Corollary 5.2] $T=(L_nS)[T,H]\leqslant L_nN$, where H is a Hall p'-subgroup of N. Hence (b) holds.

Now assume $S\lhd G$ and V is abelian. Clearly $V=[S,H]$ and so by [**4**, Theorems 5.2.3 and 5.3.5], $S=CV$, $V=[V,H]$ and $V\cap C=1$. If G is solvable, then by [**1**, Theorem 4.1] there exist $h_1,\dots,h_m\in H$ such that

$$V=\prod_{i=1}^{m}[V,h_i]=\prod_{i=1}^{m}[V,h_i,\dots,h_i].$$

Fix $h \in H$. Then for any $v \in V$, and $c_1, c_2 \in C$,

$$[vc_1, c_2h] = [v, c_2h]^{c_1}[c_1, c_2].$$

Since $\langle c_2h\rangle \geq \langle h\rangle$, $V \lhd G$ is abelian, and $[C, h] = 1$,

$$\{[v, c_2h]^{c_1} \mid v \in V\} \supset [V, h].$$

Similarly, it follows that

$$\{[vc_1, c_2h, \ldots, c_rh] \mid v \in V\} \supset [V, h, \ldots, h][c_1, \ldots, c_r] = [V, h][c_1, \ldots, c_r].$$

Hence

$$\Delta_i = \{[vc_1, c_2h_i, \ldots, c_rh_i] \mid v \in V, c_j \in C\} = [V, h_i]\Gamma_r(C).$$

Finally, note that since C normalizes $[V, h_i]$,

$$\Delta_1 \ldots \Delta_m = \left(\prod_{i=1}^{m} [V, h_i]\right)(\Gamma_rC)^m = V(L_rC).$$

Now (c) follows for $V(L_rC) = V(L_rS) = P$.

We now prove Theorem A (modulo results in section 3). So assume L_rG is finite and $P \in \mathrm{Syl}_p(L_rG)$ is abelian of rank at most 2. By [**6**, Lemma 1.4], we can assume G is finite. By Theorem 2.1b, we can also assume $P \lhd G$ and $P \cap L_\infty G = T = [T, H]$ for H a Hall p'-subgroup (note that conditions (i)–(vi) remain valid under these reductions). Now $P = T \times C_P(H)$ by [**4**, Theorem 5.2.3]. If $P = T$, then $P \subset \Gamma_rG$ by [**1**, Theorem C]. Otherwise T is cyclic and so there exists $h \in H$ with $[T, h] = T$. Moreover, since $G = N_G(S)G' = N_G(S)C_G(T)$, we can assume $h \in N_G(S)$. Consider $M = \langle S, h\rangle$. Then $L_rM = L_rS[h, T] = P$. Thus it suffices to assume $G = M$ and $P = L_rG$. Now (ii), (iii), and (iv) follow by [**6**, Theorem 3.2] and (v) follows by [**2**, Theorem 2]. As remarked before (vi) is [**7**, Theorem A]. Finally, if (i) holds but (iv) fails, then $T = 1$, $P = L_rS$ and Theorem 3.6 applies.

3. The case $L_rG = P$. We need some commutator calculus. Suppose $g \in G$. Define $\gamma_0(g) = \langle g\rangle$ and $\gamma_{i+1}(g) = [\gamma_i(g), G]$. Note that $\gamma_i(g) \lhd G$ for $i > 0$, and by the three subgroup lemma,

$$[\gamma_i(g), L_rG] \leq \gamma_{i+r}(g), \tag{3.1}$$

$$[\gamma_i(g), \gamma_j(g)] \leq \gamma_{i+j+1}(g). \tag{3.2}$$

PROPOSITION 3.3.

(a) $$[s, tu, x_1, \ldots, x_r] \equiv [s, t, x_1, \ldots, x_r][s, u, x_1, \ldots, x_r] \times [s, t, u, x_1, \ldots, x_r] \bmod \gamma_{r+2}([s, t]).$$

(b) $$[s, t^i, x_1, \ldots, x_r] \equiv [s, t, x_1, \ldots, x_r]^i \times [s, t, t, x_1, \ldots, x_r]^{i(i-1)/2} \bmod \gamma_{r+2}([s, t]).$$

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Proof. (a) Set $y=[s,t]$. Induct on r. If $r=0$, then

$$[s,tu]=[s,u][s,t][s,t,u]. \qquad (*)$$

The result holds in this case since $[s,t,G']\leq\gamma_2(y)$. Now assume $r>0$ and $\gamma_{r+2}(y)=1$. By induction

$$[s,tu,x_1,\ldots,x_{r-1}]=abcd,$$

where $a=[s,t,x_1,\ldots,x_{r-1}]$, $b=[s,u,x_1,\ldots,x_{r-1}]$, $c=[s,t,u,x_1,\ldots,x_{r-1}]$ and $d\in\gamma_{r+1}(y)\leq Z(G)$. Thus,

$$[s,t,u,x_1,\ldots,x_r]=[abcd,x_r]=[abc,x_r].$$

However, by $(*)$ (or its inverse),

$$\begin{aligned}[abc,x_r]&=[ab,x_r][ab,x_r,c][c,x_r]\\&=[a,x_r][a,x_r,b][b,x_r][ab,x_r,c][c,x_r].\end{aligned}$$

Now (a) follows by noting that $[ab,x_r,c]\in[G',\gamma_r(y)]\leq\gamma_{r+2}(y)$ and $[a,x_r,b]\in[\gamma_{r-1}(y),G,G']\leq\gamma_{r+2}(y)$.

(b) follows from (a) by a straightforward induction argument.

LEMMA 3.4. *Suppose L_rG is an abelian p-group of rank at most two and $L_{r+1}G\leq\mho^1L_rG$.*

(a) *There exist j, $1\leq j\leq r$, and u_i, $1\leq i\neq j\leq r$, such that*

$$L_rG=[u_1,\ldots,u_{j-1},G,u_{j+1},\ldots,u_r].$$

(b) *Moreover, if $p>2$, then*

$$L_rG=\{[u_1,\ldots,u_{j-1},g,u_{j+1},\ldots,u_r]\mid g\in G\}=\Sigma.$$

Proof. (a) Without loss of generality, $\mho^1L_rG=L_{r+1}G=1$. Choose $J\subset I=\{1,\ldots,r\}$ maximal so that there exist u_j, $j\in J$ with $L_rG=[E_1,\ldots,E_r]$ where $E_j=u_j$ if $j\in J$ and $E_j=G$ otherwise. Assume $k<l\in I-J$. Hence $L_rG=\langle[x_1,\ldots,x_r],[y_1,\ldots,y_r]\rangle$ where $x_j=y_j=u_j$ if $j\in J$. Suppose $z=[x_1,\ldots,y_k,\ldots,x_r]\neq1$. Then either

$$L_rG=\langle[x_1,\ldots,x_r],[x_1,\ldots,y_j,\ldots,x_r]\rangle$$

or

$$L_rG=\langle[x_1,\ldots,y_k,\ldots,x_r],[y_1,\ldots,y_r]\rangle.$$

In the first case $I-\{k\}$ satisfies the conclusion, and in the second case $J\cup\{j\}$ satisfies the same condition as J. This contradicts the maximality of J. So $z=1$. Similarly $[y_1,\ldots,x_l,\ldots,y_r]=1$; but now $J\cup\{k\}$ satisfies the condition with $u_k=x_ky_k$.

(b) We note that we can assume $j\neq1$ (for if $j=1$ then $j=2$ will also satisfy the conclusion in (a)). Set $s=[u_1,\ldots,u_{j-1}]$ and define $\phi:G\to\Sigma$ by $\phi(g)=[s,g,u_{j+1},\ldots,u_r]$. First we shall show that for any $g,v_{j+1},\ldots,v_r\in G$ we have

$$y=[s,g^{ep^i},v_{j+1},\ldots,v_r]\equiv[s,g,v_{j+1},\ldots,v_r]^{ep^i}\bmod\mho^{i+1}L_rG. \qquad (**)$$

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Induct on i. If $i=0$, this follows from Proposition 3.3b and the fact that $L_{r+1}G \leqslant \mho^1 L_r G$. So assume $i>0$. Again by Proposition 3.3b,

$$y \equiv [s, g^{p^{i-1}}, v_{j+1}, \ldots, v_r]^{ep} \bmod B,$$

where $B = \gamma_{r-j+1}([s, g^{p^{i-1}}])^p \gamma_{r-j+2}([s, g^{p^{i-1}}])$. By induction,

$$\gamma_{r-j}([s, g^{p^{i-1}}]) \leqslant \mho^{i-1} L_r G.$$

Since $L_{r+1}G \leqslant \mho^1 L_r G$ it follows that $[G, \mho^k L_r G] \leqslant \mho^{k+1} L_r G$ and so $B \leqslant \mho^{i+1} L_r G$. Thus $(**)$ holds.

Suppose now that $\mho^k L_r G \neq 1 = \mho^{k+1} L_r G$. By (a), there exists $g \in G$ such that $\phi(g)$ has order p^{k+1} and so by $(**)$, $z = \phi(g^{p^k}) = \phi(g)^{p^k}$. In particular, $z \in \mho^k L_r G \leqslant Z(G)$. By induction on $|L_r G|$, if $x \in L_r G$ then $xz^e = \phi(h)$ for some positive integer e and $h \in G$. By Proposition 3.3a,

$$\phi(g^{-ep^k} h) \equiv \phi(g^{-ep^k}) \phi(h) \bmod \gamma_{r-j+1}([s, g^{-ep^k}]).$$

However by $(**)$,

$$\phi(g^{-ep^k}) = z^{-e} \quad \text{and} \quad \gamma_{r-j}([s, g^{-ep^k}]) \leqslant \mho^k L_r G \leqslant Z(G).$$

Thus

$$\phi(g^{-ep^k} h) = z^{-e} z^e x = x,$$

and so ϕ is surjective as desired.

The reason we assume $p>3$ is apparent in the next lemma. If $p>3$, then

$$\sum_{i=0}^{p-1} i^2 \equiv 0 \bmod p.$$

LEMMA 3.5. *Let G be a p-group with $p>3$. If $x, y \in G$ with $a=[x, y]$, $\langle a^G \rangle = \langle a, b \rangle$ abelian and $[a, G] \leqslant \langle a^p, b \rangle$, then $[x, y^p] \equiv a^p \bmod B$, where $B = \langle a^{p^2}, b^p \rangle$.*

Proof. Set $A = \langle a^G \rangle$. Then $[A, G] \leqslant \langle a^p, b \rangle$. If $c = [a, y] \in \mho^1 A$, then $[y, \mho^1 A] \leqslant \mho^2 A$, and so

$$[x, y^i] \equiv a^i c^{i(i-1)/2} \bmod \mho^2 A$$

by Proposition 3.3(b). Hence as $p \neq 2$, $[x, y^p] \equiv a^p \bmod B$. Otherwise, we can take $b = [a, y]$.

Now $[b, y] = a^{\alpha p} b^{\beta p}$. Then a straightforward tedious computation shows that modulo B

$$[x, y^p] \equiv a^p \prod_{i=1}^{p-1} [a, y^i]$$

$$\equiv a^p \prod_{j=1}^{p-1} a^{\alpha p j(j-1)/2} b^j \equiv a^p$$

since $p>3$.

THEOREM 3.6. *If $L_r G$ is a rank 2 abelian p-group with $p>3$, then $L_r G = \Gamma_r G$.*

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Proof. As in Section 2, we can assume G is a p-group. Let G be a counterexample with $|L_rG|$ minimal. Choose $a \in L_rG - \Gamma_rG$. Set $A = \langle a^G \rangle = \langle a, b = [a, g] \rangle$ for some $g \in G$. Let p^α and p^β denote the orders of a and b respectively. Note that $\alpha \geqslant \beta$. First assume $\beta < \alpha$. Then $B = \mho^{\alpha-1}A = \langle a^{p^{\alpha-1}} \rangle \leqslant Z(G)$. If $\alpha > 1$, then by passing to G/B, some generator of $\langle a \rangle$ is in Γ_rG, and thus $a \in \Gamma_rG$ by [**6**, Lemma 1.3]. So $\alpha = 1$ and $\beta = 0$. If $a = c^p$ with $c \in L_rG$, then as above $c \in \Gamma_rG$. Say $c = [d, h]$ with $d \in \Gamma_{r-1}G$ and set $e = [c, h]$. Then $e^p = [c^p, h] = 1$. Moreover as $e \in \Omega_1 L_{r+1}G$ and $a \notin L_{r+1}G$ (as $L_{r+1}G = \Gamma_{r+1}G \subset \Gamma_rG$ since $|L_{r+1}G| < |L_rG|$), $e \in Z(G)$. Thus, $[d, h^p] = c^p e^{p(p-1)/2} = a$ since $p \neq 2$. Thus, $a \notin \mho^1 L_rG$ and so $L_rG = \langle a, x \rangle$. We can assume $x \in \Gamma_rG$ and so $x = [g, y]$ for some $g \in G$ and $y \in \Gamma_{r-1}G$. If $[G, y] = \langle x \rangle$, then $\langle x \rangle \lhd G$ and so $L_{r+1}G = [G, x] \leqslant \mho^1 L_rG$. Then $L_rG = \Gamma_rG$ by Lemma 3.4b. Otherwise $[G, y] = L_rG$. We claim $a = [h, y]$ for some $h \in G$. Induct on the order of x. If $x^p = 1$, then the map $h \to [h, y] \in Z(G)$ is an endomorphism from G onto L_rG. So assume x has order p^{k+1}. By induction ($x^{p^k} \in Z(G)$), we see that $[h, y] = ax^{ep^k}$ for some integer e. By Lemma 3.5, $[g^p, y] \equiv x^p \bmod \langle x^{p^2} \rangle$, and continuing we see that $[g^{p^k}, y] = x^{p^k}$. Hence $[g^{-ep^k}h, y] = a \in \Gamma_rG$.

So $\beta = \alpha$. Thus $B = \langle b^{p^{\alpha-1}} \rangle = \mho^{\alpha-1}[G, A] \leqslant Z(G)$. Hence by passing to G/B, $a(b^{\lambda p^{\alpha-1}}) \in \Gamma_rG$ for some λ. By Lemma 3.5, $[a, g^p] \equiv b^p \bmod \mho^2 A$, and continuing we find that $[a, g^{\lambda p^{\alpha-1}}] = b^{\lambda p^{\alpha-1}}$ and so a is conjugate to $ab^{\lambda p^{\alpha-1}}$. This completes the proof.

By Example 3.1 in [**6**], $p > 2$ is necessary. If $r = 2$, and $p > 3$, one can replace rank 2 by rank 3 [**7**, Theorem B]. This would seem to provide some evidence that there is a counterexample with $p = 3$ since there is such for $r = 2$ with L_2G of rank 3.

4. Lower bounds for f. Many examples have been given with $L_rG \neq (\Gamma_rG)^k$, particularly for $r = 2$ or for $k = 1$ (see [**1**], [**2**], [**5**], [**6**], [**7**], [**8**], [**10**] and [**11**]). We construct one which gives a good lower bound for $f = f(p, r, d)$. First note that for $r = 2$, it follows from [**5**] and [**7**] that:

PROPOSITION 4.1.
(a) $f(d) < \sqrt{d} - 1$.
(b) $f(3) = 1 \Leftrightarrow p > 3$.
(c) $f(2) = 1$.

For the rest of the section, assume $r > 2$. By Theorem A, $f(1) = 1$ for all p and $f(2) = 1$ for $p > 3$. Also by [**1**], $f(d) > \log_2(d+1) - 1$.

Now fix a prime p and $r > 2$. Let F be the free group on n generators. Set $H = F/(L_rF)^p L_{r+1}F$. By Witt's formula, L_rH is a free elementary abelian p-group of rank t, where

$$t = \frac{1}{r} \sum_{k \mid r} \mu(k) n^{r/k},$$

and μ is the Moebius function. It follows easily that $rt \geqslant n^{r-1}$. Now suppose d is a positive integer with $(n-1)^s \leqslant dr < n^s$, $s = r - 1$. Choose a subgroup M of L_rH of index p^d in L_rH. Set $G = H/M$. Then G is nilpotent, $L_{r+1}G = 1$, and L_rG is elementary abelian of rank d. By Proposition 3.3, the r-fold commutator is multiplicative in each variable. Thus

$|\Gamma_r G| < p^{nr}$. Hence if $f = f(p, r, d)$, then $p^d = |L_r G| = |(\Gamma_r G)^f| < |\Gamma_r G|^f < p^{nrf}$. So $f > d/nr$. Since

$$d^{1/s} \geq (n-1)r^{-1/s} \geq \tfrac{1}{2}r^{-1-(1/s)}(nr),$$

it follows that

$$f > d/nr \geq \tfrac{1}{2}r^{-1-(1/s)}d^{(s-1)/s} \geq d^{(s-1)/s}/6r.$$

We remark that for $r = 3$, one can construct a group G similar to that in [**2**, Proposition 3] in which L_3G is elementary abelian of rank n^2 and $L_3G \neq (\Gamma_3 G)^n$. Thus $f(p, 3, d) \geq \sqrt{d} - 1$. Also as r gets large, we can replace 6 by numbers tending to 2. We conjecture that $f(d) \geq d^{(r-1)/r} - 1$ (this is true for $r = 2$).

5. Theorems B–E. Theorem B follows as an easy consequence of Theorem 2.1 and a result of Gallagher.

THEOREM B. *Suppose G' is a p-group of order p^k with $k < n(n+1)$.*
(a) *If $L_\infty G$ is abelian then $G' = (\Gamma_2 G)^n$.*
(b) $G' = (\Gamma_2 G)^{2n}$.

Proof. Note that (b) follows from (a) and [**3**, Theorem 1b] by considering $G/(L_\infty G)'$. As usual, we assume G is finite. For (a), note that if $S \in \mathrm{Syl}_p(G)$, and C is as in Theorem 2.1c then $(\Gamma_2 C)^n = C'$ by Gallagher [**3**, Theorem 2]. The result now follows by Theorem 2.1c (for $n(n+1) < 2^{n+1} - 1$).

Theorem C also follows from the same result of Gallagher and the next lemma.

LEMMA 5.1. *Let G be a p-group with $G' \leq Z(G)$ and $G' \neq (\Gamma_2 G)^n$. Moreover, assume $|G|$ is minimal with respect to this property (for a fixed p and n). Then G' has exponent p.*

Proof. Choose $a \in L_2G - (\Gamma_2 G)^n$. Suppose $d = [u, v]^p = [u, v^p] \neq 1$ for some $u, v \in G$. As $|G| > |G/\langle d\rangle|$, $ad^i \in (\Gamma_2 G)^n$ for some i. By replacing d by d^i, we may assume that $ad \in (\Gamma_2 G)^n$. Say

$$ad = \prod_{i=1}^{n} [s_i, t_i].$$

Then

$$a = \prod_{i=1}^{n} [s_i, t_i][u, v^{-p}].$$

Since $a \in H'$, $H = \langle s_1, t_1, \ldots, s_n, t_n, u, v^p\rangle = G$. However, v^p is an element of the Frattini subgroup of G, and so G can be generated by $2n+1$ elements. However, this implies $G' = (\Gamma_2 G)^n$ by [**5**, Theorem 5.2]. So $[u, v]^p = 1$ for all $u, v \in G$, proving the lemma.

Now Theorem C follows from Theorem B. Moreover, by [**5**, Section 5], $n(n+1)$ can not be replaced by $(n+1)^2 + 1$ in either Theorem B or C. Recall that $f = f(p, r, d)$ is defined as follows: if $P = L_r G$ is an abelian p-group of rank d then $P = (\Gamma_r G)^f$ and f is as small as possible. We first obtain an upper bound for f.

PROPOSITION 5.2. *Let G be a finite group.*

(a) *If $\langle x\rangle = \langle y\rangle$ and $x \in (\Gamma_r G)^k$, then $y \in (\Gamma_r G)^k$.*

(b) *If $x \in (\Gamma_r G)^k$, then $\langle x\rangle \subset (\Gamma_r G)^{3k}$.*

(c) *If $x \in (\Gamma_r G)^k$ is a p-element, then $\langle x\rangle \subset (\Gamma_r G)^{2k}$.*

(d) *If $P \in \mathrm{Syl}_p(L_r G)$ is abelian of rank d, then $P \subset (\Gamma_r G)^{2d}$.*

Proof. As remarked before, (a) is [**6**, Lemma 1.3]. Now (b) and (c) follow by noting that if $y \in \langle x\rangle$, then $y = ab$ or abc, where a, b and c are some generators for $\langle x\rangle$. Moreover, if x is a p-element, then either $\langle y\rangle = \langle x\rangle$ or $y = ab$. Now (d) follows from (c) and the observation that if P has rank d, it can be generated by d elements of $\Gamma_r G$ (see Theorem 2.1).

Theorem D now follows from Proposition 5.2(d) and the results in Section 4. We note that $f(1) = f(2) = 1$ (for $p > 3$ or $r = 2$) and $f(3) \geqslant 2$ (for $r > 2$ or $r = 2$ and $p \leqslant 3$.).

Finally, we shall prove Theorem E. Note if $|L_r G| < 8$, then $L_r G = \Gamma_r G$ by Theorem A and [**6**] unless perhaps $L_r G \simeq S_3$. However if $r \geqslant 2$, then as $A = (L_r G)' \lhd G$, $L_r G \leqslant G' \leqslant C_G(A)$, a contradiction. So $L_r G \neq S_3$. Now assume $|G| < 96$. Suppose G is a counterexample. Then $r \geqslant 3$ by [**7**, Theorem D]. If G/G' is cyclic, then $\Gamma_2 G = \{[g, h] \mid g \in G', h \in G\}$, and $G' = L_r G$ for $r \geqslant 2$. Since $|G| < 96$, $G' = \Gamma_2 G$, and so by induction, we see that

$$\Gamma_{r+1}(G) = \{[g, h] \mid g \in \Gamma_r G, h \in G\} = \Gamma_2 G = G' = L_{r+1} G.$$

Since $|L_3 G| \geqslant 8$ and indeed by an argument similar to the one above $|L_3 G|$ is divisible by at least three primes, the only possibilities remaining are that $[G : G'] = 4$ and $|L_r G| = 8, 12$, or 18 or that $[G : G'] = 9$ and $|G'| = |L_r G| = 8$. The last possibility is easily eliminated by inspection. If $[G : G'] = 4$ and $|L_r G| = 8$, then G is a 2-group and G' is cyclic. Thus Theorem A applies. If $[G : G'] = 4$ and $|L_r G| = 12$ or 18, then either $L_r G$ is the union of its Sylow subgroups (and so $L_r G = \Gamma_r G$ by Theorem A) or $G' = L_r G$ is abelian. It then follows that $L_r G \neq G'$, and this contradiction completes the proof.

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