ON PERMUTATIONAL PRODUCTS OF GROUPS

PART 2 – AMALGAMATED PRODUCTS

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

The standard methods of constructing generalized free products of groups (with a single amalgamated subgroup) and permutational products of groups are to consider groups of permutations on *sets*. Although there is an apparent similarity between these two constructions, the exact nature of the relationship is not clear. The following addendum to [4] grew out of an attempt to determine this relationship. By noting that the original construction of permutational products (B. H. Neumann [7]) deals with a group of permutations on a *group* (although the group structure has been previously ignored; see [7], [8]) we here give an extension of the original permutational product-construction which yields both the generalized free product and the permutational products as groups of permutations on *groups*. A generalized free product is represented as a group of permutations on the ordinary free product of the constituents of the underlying group amalgam and a permutational product is a group of permutations on the direct product of the constituents of the amalgam.

It is also shown that this construction can be extended to other groups G containing the constituents of the amalgam provided certain conditions hold; to differentiate the general case from ordinary permutational products we call the groups of permutations so obtained *amalgamated products*.

As in [4] an epimorphism can be constructed between suitable amalgamated products and the wreath product embeddings of permutational products given in [4] can then be extended to certain amalgamated products.

Finally, this construction also yields a class of related generalized regular products (Theorem 4.6), which, so far as we know, is the only such class known, besides ordinary permutational products (Allenby [2]) and some classes which have been shown to exist by Wiegold [12].

2. Preliminaries

If x and y are elements of a group G, write $y^{-1}xy = x^y$ and $x^{-1}y^{-1}xy = [x, y]$. Note that all mappings act on the right. If X_i $(i \in I)$ are subgroups of G, then $[X_i]$ denotes the subgroup of G generated by $\{[x_i, x_j] | x_i \in X_i, x_j \in X_j, i \neq j, i, j \in I\}$ and X_i^G the normal closure $X_i\,[X_i, G]$ of X_i in G. We shall say that a group G generated by subgroups X_i $(i \in I)$ is a *regular product* of the X_i, if $G \cong F/N$, where $F = \Pi^*\{X_i | i \in I\}$ is the (ordinary) free product of the X_i and $N \subseteq [X_i]^F$ (Golovin [3]). Assume now that the index set I is ordered.

THEOREM 2.1 [3]. *If a group G is generated by subgroups X_i $(i \in I)$, then G is a regular product of the X_i if and only if every element g if G has a unique regular representation*

$$g = x_1 x_2 \cdots x_n u,$$

where $x_k \in X_{i_k}$, $u \in [X_i]^G$ *and*

$$i_1 < i_2 < \cdots < i_n.$$

If V is a set of words, let $V(G)$ denote the V-verbal subgroup of G, i.e., the subgroup of G generated by all values of the words of V in G.

DEFINITION 2.2 (*Moran* [5]). *Let V be a set of words. The V-verbal product* $\Pi_V^*\{X_i | i \in I\}$ *of groups* X_i *is* $F/V(F) \cap [X_i]^F$, *where* $F = \Pi^*\{X_i | i \in I\}$.

THEOREM 2.3 [5]. *If* $G = \Pi_V^*\{X_i | i \in I\}$ *and* $I = I_1 \cup I_2$, *where* $I_1 \cap I_2$ *is empty, then the subgroups generated by the* X_i $(i \in I_1)$ *and* X_j $(j \in I_2)$ *are, respectively,* $G_1 = \Pi_V^*\{X_i | i \in I_1\}$ *and* $G_2 = \Pi_V^*\{X_j | j \in I_2\}$, *and* $G = G_1 *_V G_2$.

THEOREM 2.4 [6]. *If* X_i $(i \in I)$ *are groups and* ϕ_i *is a homomorphism of the group* X_i *for each* $i \in I$, *then there exists a homomorphic mapping* ϕ *of* $\Pi_V^*\{X_i | i \in I\}$ *onto* $\Pi_V^*\{X_i \phi_i | i \in I\}$ *whose restriction to the group* X_i *is* ϕ_i *for every* $i \in I$.

Suppose for each $i \in I$, A_i is a group containing a subgroup H_i which is isomorphic to a fixed group H, say $\psi_i : H_i \cong H$. Let $\psi_{ij} = \psi_i \psi_j^{-1}$. We change the notation of [4] and define an amalgam of the A_i amalgamating the H_i according to the ψ_{ij} to be the system $(A_i, H_i, \psi_{ij}; i, j \in I)$. We denote this amalgam by $\mathscr{A} = Am(A_i, H_i, \psi_{ij}; i, j \in I)$ and ordinarily think of the H_i as being identified by the ψ_{ij} so the amalgam becomes the union of the A_i intersecting in H (or H_1). The A_i are called the *constituents* of the amalgam and H is the *amalgamated subgroup*.

A group G *embeds* the amalgam $\mathscr{A}$ if there exist isomorphisms $\phi_i : A_i \to A_i' \subseteq G$ such that (i) $A_i' \cap A_j' = H' \subseteq G$, (ii) if $h \in H$, then $h\psi_i^{-1}\phi_i = h\psi_j^{-1}\phi_j$ and (iii) if $h' \in H'$, then $h'\phi_i^{-1} \in H_i$ and $h'\phi_i^{-1}\psi_i = h'\phi_j^{-1}\psi_j$ $(i, j \in I)$.

The group G will be said to be *generated by the amalgam* $\mathscr{A}$, if G embeds $\mathscr{A}$ and is generated by the embedded copy of $\mathscr{A}$.

If G is the generalized free product on $\mathscr{A}$ (this can be defined as the group constructed in the following Example (3.6) (2)), then K is called a *generalized regular product* on $\mathscr{A}$, if K embeds $\mathscr{A}$ and $K \cong G/N$, where N is a normal subgroup of G contained in $[A_i]^G$ (Wiegold [12]).

Definition 2.5 [11]. *If V is a set of words, the group G is a generalized V-verbal product of its subgroups G_α $(\alpha \in M)$ with amalgamations $G_\alpha \cap G_\beta = H_{\alpha\beta}$ $(\alpha \neq \beta)$, if*

(i) *G is generated by the G_α $(\alpha \in M)$ and*
(ii) $V(G) \cap [G_\alpha]^G = \{1\}$.

Theorem 2.6 [12]. *If the free generalized V-product of A and B amalgamating H and $H\phi$ according to ϕ exists it is G_0/N, where*

(i) *G_0 is the V-verbal product of A and B and*
(ii) *N is the normal closure in G_0 of the set of all elements of the form $h^{-1}(h\phi)$, where h ranges over H.*

Lemma 2.7. ([11], Lemma 7.9). *Let G be any group and $g, d \in G$ such that $[d^2, g] = 1$. Then for each $r \geqq 0$ $[g^{2^r}, d]$ is in the $(r+1)$-st term of the lower central series of G, $G_{(r+1)}$.*

3. The construction

For simplicity we deal with only two groups here; an extension to an arbitrary number of groups will be indicated later.

Let $Am(A_1, A_2; H_1, H_2; \psi)$, $\psi_{12} = \psi$, be a given group amalgam. Suppose G is any group containing isomorphic copies A_i^* of A_i $(i = 1, 2)$, where $\phi_i : A_i \cong A_i^*$ $(i = 1, 2)$, such that

(i) $A_1^* \cap A_2^* = H_1^* \cap H_2^*$, where $H_i^* = H_i\phi_i$ $(i = 1, 2)$, and
(ii) the isomorphism $\psi^* = \phi_1^{-1}|_{H_1^*}\psi\phi_2|_{H_2}$ from H_1^* onto H_2^* acts as the identity when restricted to $H_1^* \cap H_2^*$.

Let H be the subgroup of G generated by H_1^* and H_2^* and suppose $H \cap A_2^* = H_2^*$. Let $G = \cup zH$, $z \in Z$, be a coset decomposition of G relative to H. Assume further that there is an automorphism τ (called a *switching map*) of H such that $\tau|H_1^* = \psi^*$ and $\tau|H_2^* = \psi^{*-1}$. Note that $\tau^2 = 1$. Next let σ be any permutation on Z of order two which fixes the coset representative of H and such that for all $z \in Z$, if $A_2^*\setminus H_2^*$ meets zH, then $A_1^* \cap (z\sigma)H$ is empty. (E.g., see Example 3.6 (1) which follows.) Define a function π on G by

$$(3.1) \qquad (zh)\pi = (z\sigma)(h\tau), \qquad \text{for } z \in Z, h \in H.$$

Clearly $\pi^2 = 1$, so $\pi = \pi^{-1}$ and $\pi \in \mathscr{S}(G)$, the group of all permutations on G. Finally, assume $(1\pi)A_2^* \subseteq A_2^*H$.

Let $\rho : G \to \mathscr{S}(G)$ be the right regular representation of G. We shall now prove that the amalgam $\mathscr{A} = A_1^*\rho \cup \pi^{-1}(A_2^*\rho)\pi$ is a copy of $Am(A_1, A_2; H_1, H_2; \psi)$ embedded in $\mathscr{S}(G)$. The subgroup of $\mathscr{S}(G)$ generated by the amalgam $\mathscr{A}$ will be the required product.

Clearly $A_1^*\rho \cong A_1$ and $\pi^{-1}(A_2^*\rho)\pi \cong A_2^*\rho \cong A_2$. We first show that $H_1^*\rho = \pi(H_2^*\rho)\pi$ (recall $\pi = \pi^{-1}$).

Let $h_1 \in H_1^*$ and denote the image of h_1 under ρ by ρ_{h_1}. Then

$$(3.2) \qquad \rho_{h_1} = \pi\rho_u\pi, \qquad \text{where } u = h_1\psi^*,$$

for if $zh \in G$, $z \in Z$, $h \in H$, then

$$\begin{aligned}(zh)\pi\rho_u\pi &= (z\sigma\, h\tau)\rho_u\pi \\ &= (z\sigma\, h\tau\, h_1\psi^*)\pi \\ &= (z\sigma(hh_1)\tau)\pi \qquad (\psi^* = \tau|H_1^*) \\ &= z(hh_1) \qquad\qquad (\sigma^2 = \tau^2 = 1) \\ &= (zh)\rho_{h_1}.\end{aligned}$$

Now let

$$(3.3) \qquad \rho_{a_1} = \pi\rho_{a_2}\pi \in A_1^*\rho \cap \pi(A_2^*\rho)\pi.$$

where $a_1 \in A_1^*$ and $a_2 \in A_2^*$. Then

$$(3.4) \qquad 1\rho_{a_1} = a_1 \in A_1^*.$$

Let $1 = zh$, where z represents H and $h \in H$. Note $1\pi = (z\sigma)(h\tau) = z(h\tau) \in H$ and

$$\begin{aligned}(1)\pi\rho_{a_2}\pi &= (1\pi a_2)\pi \\ &= (a_2'h')\pi,\ a_2' \in A_2^*,\ h' \in H \qquad ((1\pi)A_2^* \subseteq A_2^*H) \\ &= (z'\sigma)(h''\tau),\end{aligned}$$

where $a_2'h' = z'h''$, $z' \in Z$, $h'' \in H$. By (3.3) and (3.4),

$$(3.5) \qquad (z'\sigma)(h''\tau) = a_1 \in A_1^*,$$

so $(z'\sigma)H$ meets A_1^*; hence no element of $A_2^*\setminus H_2^*$ can be written as $z'h^*$, $h^* \in H$, that is, $a_2' \in H_2^*$. Since $1\pi \in H$, $a_2 = (1\pi)^{-1}a_2'h'$ must also be in $H \cap A_2^* = H_2^*$. If $a_2 = a_1^*\psi^*$, for some $a_1^* \in H_1^*$, then $1\pi a_2 = z(h\tau)(a_1^*\psi^*) = z(ha_1^*)\tau$ and $(1\pi a_2)\pi = (zh)a_1^*$. But $a_1^* = a_1$ by (3.5), so $a_1 \in H_1^*$ and as in (3.2) $a_2 = a_1\psi^*$. Hence $H_1^*\rho = A_1^*\rho \cap \pi(A_2^*\rho)\pi$, as required.

The group $P(G, Z, \sigma)$ generated by $\mathscr{A}$ in $\mathscr{S}(G)$ will be called a (G, Z, σ)-*amalgamated product* on $\mathscr{A}$, or more briefly, a (G, Z, σ)-*product* on $\mathscr{A}$.

(3.6) Examples. Throughout the following examples we consider the given amalgam $\mathscr{A} = Am(A_1, A_2; H_1, H_2; \psi)$.

Let $G = A_1 * A_2/N$ be any ordinary regular product of A_1 and A_2 (we assume here that $A_i^* = A_i$ $(i = 1, 2)$). Then (i) and (ii) hold trivially because $H_1 \cap H_2 = 1$. It follows from the unique normal form for elements of G that $H \cap A_2 = H_2$, for, if $a_2 \in A_2 \cap H$ and $a_2 = h_1h_2c$, $h_1 \in H_1$, $h_2 \in H_2$, $c \in [A_1, A_2]$, then

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$$1 = h_1 a_2^{-1}[a_2^{-1}, h_1]h_2 c = h_1(a_2^{-1}h_2)c',\ c' \in [A_1, A_2], \qquad \text{so } a_2^{-1}h_2 = 1,$$

that is, $a_2 \in H_2$.

Choose a transversal Z of H. There is always at least one permutation σ satisfying the hypotheses of the construction, namely, the identity i on Z. For, if zH meets $A_2 \setminus H_2$, then $zh^* = a_2 \in A_2 \setminus H_2$, so $z = h_1^* a_2^* c$, $h_1^* \in H_1$, $a_2^* \in A_2 \setminus H_2$, $c \in [A_1, A_2]$. If $zh = a_1 \in A_1$, then $h_1^* a_2^* ch = a_1$, that is, $h_1^{**} a_2^{**} c^* = a_1$, $h_1^{**} \in H_1$, $a_2^{**} \in A_2 \setminus H_2$, $c^* \in [A_1, A_2]$, which is impossible by the uniqueness of the normal form for elements. Thus $zH \cap A_1$ is empty. Finally, if τ exists and $1 \in Z$, then $1\pi = 1$, so the condition $(1\pi)A_2 \subseteq A_2 H$ can also always be satisfied here. The main problem is, of course, the existence of τ. We now consider some important cases where τ can be shown to exist.

(1) PERMUTATIONAL PRODUCTS. Let $G = A_1 \times A_2$; then $H = H_1 \times H_2$ and the switching map τ exists, since it merely sends $(h_1, h_2) \in H$ to $(h_2\psi^{-1}, h_1\psi)$ which evidently defines an automorphism on H. Let S_i be any transversal of H_i in A_i $(i = 1, 2)$, choose $S_1 \times S_2$ as the transversal Z of H in G and let σ be the identity on Z. Note $HA_2 = A_2 H$, so $(1\pi)A_2 \subseteq A_2 H$. The group $P = P(A_1 \times A_2, S_1 \times S_2, i)$ is a *permutational product* on $\mathscr{A}$ as originally described by B. H. Neumann [7] in 1954. (The term 'permutational product' was given by B. H. Neumann in 1960 ([8]) to a certain permutation group on $S_1 \times S_2 \times H_1$ which is isomorphic to P above.)

(2) GENERALIZED FREE PRODUCTS. Let $G = A_1 * A_2$, the ordinary free product on A_1 and A_2; then $H = H_1 * H_2$. By the fundamental property of free products the isomorphisms $\psi : H_1 \to H_2 \subseteq H$ and $\psi^{-1} : H_2 \to H_1 \subseteq H$ can be extended to a homomorphism τ from H onto H. Since $\tau^2 = 1$, τ is an automorphism on H, i.e., the switching map exists. Let Z be any transversal of H in G containing 1 and let σ be i. If $\chi \in P = P(A_1 * A_2, Z, i)$, then without loss of generality

$$\chi = \rho_{a_1} \rho_{a_2}^{\pi} \cdots \rho_{a_{n-1}}^{\pi} \rho_{a_n} \tag{3.7}$$

where, if $\chi \notin H_1 \rho$, then it can be assumed that $a_i \in A_1 \setminus H_1$, $i = 1, 3, \cdots, n$ and $a_i \in A_2 \setminus H_2$, $i = 2, 4, \cdots, n-1$. If $n \geqq 1$, and $\chi \notin H_1\rho$, χ is said to have length n; otherwise χ has length zero.

In order to show P is the generalized free product on $\mathscr{A}$ it suffices to show that χ is non-trivial whenever the length $n \geqq 1$. The action of ρ_{a_1} on $1 \in A_1 * A_2$ is $1\rho_{a_1} = a_1 = (a_1 h_1^{-1})h_1$, where $a_1 h_1^{-1} \in Z$, $h_1 \in H$, so

$$\begin{aligned}(1)\rho_{a_1}\pi\rho_{a_2} &= a_1 h_1^{-1} h_1^{\tau} a_2 \\ &= (a_1 h_1^{-1} h_1^{\tau} a_2 h_2^{-1})h_2\end{aligned}$$

where $a_1 h_1^{-1} h_1^{\tau} a_2 h_2^{-1} \in Z$, $h_2 \in H$ and $(1)\rho_{a_1}\rho_{a_2}^{\pi} = a_1 h_1^{-1} h_1^{\tau} a_2 h_2^{-1} h_2^{\tau}$.

Continuing this process,

$$1\chi = a_1 h_1^{-1} h_1^{\tau} a_2 h_2^{-1} h_2^{\tau} \cdots a_n \in A_1 * A_2.$$

Assume all pairs $h_j^{-1}h_j^{\tau}$ are written in normal form as elements of $A_1 * A_2$. Suppose $a_j \in A_1$. Since $a_j \notin H$, $h_1^* a_j h_2^* \notin H$ for any $h_1^*, h_2^* \in H$; in particular $h_1^* a_j h_2^* \notin H_1$ for any $h_1^*, h_2^* \in H_1$. Therefore only contractions, but no cancellation, can occur between the $h_j^{-1}h_j^{\tau}$ and a_j when reducing 1χ to normal form. Thus, $1\chi \neq 1$, which was to be shown.

(3) A *retraction* ϕ of a group G is an idempotent endomorphism of G, i.e. $\phi^2 = \phi : G \to G$. If $H = G\phi$, then H is called a *retract* of G.

LEMMA 3.8 (*Šmel'kin* [10]). *Let $G = A_1 *_V A_2$ be a V-verbal product of A_1 and A_2. Suppose ϕ_i is a retraction of A_i, $(i = 1, 2)$. Then the subgroup H of G generated by the retracts $H_i = A_i\phi_i$, $(i = 1, 2)$, is the V-verbal product of H_1 and H_2.*

Now suppose H_1 and H_2 are retracts of A_1 and A_2, i.e., H_1 and H_2 have normal complements in A_1 and A_2 (in particular, suppose A_i is a regular product $A_i' * H_i/N_i$, $i = (1, 2)$). Let V be a verbal subgroup of $A_1 * A_2$ and let $G = A_1 *_V A_2$ be the V-verbal product of A_1 and A_2. By the above Lemma 3.8 $H = H_1 *_V H_2$, so τ exists by an argument similar to that given in (2). That is, by Theorem 2.4 an epimorphism $\tau : H \to H$ exists such that $\tau|H_1 = \psi$ and $\tau|H_2 = \psi^{-1}$. Finally, τ is an isomorphism because $\tau^2 = 1$.

Before continuing with further examples consider the following special case of Example (3) which shows that the amalgamated products will, in general, be different from each other as V varies. (Of course, not always. Some amalgams can only generate their generalized free products; see Example 4.12 [4].)

Suppose H_1 and H_2 are V-verbal factors of A_1 and A_2, say $A_1 = A_1' *_V H_1$ and $A_2 = A_2' *_V H_2$. Let $G = A_1 *_V A_2$. Then $H = H_1 *_V H_2$ is a V-verbal factor of G, $G = (A_1' *_V A_2') *_V H$, by the properties of V-verbal multiplication. Furthermore, the switching map τ as defined above can be extended to an automorphism τ' of G of order two such that $\tau'|A_1' *_V A_2'$ is the identity on $A_1' *_V A_2'$ by Theorem 2.4. Choose Z to be the normal complement $(A_1' *_V A_2')^G$ of H in G, and let $\sigma = \tau'|Z$. Since $Z = (A_1' *_V A_2')[A_1' *_V A_2', G]$, σ is a permutation on Z. It must be verified that if A_1 meets $(z\tau')H$, then $(A_2 \setminus H_2) \cap zH$ is empty. Suppose $z\tau' h = a_1 \in A_1$ for some $z \in Z$. Applying τ' to both sides of this equation, $z(h\tau') = a_1\tau' \in A_1 *_V H_2$. Thus $z = a_1^* h_2^* c^*$ for some $a_1^* \in A_1$, $h_2^* \in H_2$, $c^* \in [A_1, A_2]$. If also $zh^* = a_2 \in A_2 \setminus H_2$, then $z = h_1' a_2' c'$, where $h_1' \in H_1$, $a_2' \in A_2 \setminus H_2$ and $c' \in [A_1, A_2]$. This would imply the contradiction $a_2' = h_2^* \in H_2$. Thus $(A_2 \setminus H_2) \cap zH$ is empty.

Now we show

$$P = P(G, Z, \tau'|Z) \cong A_1' *_V H_1 *_V A_2'. \tag{3.9}$$

Let $u \in A_2'$ and $zh \in G$, where $z \in Z$, $h \in H$. Then

$$\begin{aligned} (zh)\rho_u^{\pi} &= ((zh)\tau' u)\pi && (\sigma = \tau'|Z) \\ &= ((zhu)\tau')\tau' && (\tau'|A_2' \text{ is the identity on } A_2') \\ &= (zh)\rho_u, \end{aligned}$$

that is, P is generated by $A_1\rho$ and $(A'_2\rho)^\pi = A'_2\rho$. But these are just the right regular representations of A_1 and A'_2 over G, which generate the right regular representation of $A_1 *_V A'_2$ over G, conpleting the proof of (3.9).

Note the condition that H_1 and H_2 be retracts in (3) is not necessary in order that τ exist; for example, Šmel'kin [10] proved that if A_1 and A_2 are torsion free abelian groups and V is the verbal subgroup of $A_1 * A_2$ corresponding to the variety of nilpotent groups of class at most n, then $H = H_1 *_V H_2 \subseteq A_1 *_V A_2$.

(4) Isomorphic Constituents. Suppose A_1 and A_2 are isomorphic, say $\gamma : A_1 \cong A_2$, $\psi = \gamma | H_1$, and consider the V-verbal product $A_1 *_V A_2$. Then τ exists, for there is an isomorphism τ' of order two from $A_1 *_V A_2$ onto $A_1 *_V A_2$ such that $\tau' | A_1 = \gamma$ and $\tau' | A_2 = \gamma^{-1}$. Take $\tau = \tau' | H$.

(5) Right Regular Representation. So far in the examples $A_1 \cap A_2 = \{1\} \subseteq G$. At the other extreme, let $\mathscr{A}$ generate G, $H_1 = H_2 = H \subseteq G$, take τ as the identity on H; let Z be any transversal of H in G containing 1 and let σ be the identity on Z. Clearly π is the identity and $P(G, H, i)$ is just the right regular representation of G. In particular, an amalgam $\mathscr{A}$ can generate a group G if and only if G is isomorphic to some amalgamated product on $\mathscr{A}$.

(3.11) The General Case. Suppose now that the amalgam has more than two constituents. Suppose that for each $i \in I$, A_i is a group having a subgroup H_i which is isomorphic to a fixed group H', say $\psi_i : H_i \cong H'$ and set $\psi_{ij} = \psi_i \psi_j^{-1} : H_i \cong H_j$, $i, j \in I$, $i \neq j$. Let G be any group containing isomorphic copies A_i^* of A_i, say $\phi_i : A_i \cong A_i^*$, $(i \in I)$, and suppose $A_i^* \cap A_j^* = H_i^* \cap H_j^*$ and $\phi_i^{-1}\psi_{ij}\phi_j$ acts as the identity on $H_i^* \cap H_j^*$, $(i, j \in I, i \neq j)$. Let H be the subgroup of G generated by the H_i^*, $(i \in I)$, and assume $H \cap A_j^* = H_j^*, j \in I \diagdown \{1\}$. Choose a transversal Z of H in G and assume automorphisms τ_j can be defined on H such that $\tau_j | H_1^* = \psi_{1j}^*$, $\tau_j | H_j^* = \psi_{1j}^{-1}$ and $\tau_j | H_k^*$ acts as the identity on H_k^*, $(j, k \in I \diagdown \{1\}, k \neq j)$. Define a permutation σ on Z as before, except assume for all $i, j \in I$, $i \neq j$, if A_j^* meets zH, then both $(A_i^* \diagdown H_i^*) \cap (z\sigma)H$ and $(A_i^* \diagdown H_i^*) \cap zH$ are empty. Finally, for each $j \in I \diagdown \{1\}$, let π_j be a permutation on G given by $(zh)\pi_j = (z\sigma)(h\tau_j)$, $z \in Z$, $h \in H$; assume also that $(1\pi_j)A_j = A_j H$, $(j \in I \diagdown \{1\})$. Then, as before, the amalgam is isomorphic to $\cup \{(A_i^*\rho)^{\pi_i} | i \in I\}$, where π_1 is defined to be the identity on G. The details are omitted.

4. An epimorphism

Let $\mathscr{A} = Am(A_i, H_i; \psi_{ij}; i, j \in I)$ be an amalgam and let G be a group containing copies A_i^* of the A_i as in Section (3.11).

Assume further that G is generated by the A_i^* and let $P = P(G, Z, \sigma)$ be an amalgamated product on $\mathscr{A}$. A homomorphism θ of G will be called a (G, Z, σ)-*homomorphism*, if the following conditions are satisfied:

(i′) there exist isomorphisms $\psi'_{ij} : H_i\theta \cong H_j\theta$ such that $\theta\psi'_{ij} = \psi_{ij}\theta$, on H_i $(i, j \in I,\ i \neq j)$.

(ii′) $Z\theta$ is a transversal of $H\theta = \langle H_i\theta | i \in I\rangle$ in $G\theta$.

(iii′) a permutation $\sigma' : Z\theta \to Z\theta$ exists as required in order to construct a $(G\theta, Z\theta, \sigma')$-product on the factor amalgam $\mathscr{F} = Am(A_i\theta, H_i\theta, \psi'_{ij} | i, j \in I, i \neq j)$, such that in addition $\theta\sigma' = \sigma\theta$ on Z, and

(iv′) for all $j \in I, j \neq 1$, switching maps $\tau'_j : H\theta \to H\theta$ exist such that $\tau'_j | H_1\theta = \psi'_{1j}$ and $\tau'_j | H_j\theta = (\psi'_{1j})^{-1}$.

Now suppose θ is such a (G, Z, σ)-homomorphism; then, since H is generated by the H_i, $\theta\tau'_j = \tau_j\theta$ on H. Furthermore, permutations $\pi'_j : G\theta \to G\theta$ can be constructed as in (3.11) using σ' and τ'_j and

$$\theta\pi'_j = \pi_j\theta, \qquad (j \in I, j \neq 1). \tag{4.1}$$

Thus $(1\pi'_j)A_j\theta = ((1\pi_j)A_j)\theta \subseteq (A_jH)\theta = A_j\theta H\theta$, which is required to construct a $(G\theta, Z\theta, \sigma')$-amalgamated product on $\mathscr{F}$ using the switching maps τ'_j. Denote the product depending on the ψ'_{ij} by $P'(G\theta; Z\theta, \sigma', \psi'_{ij})$ or merely by P'.

THEOREM 4.2. *Let $\mathscr{A}$ and G be as above and suppose θ is a (G, Z, σ)-homomorphism of G. Then there exists an epimorphism f from $P = P(G, Z, \sigma)$ onto $P' = P'(G\theta, Z\theta, \sigma', \psi'_{ij})$ extending the canonical epimorphisms $(A_i\rho)^{\pi_i} \to (A_i\theta\rho)^{\pi'_i}$, $(i \in I)$.*

PROOF. The function θ is an epimorphism. It follows from (4.1) that for each $a_j \in A_j^*$

$$\pi_j P_{a_j} \pi_j \theta = \theta\pi'_j P_{a_j\theta}\pi'_j \qquad (j \in I) \tag{4.3}$$

where, as in Section (3.11), π_1 and π'_1 are the identities on G and $G\theta$ respectively. Thus, since P is generated by the $(A_i^*\rho)^{\pi_i}$, to each $x \in P$, there exists a unique $xf \in P'$ such that $x\theta = \theta(xf)$; xf is unique because θ is an epimorphism. The required epimorphism f is given by $f : x \to xf$. (cf. Theorem 3.1, [4]).

We shall call f the *natural homomorphism* from P onto P' when it exists.

The usual proof of the following well-known result uses directly the uniqueness of the normal form in the generalized free product.

COROLLARY 4.4. *Let G be any group generated by $\mathscr{A}$. Then there exists a natural homomorphism from the generalized free product on $\mathscr{A}$ onto G which acts as the identity on the A_i, $(i \in I)$.*

PROOF. (See Example (3.6), (2) and (5).) Consider the right regular representation of G, $G\rho$ as a product on G. There is a natural homomorphism θ from $F = \pi^*\{A_i | i \in I\}$ onto G extending the maps $A_i \to A_i \subseteq G$. Let $Z = Z_1Z_2$ where Z_2 is a transversal of H in $H \ker\theta$ such that $1 \in Z_2 \subseteq \ker\theta$ and Z_1 is a transversal of $H \ker\theta$ in F, $1 \in Z_1$. Then Z is a transversal of H in F which maps onto a transversal $Z\theta$ of $H\theta$ in $F\theta = G$. Let σ be the identity on Z. Then if σ', ψ'_{ij} and

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τ_j' are taken to be identity maps, θ is a (G, Z, σ)-homomorphism, so the result follows by Theorem 4.2.

NOTE. Many times it will be convenient to choose Z as above in Corollary 4.4; this will be denoted by a remark such as 'let $Z = Z_1 Z_2 \cdots$', if no further explanation is required. If no mention of σ is made it will be assumed to be the identity on Z.

Now consider an amalgam on two groups A_1 and A_2. Let $G = A_1 *_V A_2$ be a verbal product. Choose a transversal $Z_1 Z_2$ of H in G as follows: let Z_2 be a transversal of H in HN, $1 \in Z_2 \subseteq N$, where N is the normal closure of the amalgamating relations $\{h_1^{-1}(h_1 \psi) | h_1 \in H_1\}$ in G and let Z_1 be a transversal of HN in G, $1 \in Z_1$. (See Theorem 2.6.)

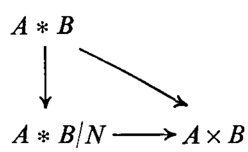
COROLLARY 4.5. *Let $G = A_1 *_V A_2$ and $Z_1 Z_2$ be as above. If some $P = P(G, Z_1 Z_2, \sigma)$ exists which is a generalized V-verbal product on $\mathscr{A}$, then P is the free generalized V-verbal product on $\mathscr{A}$.*

PROOF. Let K be the free generalized V-verbal product on $\mathscr{A}$ and let $\theta : G \to K$ be the natural epimorphism from G onto K. Then $Z\theta$ is a transversal of H_1 in K.

Thus there is a natural epimorphism f from P onto $K\rho$. If ψ is the canonical epimorphism from $K\rho$ onto P, then ψf is the identity, so $P \cong K$ which was to be shown.

THEOREM 4.6. *Let $G = A_1 * A_2/N$ be any regular product. If any amalgamated product exists on G which is generated by the amalgam $\mathscr{A}$, then a (G, Z, i)-amalgamated product exists which is a generalized regular product on $\mathscr{A}$.*

PROOF. Since at least one amalgamated product exists, the switching map exists. Let Z be any transversal of H in G which maps onto a transversal $Z\theta = S \times T$ of $H_1 \times H_2$ in $A_1 \times A_2$, where θ is the canonical epimorphism from G onto $A_1 \times A_2$. Then an amalgamated product $P = P(G, Z, i)$ exists and maps onto the permutational product $P' = P(A \times B; S \times T)$, say $\phi : P \to P'$. Let f and f' be the natural epimorphisms from the generalized free product on the amalgam onto P and P', respectively. Since



is a commutative diagram (where the maps are the canonical epimorphisms) it follows from Theorem 4.2 that $f' = f\phi$, so $\ker f \subseteq \ker f'$. Allenby [2] has shown that any permutational product is a generalized regular product, hence P is itself a generalized regular product on the amalgam.

It is known that if the generalized direct product D on $\mathscr{A} = Am(A, B; H_1,$

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$H_2; \psi)$ exists, then all permutational products must be isomorphic to D, that is, D is the free generalized abelian product on $\mathscr{A}$. The following examples show that even though the free generalized V-product generated by $\mathscr{A}$, say K, exists, and an amalgamated product $P = P(G, Z_1 Z_2, i)$ exists on $A *_V B$ where the transversal $Z_1 Z_2$ is chosen as in Corollary 4.5 (so P is a generalized regular product mapping onto K), P may not be isomorphic to K. (In this example K will exist, because the generalized direct product does; see Wiegold [11], Theorem 4.6.)

Let N_c stand for the verbal subgroup of $A * B$ corresponding to the class of nilpotent groups of class at most c.

Let $A \otimes B$ denote the tensor product of the groups A and B. The regular N_2-product of groups A and B can be faithfully represented by

$$G = \{(a, b, c) | a \in A, b \in B, c \in A \otimes B\},$$

where

$$(a, b, c)(a_1, b_1, c_1) = (aa_1, bb_1, cc_1 a_1^{-1} \otimes b)$$

and

$$A \cong \{(a, 0, 0) | a \in A\}, \qquad B \cong \{(0, b, 0) | b \in B\}$$

(Wiegold [11], p. 154).

Example (4.7). If A and B are copies of the additive group of rational numbers, Q, then (using additive notation)

$$G = \{(s, t, u) | s, t, u \in Q\}$$

where

$$(s, t, u)(s_1, t_1, u_1) = (s+s_1, t+t_1, u+u_1-ts_1),$$

and

$$(s, t, u)^{-1} = (-s, -t, -u-ts).$$

Let

$$H_1 = \{(2n, 0, 0) | n \in I\}, \qquad H_2 = \{(0, 3m, 0) | m \in I\},$$

where I is the integers, and assume the amalgamating isomorphism ψ is given by $(2n, 0, 0) = (0, 3n, 0)$, $n \in I$. Now

$$(4.8) \qquad [(2, 0, 0), (0, 3, 0)] = (0, 0, 6)$$

so

$$\langle H_1, H_2 \rangle = \{(2n, 3m, 6p) | n, m, p \in I\}.$$

The switching map τ exists by the remark at the end of (3.6) (3).

If $h_1 = (2n, 0, 0)$, $n \in I$ then $h_1(h_1^{-1}\psi) = (2n, -3n, 0) \in N$, where N is the normal closure of $\{h_1(h_1^{-1}\psi) | h_1 \in H_1\}$ in G,

$$(2n, -3n, 0)^{(s, t, u)} = (2n, -3n, 2nt+3ns) \in N$$

and

$$(2, -3, u)(-2, 3, 0) = (0, 0, u-6) \in N,$$

where $s, t, u, \in Q$.

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Thus

$$N = \{(2n, -3n, u) | n \in I, u \in Q\}$$

and

$$HN = \{(2n, 3m, u) | n, m \in I, u \in Q\}.$$

If $u \in Q$, then u can be uniquely written $u = 6k + u'$, $0 \leqq u' < 6$, $k \in I$, $u' \in Q$. Choose the transversal Z_2 of H in HN to be

$$Z_2 = \{(0, 0, u') | 0 \leqq u' < 6, u' \in Q\}.$$

Similarly choose a transversal Z_1, of HN in G; let

$$Z_1 = \{(s', t', 0) | 0 \leqq s' < 2, 0 \leqq t' < 3, s', t' \in Q\}.$$

Then

$$Z_1 Z_2 = \{(s', t', u') | 0 \leqq s' < 2, 0 \leqq \tau' < 3, 0 \leqq u' < 6, s', \tau', u' \in Q\}$$

is a transversal of H in G chosen as required in Corollary 4.5.

If $z = (0, 0, 6p) = (0, 0, 6)^p \in H$, then by (4.8) $z\tau = (0, 0, 6)\tau^p = [(0, 3, 0), (2, 0, 0)]^p = (0, 0, -6)^p$. Thus if $(2m, 3m, 6p) \in H$,

$$(2n, 3m, 6p)\tau = (2m, 3n, -6p - 6mn).$$

Since $\sigma = i$ on $Z_1 Z_2$, $(s, t, u)\pi$ can now be calculated for any $(s, t, u) \in G$.

Let $a' = (1, 0, 0)\rho$ and $b' = (0, 1, 0)\rho^\pi$. Then $(a')^2 \in H\rho \subseteq Z(P)$; set d equal to a' in Lemma 2.7 and $g = (b')^{\frac{1}{2}^r} = (0, \frac{1}{2}^r, 0)\rho^\pi$. Then $[b', a'] \in G_{(r+1)}$, $r \geqq 0$. Calculating,

$$(\tfrac{1}{2}, \tfrac{5}{2}, 5)[b', a'] = (\tfrac{1}{2}, \tfrac{5}{2}, 6).$$

Thus P is not nilpotent of any class, so P is not isomorphic to the free generalized nilpotent product of class 2, K.

Suppose now the generalized N_2-product of an amalgam $\mathscr{A}$ exists. Does the existence of this product force the switching automorphism to exist in $A *_{N_2} B$?

The following example due to Dr L. G. Kovács shows this is not the case. Let $A = C_2 \times C_4$ and $B = C_2 \times C_2$, where C_n is the cyclic group of order n; let these cyclic groups be generated by a, b, c and d, respectively. Amalgamate $\langle a, b^2 \rangle$ with B via $a \leftrightarrow c$, $b^2 \leftrightarrow d$. Then in $G = A *_{N_2} B$ we have $[b^2, e] = [b, e^2] = 1$, $e \in \{c, d\}$, so b^2 is in the centre of G and thus of H. A simple calculation using Wiegold's representation of G above shows that d does not commute with a. Thus a switching automorphism does not exist. I thank Dr. Kovács for allowing me to use this example.

5. A wreath product embedding

It is convenient to generalize and unify the embeddings given in Theorems 4.1, 5.2 and 6.1 of [4] in the following way.

Assume that an amalgam $\mathscr{A}$ is given as in (3.11) and that some amalgamated product $P = P(G, Z, \sigma)$ on a group G exists generated by $\mathscr{A}$. Let θ be a (G, Z, σ)-homomorphism, $P' = P'(G\theta, Z\theta, \sigma')$ and $f : P \to P'$ the natural homomorphism. Choose a set W of coset representatives of $\ker\theta$ in G. Thus, if $d \in G$, then $d = w\lambda$, $w \in W$, $\lambda \in \ker\theta$ and $d\theta = w\theta$. Define $[d\theta] = w$ and note $[d\theta]\theta = d\theta$.

THEOREM 5.1. *Suppose there exist homomorphisms* $\alpha : P \to \operatorname{Aut}(\ker\theta)$ *and* $r : \ker\theta \to \mathscr{S}(G)$ *such that*

(1) *if* $g \in G$, *then there exists a unique* $\lambda r \in (\ker\theta)r = R$ *such that* $g = [g\theta]\lambda r$,
(2) *if* $y \in P$, *then*

$$y^{-1}(\lambda r)y = (\lambda^{y\alpha})r \tag{5.2}$$

and $(\ker r)^{y\alpha} \subseteq \ker r$ $(y \in P)$.

Then there exists a monomorphism from P *into the unrestricted permutational wreath product*

$$P\beta(\ker\theta)r\ Wr(P'; G\theta), \tag{5.3}$$

where the homomorphism $\beta : P \to \operatorname{Aut}((\ker\theta)r)$ *is given by*

$$\lambda r^{y\beta} = (\lambda^{y\alpha})r \qquad (\lambda \in \ker\theta,\ y \in P). \tag{5.4}$$

PROOF. First note that (5.4) determines a homomorphism β as required.
Now let $x \in P$. It follows from the proof of Theorem 4.2 if $d \in G$, then

$$\mathrm{d}x\theta = d\theta xf, \tag{5.5}$$

so $[d\theta]x\theta = d\theta xf$. Thus, by (1), if $d \in G$, there exists a unique $(\lambda_x(d\theta))r \in (\ker\theta)r$ such that

$$[d\theta]x = [d\theta xf](\lambda_x(d\theta))r. \tag{5.6}$$

Define an element e_x in the direct power of $|G\theta|$ copies of $P\beta(\ker\theta)r$, $(P\beta(\ker\theta)r)^{G\theta}$, by

$$e_x(d\theta) = x\beta(\lambda_x^{-1}(d\theta))r. \qquad (d\theta \in G\theta) \tag{5.7}$$

LEMMA 5.8. *The required monomorphism is given by*

$$x \to xf\,e_x^{xf} = e_x xf \qquad (x \in P). \tag{5.9}$$

PROOF. It must be shown that

$$e_{xy}(xy)f = e_x xf\,e_y\,yf$$

or

$$e_{xy} = e_x e_y^{xf^{-1}}$$

which by the definition of conjugation in wreath products is equivalent to

$$e_{xy}(d\theta) = e_x(d\theta)e_y(d\theta xf) \qquad (d\theta \in G\theta). \tag{5.10}$$

Now by (1) and repreated use of (5.6), if $d = [d\theta]\lambda r \in G$, with $\lambda \in \ker\theta$, then

$$\begin{aligned}([d\theta]\lambda r)xy &= [d\theta xfyf](\Lambda_1)r \\ &= [d\theta xf]y(\Lambda_2)r \\ &= [d\theta]x(\lambda_x^{-1}(d\theta))ry(\Lambda_2)r \\ &= [d\theta]xy(\Lambda_3)r \\ &= ([d\theta]\lambda r)(\lambda^{-1})rxy(\Lambda_3)r \\ &= ([d\theta]\lambda r)xy(\Lambda_4)r,\end{aligned}$$

where

$$\begin{aligned}\Lambda_1 &= \lambda_{xy}(d\theta)\lambda^{(xy)\alpha} \\ \Lambda_2 &= \lambda_y^{-1}(d\theta xf)\Lambda_1 \\ \Lambda_3 &= \lambda_x^{-y\alpha}(d\theta)\Lambda_2\end{aligned}$$

and $\Lambda_4 = \lambda^{-(xy)\alpha}\Lambda_3 = \lambda^{-(xy)\alpha}\lambda_x^{-y\alpha}(d\theta)\lambda_y^{-1}(d\theta xf)\lambda_{xy}(d\theta)\lambda^{(xy)\alpha}$.

Thus $(\Lambda_4)r = 1$, so

$$(\lambda_x^{-y\alpha}(d\theta))r(\lambda_y^{-1}(d\theta xf))r(\lambda_{xy}(d\theta))r = 1$$

from which (5.10) follows.

To complete the proof suppose $e_x xf = 1$. Then $xf = 1$, $x\beta = 1$ and for each $d\theta \in G\theta$, $(\lambda_{xy}(d\theta))r = 1$. Let $d = [d\theta]\lambda r \in G$. Then

$$\begin{aligned}([d\theta]\lambda r)x &= [d\theta]x\lambda r^{x\beta} \\ &= [d\theta xf](\lambda_x(d\theta))r\lambda r^{x\beta} \\ &= [d\theta]\lambda r.\end{aligned}$$

Thus $x = 1$ completing the proof.

For example, if Theorem 5.1 is applied to permutational products where $H_1 \lhd A$ and $H_2 \lhd B$, then $\ker\theta = H_1 \times H_2$, r can be chosen to be the restriction of the right regular representation of G to $H_1 \times H_2$ and if $y = \rho_{a_1}\rho_{b_2}^{\pi} \cdots \rho_{a_n} \in P$, where $a_i \in A$, $b_j \in B$, then α is given by the equations $y^{-1}(\rho_{h_1h_2})y = \rho_u$, where $u = h_1^z h_2$, $h_1h_2 \in H_1 \times H_2$, and (with the obvious meaning), $z = a_1b_2 \cdots a_n$. This is essentially the embedding Theorem 4.1 of [4] mentioned at the beginning of this section. It can be shown that, in general, the term $P\beta$ is needed for permutational products. On the other hand, the following shows why r is not always set equal to ρ as above.

If G^* is the generalized free product of A and B above (H normal in each), then there is a homomorphism $\theta : G^* \to A/H * B/H$ such that $\ker\theta = H$. Considering both the right regular representations of G^* and $A/H * B/H$ as amalgamated products on G^* and $A/H * B/H$, and taking $r = \rho$ as above, G^* can be embedded in $P\beta H\, Wr\, A/H * B/H$, where $P\beta$ is the group of automorphisms G^* induces on H, $G^*/C_{G^*}(H)$. This is not as good as the standard wreath product embedding of G^*, $H\ Wr\ A/H * B/H$. Instead if $g^* \in G^*$, define $(g^*)\lambda r = \lambda^{-1}g^*$. Then Hr commutes with P in $\mathscr{S}(G)$. This choice of r in (5.1) thus gives the expected embedding of G^*.

It is also not difficult to see that Theorem 6.10 of [4] can also be extended to amalgamated products. That is, suppose $P(G, Z, \sigma)$ is an amalgamated product on $\mathscr{A}$, $H_1 \subseteq U_1 \subseteq A_1$, $H_2 \subseteq U_2 \subseteq A_2$, and assume Z is chosen as in [4], i.e., $Z = Z_1 Z_2$, where Z_1 is a transversal of U in G, where $U = \langle U_1, U_2 \rangle$, and Z_2 is a transversal of H in U, $1 \in Z_1 \cap Z_2$.

Then, if σ sends $z_1 z_2$ to $z_1 z_2'$, $z_1 \in Z_1$, $z_2, z_2' \in Z_2$, the subgroup U^* of P generated by $U_1 \rho$ and $(U_2 \rho)^\pi$ is isomorphic to $P_1(U, Z_2, \sigma | Z_2)$.

We conclude by stating two of the many problems which suggest themselves here and which we have not been able to answer.

(1) It is known that not every subgroup U^* of a permutational product (i.e., an amalgamated product on $A \times B$) need again be a permutational product even though it is generated by $U_1 \subseteq A_1$ and $U_2 \subseteq A_2$, where $U_1 \cap H_1 = U_2 \cap H_1$ ([9]). Suppose U_1 and U_2 are so chosen in an amalgamated product P on a regular product $A * B/N$, such that (i) P is a generalized regular product and (ii) the subgroup U^* of P is a generalized regular product (Allenby [1] gives some general criteria for this to happen). When must the subgroup U^* be an amalgamated product on a regular product $U_1 * U_2/N_1$ (where N_1 is possibly different from N)?

(2) Determine some classes of amalgamated products on verbal products $A *_V B$ which are generalized V-verbal products, other than those on $A * B$ and $A \times B$.

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