Sixth Meeting, May 8th, 1896.

Dr PEDDIE, President, in the Chair.

Note on the Formula for tan (A+B).

By Professor STEGGALL.

FIGURE 43.

From the figure we have at once

$$\tan\theta + \tan\phi = \frac{BC}{AE} = \frac{2BN}{AE}$$

$$\tan\theta\tan\phi = \frac{BE}{AE}\cdot\frac{ED}{BE} = \frac{ED}{AE}$$

$$1 - \tan\theta\tan\phi = \frac{2ON}{AE}$$

$$\frac{BN}{ON} = \tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi};$$

similarly
$$\frac{OK}{KD} = \tan(\phi-\theta) = \frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}.$$

On the envelope of the Simson line of a polygon.

By Professor STEGGALL.

It is known that if from a given point perpendiculars be let fall on the four Simson lines formed from the four triangles made by taking every three of four points concyclic with the first, the feet of these perpendiculars lie on a straight line proposed to be called the Simson line of the quadrangle formed by the four points; it is

https://doi.org/10.1017/S0013091500031874 Published online by Cambridge University Press

also known that this process can be extended. I propose to examine various results connected with these lines.

Let the angular coordinate of any points on a circle be 2α, 2β, 2γ, . . the radius a, and let any other point have an angular coordinate $\pi - 2\theta$: the chord (α, β) is

$$x\cos(\alpha+\beta) + y\sin(\alpha+\beta) = a\cos(\alpha-\beta)$$

the axes being through the centre as usual. This equation may be written

$$(x + a\cos 2\theta)\cos(\alpha+\beta) + (y - a\sin 2\theta)\sin(\alpha+\beta)$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta) \quad - \quad - \quad - \quad (1)$$

The perpendicular from the point $(\pi - 2\theta)$ has for equation

$$(x + a\cos 2\theta)\sin(\alpha+\beta) - (y - a\sin 2\theta)\cos(\alpha+\beta) = 0 \quad - \quad (2)$$

and the line

$$(x + a\cos 2\theta)\cos(\alpha+\beta+\gamma+\theta) + (y - a\sin 2\theta)\sin(\alpha+\beta+\gamma+\theta)$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta)\cos(\gamma+\theta) \quad - \quad - \quad (3)$$

passes through the intersection of (1) and (2). But its symmetry at once shows that it is the pedal line of the triangle $(2\alpha, 2\beta, 2\gamma)$.

Proceeding to a fourth point

$$(x + a\cos 2\theta)\cos(\alpha+\beta+\gamma+\delta+2\theta)$$
$$+ (y - a\sin 2\theta)\sin(\alpha+\beta+\gamma+\delta+2\theta)$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta)\cos(\gamma+\theta)\cos(\delta+\theta) \quad - \quad (4)$$

is the Simson line of the quadrangle 2α, 2β, 2γ, 2δ, and so on.

If we choose the original axis of x to coincide in direction with the mean of the angular directions 2α, 2β,

$$\alpha + \beta + \gamma + \quad . \quad . \quad = 0$$

and our line becomes in the general case

$$(x + a\cos 2\theta)\cos(n-2)\theta + (y - a\sin 2\theta)\sin(n-2)\theta$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta) \quad . \quad . \quad .$$

or

$$x\cos(n-2)\theta + y\sin(n-2)\theta$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta) \quad . \quad . \quad - a\cos n\theta$$
$$= a\left(\frac{\Sigma\cos(\overline{\alpha+\theta} \pm \overline{\beta+\theta} \pm \overline{\gamma+\theta})}{2^{n-2}} - \cos n\theta\right) \quad - \quad - \quad (5)$$

https://doi.org/10.1017/S0013091500031874 Published online by Cambridge University Press

Now in the case of a triangle, if θ varies, this gives

$$x\cos\theta + y\sin\theta = a\left(\frac{\cos(\theta - 2a) + \;.\;.\; - \cos 3\theta}{2}\right)$$

or
$$\left(x - \frac{\Sigma\cos 2a}{2}\right)\cos\theta + \left(y - \frac{\Sigma\sin 2a}{2}\right) = \frac{a\cos 3\theta}{2}$$

a line enveloping a three-cusped hypocycloid whose centre is at the nine-points centre - - - - - - - - (6)

In the case of a quadrangle

$$x\cos 2\theta + y\sin 2\theta = a\frac{\cos 2(a+\beta) + \;\;.\;\;.}{4}$$
$$+ a\frac{\cos(2\theta - 2a) + \;\;.\;\;.}{4}$$
$$+ \qquad \frac{3a\cos 4\theta}{4}$$

which only envelopes a four-cusped hypocycloid if

$$\cos 2(a+\beta) + \;\;.\;\;.\;\; + \;\;.\;\;.\;\; = 0.$$

If we consider the given point as fixed, we may take $\theta = 0$, and change the axes so that it is at the origin. The Simson line now becomes from (4)

$$x\cos(a + \beta + \gamma + \;\;.\;\;.\;\;) + y\sin(a + \beta + \gamma + \;\;.\;\;.\;\;)$$
$$= 2a\cos a\cos\beta\cos\gamma\cos\delta \;\;.\;\;.\;\;.$$

and if 2θ be the inclination of the mean line and $2a'$, $2\beta'$, 2γ, . . inclinations of the radii a, β, γ . . to it, we may write the equation

$$x\cos(a' + \beta' + \;\;.\;\;.\; + n\theta) + y\sin(a' + \beta' + \;\;.\;\;.\;\; + n\theta)$$
$$= 2a\cos(a' + \theta)\cos(\beta' + \theta) \;\;.\;\;.$$

where
$$a' + \beta' + \gamma' + \;\;.\;\;.\;\; = 0$$

we may now drop accents, and we get

$$x\cos n\theta + y\sin\theta$$
$$= 2a\cos(a + \theta)\cos(\beta + \theta)\cos(\gamma + \theta) \qquad - \quad - \quad (7)$$

For a triangle

$$x\cos 3\theta + y\sin 3\theta$$
$$= \frac{a}{2}\{\cos\overline{\theta - 2a} + \;\;.\;\;\; + \cos 3\theta\}$$

which envelopes a cardioide.

https://doi.org/10.1017/S0013091500031874 Published online by Cambridge University Press

For a quadrangle

$$x\cos 4\theta + y\sin 4\theta = \frac{a}{4}\{\cos 2(\alpha+\beta) + \quad . \quad . \quad . \quad .$$
$$+ \cos 2(\theta - \alpha) + \quad . \quad . \quad .$$
$$+ \cos 4\theta\}.$$

In the case of regular figures these results are much simplified, and it will be found best to start *de novo* from the equation (4): we may take the axis of x to pass through a corner when n is odd, and through a mid point of a side if n is even: so that

$$n \text{ odd}, \qquad \alpha = \frac{0}{n}, \qquad \beta = \frac{2\pi}{n}, \qquad \gamma = \frac{2\pi}{n} \quad . \quad .$$

$$n \text{ even}, \qquad \alpha = \frac{\pi}{2n}, \qquad \beta = \frac{3\pi}{2n}, \qquad \gamma = \frac{5\pi}{2n} \quad . \quad .$$

The lines are respectively

$$x\cos\left(\frac{n-1}{2}\pi + (n-2)\theta\right) + y\sin\left(\frac{n-1}{2}\pi + (n-2)\theta\right)$$
$$= -a\cos\left(\frac{n-1}{2}\pi + n\theta\right) + 2a\cos\theta\cos\left(\frac{\pi}{n} + \theta\right) \quad . \quad . \quad . \quad .$$

and

$$x\cos\left(\frac{n\pi}{2} + (n-2)\theta\right) + y\sin\left(\frac{n\pi}{2} + (n-2)\theta\right)$$
$$= -a\cos\left(\frac{n\pi}{2} + n\theta\right) + 2a\cos\left(\frac{\pi}{2n} + \theta\right)\cos\left(\frac{3\pi}{2n} + \theta\right) \quad . \quad . \quad . \quad .$$

But by a known trigonometrical formula

$$\cos\theta\cos\left(\frac{\pi}{n} + \theta\right)\cos\left(\frac{2\pi}{n} + \theta\right) \quad . \quad . \quad n \text{ factors (odd)}$$
$$= (-1)^{\frac{n-1}{2}} \cdot \frac{\cos n\theta}{2^{n-1}}$$

$$\cos\left(\theta + \frac{\pi}{2n}\right)\cos\left(\theta + \frac{3\pi}{2n}\right) \quad . \quad . \quad = \frac{\cos n\left(\theta + \frac{\pi}{2}\right)}{2^{n-1}}$$

whence the line, in either case, becomes

$$x\cos(n-2)\theta + y\sin(n-2)\theta$$
$$= a\cos n\theta \cdot \frac{1-2^{n-2}}{2^{n-2}} \quad - \quad - \quad - \qquad (8)$$

which envelopes an n-cusped hypocycloid.

https://doi.org/10.1017/S0013091500031874 Published online by Cambridge University Press

In the next place, if the point is fixed, transfer to it as orgin and we get as in equation (7)

$$x\cos n\theta + y\sin n\theta = 2a\cos(\alpha+\theta)\cos(\beta+\theta) \quad . \quad .$$
$$= \frac{a\cos\theta}{2^{n-2}}$$

which passes through the fixed point $x = a/2^{n-2}$, $y = 0$

Hence the Simson line of a regular polygon with respect to a moving point envelopes a hypocycloid and the Simson line with respect to a fixed point of a regular polygon that slides round a circle passes through another fixed point.