Ergod. Th. & Dynam. Sys. (1985), **5**, 273–283
Printed in Great Britain

The entropy of C^2 surface diffeomorphisms in terms of Hausdorff dimension and a Lyapunov exponent

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(*Received* 5 *April* 1984 *and revised* 10 *October* 1984)

Abstract. In this paper we prove that if the entropy of an ergodic measure preserved by a C^2 surface diffeomorphism is positive then it is equal to the product of the Hausdorff dimension of the quotient measure defined by the family of stable manifolds and the positive Lyapunov exponent.

0. *Introduction*

In [**6**] Manning proved that for an Axiom A diffeomorphism f of a surface M, which preserves an ergodic Borel probability measure μ, its entropy $h_\mu(f)$ satisfies

$$h_\mu(f) = \delta_\mu \chi^+_\mu, \tag{1}$$

where δ_μ is the Hausdorff dimension of the intersection of an interval of an unstable manifold with the set of generic points of μ, and χ^+_μ is the positive Lyapunov exponent for μ. The number δ_μ could be reinterpreted as follows: Take $x \in$ support μ, let Λ be the basic set supporting μ, choose $\varepsilon > 0$ small enough and put

$$R = \bigcup_{y \in W^u_\varepsilon(x) \cap \Lambda} W^s_\varepsilon(y),$$

where $W^s_\varepsilon(x)$ ($W^u_\varepsilon(x)$) denotes the stable (unstable) manifold of size ε through x. Obviously $\mu(R) > 0$.

The family of stable manifolds $\{W^s_\varepsilon(y)\}_{y \in W^u_\varepsilon(x) \cap \Lambda}$ is a partition of R, thus we can define the *transverse measure* $\tilde{\mu}_x$ *on* $W^u_\varepsilon(x)$ as the quotient measure on $W^u_\varepsilon(x) \cap \Lambda$ given by the stable manifolds, i.e. if $A \subset W^u_\varepsilon(x) \cap \Lambda$ then

$$\tilde{\mu}_x(A) = \mu\left(\bigcup_{y \in A} W^s_\varepsilon(y)\right).$$

From Manning's proof of (1) it follows that

$$\delta_\mu = \inf_Y \{\delta(Y)\colon Y \subset W^u_\varepsilon(x) \cap \Lambda \quad \text{and} \quad \tilde{\mu}_x(Y) = \mu(R)\},$$

where $\delta(Y)$ denotes the Hausdorff dimension of Y. So if we define the *Hausdorff* *dimension of a Borel measure* μ on a compact metric space X as

$$\delta(\mu)=\inf\{\delta(Y)\colon Y\subset X \quad\text{and}\quad \mu(Y)=\mu(X)\},$$

then Manning's $\delta_\mu=\delta(\tilde{\mu}_x)$.

Unfortunately, the proof of (1) in [6] cannot easily be extended to the non-uniform hyperbolic case because it relies on the invariance of the basic set Λ and the continuous splitting of the tangent space. However, combining ideas of Manning [6], Mañé [5] and Young [12], in the setting of Pesin's theory [8], [9], we are able to extend (1) to any C^2 surface diffeomorphism preserving an ergodic measure.

THEOREM 1. *Let $f\colon M\to M$ be a C^2 diffeomorphism of a surface M preserving an ergodic Borel probability measure μ with $h_\mu(f)>0$. Choose $l>1$ such that the Pesin set $\Lambda_{\chi,l}$ has positive measure, consider x a μ-density point of $\Lambda_{\chi,l}$ and let $S_\chi^l(x)$ denote the family of stable manifolds through a neighbourhood of x. Let V be a transverse submanifold to $S_\chi^l(x)$ and let $\tilde{\mu}_V$ be the transverse measure on V defined on $V\cap S_\chi^l(x)$ by $S_\chi^l(x)$. Then*

$$h_\mu(f)=\delta(\tilde{\mu}_V)\chi_\mu^+,$$

where χ_μ^+ is the positive Lyapunov exponent for μ.

COROLLARY 1. *If $\tilde{\mu}_V$ is absolutely continuous with respect to the Riemannian measure on V then $h_\mu(f)=\chi_\mu^+$.*

Let us recall that a point $x\in M$ is said to be *generic for* μ if for every continuous function $\Psi\colon M\to\mathbb{R}$

$$\frac{1}{n}\sum_{i=0}^{n}\Psi(f^i(x))\to\int\Psi\,d\mu.$$

COROLLARY 2. *If G_μ denotes the set of generic points for μ, then*

$$\delta(G_\mu)\geq 1+h_\mu(f)/\chi_\mu^+.$$

The proof of the above corollary follows from the fact, see §II of this paper, that for any $0<\theta<1$ the family of local stable manifolds is θ-Hölder continuous. This also implies that for any submanifold V transverse to $S_\chi^l(x)$ the dimension $\delta(\mu_V)$ is constant; let us call this number $\delta(\mu^s)$. Similarly define $\delta(\mu^u)$ for the family of local unstable manifolds $U_\chi^l(x)$.

COROLLARY 3. $\delta(\mu)=\delta(\mu^s)+\delta(\mu^u)$.

This corollary follows from [12].

The proof of theorem 1 relies on the definition and properties of the family of local stable manifolds $S_\chi^l(x)$, which we summarize in §II. Also we shall need Bowen's definition of entropy for non-compact sets [1] and local approaches to entropy [5] and Hausdoff dimension [12].

The results presented in this paper are basically contained in the author's Ph.D. thesis [7] written under the supervision of Dr. Anthony Manning, whom we thank for his guidance.

I. *Entropy and dimension*

We recall from [1] Bowen's definition of a topological entropy for a possibly non-compact subset Y of a compact metric space X and a continuous map $f: X \to X$.

Let $\mathscr{A}$ be a finite open cover of X and write $E < \mathscr{A}$ if E is contained in some member of $\mathscr{A}$. Write $n_{\mathscr{A}}(E)$ or simply $n(E)$ for the largest non-negative integer such that $f^k E < \mathscr{A}$ for $0 \le k < n(E)$.

Let $Y \subset X$, possibly non-compact. If $\mathscr{E} = \{E_1, E_2, \ldots\}$ has union containing Y, set for $\lambda > 0$

$$\mathscr{D}_{\mathscr{A}}(\mathscr{E}, \lambda) = \sum_{i=1}^{\infty} \exp(-\lambda n_{\mathscr{A}}(E_i)).$$

Define $H_{\mathscr{A},\lambda}$ by

$$H_{\mathscr{A},\lambda}(Y) = \liminf_{\varepsilon \to 0} \left\{ \mathscr{D}_{\mathscr{A}}(\mathscr{E}, \lambda) : \mathscr{E} = \{E_1, E_2, \ldots\}, \bigcup_{i=1}^{\infty} E_i \supset Y \right.$$
$$\left. \text{and } \exp - n_{\mathscr{A}}(E_i) < \varepsilon \text{ for each } i \right\},$$

and then

$$h_{\mathscr{A}}(f, Y) = \inf\{\lambda : H_{\mathscr{A},\lambda}(Y) = 0\},$$

and

$$h(f, Y) = \sup_{\mathscr{A}} h_{\mathscr{A}}(f, Y).$$

Its connection with the measure theoretic entropy, see [2] for definition, is made clear by the following proposition.

PROPOSITION 1.1 [1]. *If $f: X \to X$ is a continuous map of a compact metric space X, preserving an ergodic Borel probability measure μ, then*

$$h_\mu(f) = \inf\{h(f, Y) : Y \subset X \text{ and } \mu(Y) = 1\}.$$

LEMMA 1.2. *Let $\{\mathscr{A}_n\}_{n=1}^{\infty}$ be a collection of finite open covers of M, with* diam $\mathscr{A}_n \to 0$, *and $\{A_n\}_{n=1}^{\infty}$ a collection of sets of positive measure, then*

$$h_\mu(f) \le \sup_n h_{\mathscr{A}_n}(A_n, f).$$

Proof. It is easy to verify that for any finite open cover $\mathscr{A}$ and any set $Y \subset M$, $h_{\mathscr{A}}(Y, f) = h_{\mathscr{A}}(fY, f)$ and

$$h_{\mathscr{A}}\left(\bigcup_{k=0}^{\infty} f^k Y, f\right) = \sup_k h_{\mathscr{A}}(f^k Y, f) = h_{\mathscr{A}}(Y, f).$$

For each $n \ge 1$ let $B_n = \bigcup_{k=0}^{\infty} f^k A_n$, then by ergodicity each B_n has measure 1, and so does $B = \bigcap_{n=1}^{\infty} B_n$. Then for each $n \ge 1$,

$$h_{\mathscr{A}_n}(B, f) \le h_{\mathscr{A}_n}(B_n, f) = h_{\mathscr{A}_n}(A_n, f).$$

Since diam $\mathscr{A}_n \to 0$, as $n \to \infty$

$$h(B, f) = \sup_n h_{\mathscr{A}_n}(B, f) \le \sup_n h_{\mathscr{A}_n}(A_n, f),$$

and by proposition 1.1 $h_\mu(f) \le h(B, f)$ from which the lemma follows. $\square$

https://doi.org/10.1017/S0143385700002893 Published online by Cambridge University Press

Now suppose $\psi: X \to \mathbb{R}^+$ is a function, for $x \in X$ and $n \geq 0$ define

$$B(x, \psi, n) = \{y \in X | d(f^i x, f^i x) \leq \psi(f^i x) \quad 0 \leq i < n\}.$$

PROPOSITION 1.3 [**5**]. *Let $f: M \to M$ be a continuous map of a compact manifold M, preserving an ergodic Borel probability measure μ. Let $Y \subset M$ be of positive measure and denote by μ_Y the conditional measure on Y, then*

$$\limsup_{n\to\infty} -\frac{1}{n} \log \mu_Y(B(x, \psi, n)) \leq h_\mu(f),$$

for almost every $x \in Y$ and every $\psi: M \to (0, 1)$ such that $\log \psi$ *is μ-integrable.*

The *Hausdorff dimension* $\delta(Y)$ of $Y \subset X$, a compact metric space, is defined as follows: Let

$$m_\lambda(Y) = \lim_{\varepsilon\to 0} \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} E_i)^\lambda : \bigcup_{i=1}^{\infty} E_i \supset Y \quad \text{and} \quad \operatorname{diam} E_i < \varepsilon \quad \text{for all } i\right\};$$

define

$$\delta(Y) = \inf\{\lambda : m_\lambda(Y) = 0\}.$$

The next proposition is the dimensional analogue to proposition 1.2.

PROPOSITION 1.4 [**12**]. *Let μ be a Borel probability measure on $\mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ be measurable and of positive measure. Suppose that for every $x \in \Lambda$*

$$\limsup_{\rho\to 0} \frac{\log \mu(B(x, \rho))}{\log \rho} \leq \bar{\delta},$$

then $\delta(\Lambda) \leq \bar{\delta}$.

II. *Lyapunov exponents and Pesin's theory*

Let $f: M \to M$ be a C^2 diffeomorphism of a surface M, preserving an ergodic Borel probability measure μ. For $x \in M$ and $v \in T_x M$, the tangent space at x, define the *Lyapunov exponent of f at* (x, v) to be the number

$$\chi(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\|.$$

For each $x \in M$ the restriction of χ to $T_x M$ takes at most two values $\chi_-(x) \leq \chi_+(x)$. The exponents are f-invariant, i.e. $\chi_\pm(x) = \chi_\pm(fx)$, and by the Multiplicative Ergodic Theorem [**11**] for μ-almost every $x \in M$ $T_x M = E_x^s \oplus E_x^u$ and if $v \in E_x^{s(u)}$, $\lim_{n\to\pm\infty} (1/n) \log \|D_x f^n v\| = \chi_\mp(x)$ exists. The Subadditive Ergodic Theorem [**4**] allows us to study the growth rate of $\|D_x f^n\|$, and for μ-almost every $x \in M$

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n\| = \chi_+(x).$$

The proofs of the facts and theorems mentioned below are due to Pesin [**8**], [**9**], many of them also appear in [**3**].

Since μ is ergodic the exponents are constant almost everywhere, so let $\chi_\mu^- = \chi_-(x)$ and $\chi_\mu^+ = \chi_+(x)$. Let $\chi = \min\{-\chi_\mu^-, \chi_\mu^+\} > 0$ and $l > 1$; denote by $\Lambda_{\chi,l}$ the set of points $x \in M$ with the following properties: there exists a decomposition $T_x M = E_x^s \oplus E_x^u$

such that for every $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$ we have
for $v \in Df^m E^s_x$:

$$\|D_{f^m_x} f^n v\| \le l \exp - n\chi \exp(\chi 10^{-3}(n+|m|))\|v\|,$$

$$\|D_{f^m_x} f^{-n} v\| \ge l^{-1} \exp n\chi \exp(-\chi 10^{-3}(n+|m|))\|v\|;$$

for $v \in Df^m E^u_x$:

$$\|D_{f^m_x} f^n v\| \ge l^{-1} \exp n\chi \exp(-\chi 10^{-3}(n+|m|))\|v\|,$$

$$\|D_{f^m_x} f^{-k} v\| \le l \exp - n\chi \exp(-\chi 10^{-3}(n+|m|))\|v\|;$$

and for the angle $\gamma(x)$ between the subspaces E^s_x and E^u_x:

$$\gamma(f^m x) \ge l^{-1} \exp -\chi 10^{-3}|m|.$$

Write $\Lambda_\chi = \bigcup_{l>1} \Lambda_{\chi,l}$ and if $\delta(x)$ is a positive measurable function on Λ_χ write

$$B^{s(u)}(\delta(x)) = \{v \in E^{s(u)}: \|v\| < \delta(x)\},$$

$$B(\delta(x)) = B^s(\delta(x)) \times B^u(\delta(x)),$$

and

$$U(x, \delta(x)) = \exp_x B(\delta(x)).$$

STABLE MANIFOLD THEOREM (S.M.T.) [**8**]. *There exist measurable functions $\delta(x)$ and $K(x)$, $x \in \Lambda_\chi$, and a family of maps $\phi(x): B^s(\delta(x)) \to B^u(\delta(x))$ of class C^1 depending measurably on $x \in \Lambda_\chi$ satisfying the following conditions*:

(1) *The set $W^s(x) = \{\exp_x(v, \phi(x)v): v \in B^s(\delta(x))\}$ is a C^1 submanifold.*
(2) *$x \in W^s(x)$ and $T_x W^s(x) = E^s_x$.*
(3) *For $y \in W^s(x)$, $n \in \mathbb{Z}^+$ and $0 < \chi' < \chi$ we have*

$$d(f^n x, f^n y) \le K(x) \exp -\chi' n d(x, y).$$

The submanifold $W^s(x)$ is called the *local stable manifold through x.* If we apply the theorem to f^{-1} we get $W^u(x)$ the *local unstable manifold through x.*

PROPOSITION 2.1. (1) *The sets $\Lambda_{\chi,l}$ are closed.*
(2) *The subspaces E^s_x, E^u_x depend on x continuously on $\Lambda_{\chi,l}$.*
(3) $\mu(\Lambda_\chi) = 1$.
(4) $K_l = \sup_{x \in \Lambda_{\chi,l}} K(x) < \infty$.
(5) $\delta_{\chi,l} = \inf_{x \in \Lambda_{\chi,l}} \delta(x) > 0$.

For $x \in \Lambda_{\chi,l}$ the collection of local stable manifolds passing through $U(x, \delta_{\chi,l}/8) \cap \Lambda_{\chi,l}$ is called *the family of local stable manifolds $S^l_\chi(x)$*. Choose q, $0 < q \le \delta_{\chi,l}/8$ and put

$$\Lambda_{\chi,l}(x) = \bigcup_{y \in \Lambda_{\chi,l} \cap \overline{U(x,q)}} \overline{W^s(y)} \cap \overline{U(x, q)}$$

Let V^1 and V^2 be two C^1 submanifolds transversal to the family $S^l_\chi(x)$. There exist open sets $\tilde V^1 \subset V^1$ and $\tilde V^2 \subset V^2$ for which the *Poincaré map*

$$p: \Lambda_{\chi,l}(x) \cap \tilde V^1 \to \Lambda_{\chi,l}(x) \cap \tilde V^2,$$

defined by $p(y) = \tilde V^2 \cap W^s(w)$, for $w \in U(x, q) \cap \Lambda_{\chi,l}$ and $\{y\} = \tilde V^1 \cap W^s(w)$, is a homeomorphism. We say that the family of local stable manifolds $S^l_\chi(x)$ is *θ-Hölder continuous*, for $0 < \theta < 1$, if each Poincaré map between any two C^1 transverse submanifolds is θ-Hölder continuous with constant depending continuously on the C^1 distance between the two submanifolds.

THEOREM 2.2 [8]. *For any* $0<\theta<1$ *the family of local stable manifolds is* θ*-Hölder continuous.*

For the proof of the above theorem we refer to Pesin's proof of the absolute continuity of the family of local stable manifolds.

III. *Proof of theorem* 1.
Since $h_\mu(f)=h_\mu(f^{-1})$ by Ruelle's inequality [10] $h_\mu(f)\le \min\{-\chi_\mu^-, \chi_\mu^+\}$, therefore by hypothesis $\chi>0$.

Choose $l>1$ such that $\Lambda_{\chi,l}$ has positive μ measure, let $x\in M$ be a μ-density point of $\Lambda_{\chi,l}$ and consider the family of local stable manifolds $S_\chi^l(x)$. Let $V=W^u(x)$ be the C^1 submanifold transversal to $S_\chi^l(x)$.

From Pesin [9] there exists $h=h(l)>0$ such that for all $y\in\Lambda_{\chi,l}(x)$, $D(z(y),h)\subset W^s(y)$, where $\{z(y)\}=W^s(y)\cap V$ and $D(z(y),h)$ is a 1-disk of radius h centred at $z(y)$. For $0<q\le h$ put

$$R_q=\bigcup_{y\in\Lambda_{\chi,l}(x)} D(z(y),q).$$

Since x is a density point of $\Lambda_{\chi,l}$, $\mu(R_q)>0$ and

$$h\left(f,\bigcup_{i=0}^{\infty} f^iR_q\right)=h(f,R_q),\qquad h_\mu(f)\le h(f,R_q).$$

Let G_μ denote the of generic points of μ and for $\varepsilon>0$, $r>0$ and $n>0$ set

$$G_{\mu,n}=\left\{y\in M\,\middle|\,\left|\frac{1}{m}\sum_{k=0}^{m-1}\log(\|D_{f_y^k}f\|+r)-\int\log(\|D_zf\|+r)\,d\mu\right|\le\varepsilon\ \textit{for every } m\ge n\right\}.$$

Let $V_{\mu,n}=\{z(y)\in G_{\mu,n}|y\in\Lambda_{\chi,l}(x)\}$ and $V_\mu=\bigcup_n V_{\mu,n}$. If $\mathcal{U}_n=\{U\}$ is a cover of $V_{\mu,n}$ by sets U contained in $V_{\mu,n}$, write $\mathcal{U}_{q,n}^*=\{U_q^*\}$ where

$$U_q^*=\bigcup_{\substack{z(y)\in U\\ y\in\Lambda_{\chi,l}(x)}} D(z(y),q).$$

Now let $\mathcal{A}$ be a finite open cover of M and let L be a Lebesgue number for $\mathcal{A}$; furthermore choose $\mathcal{A}$ such that $\|D_zf\|$ does not change more than $r>0$ in each element of $\mathcal{A}$. Choose q such that $8K_lq<L$.

Suppose $y_1, y_2\in\mathcal{U}_q^*$, then by the Mean Value Theorem, the S.M.T. and proposition 2.1 there exists $K_l>0$ such that for $k>0$

$$d(f^ky_1,f^ky_2)\le 2K_lq\exp -k\chi+d(z(y_1),z(y_2))\prod_{j=0}^{k-1}(\|D_{f^jz(y_1)}f\|+r),$$

if $d(f^{k-1}z(y_1),f^{k-1}z(y_2))<L/2$.

It is clear that if $W^s(y)$ exists and y is generic then so is any $z\in W^s(y)$; therefore since $2K_l q < L/4$ and for $U_q^*\in \mathcal{U}_{q,n}^*$ diam $f^{n-1}U_q^* < L/2$, we have

$$\operatorname{diam} f^k U^* \leq \operatorname{diam} U \exp\left(\left(\int \log(\|D_z f\| + r\, d\mu + 2\varepsilon\right)k + L/4,\right.$$

so if $\mathcal{U}_{q,n}^*$ is fine enough

$$\sum \exp -n_{\mathcal{A}}(U^*)\left(\int \log(\|D_z f\| + r)\, d\mu + 2\varepsilon\right)(\delta(V_\mu)+\varepsilon)$$
$$\leq \sum (L/2)^{\delta(V_\mu)+\varepsilon} \operatorname{diam} U^{\delta(V_\mu)+\varepsilon},$$

and therefore

$$h_{\mathcal{A}}(f, R_q\cap G_{\mu,n}) \leq \delta(V_\mu)\int \log(\|D_z f\| + r)\, d\mu.$$

Since $\mathcal{A}$, r, ε and n were arbitrarily chosen then by lemma 1.2

$$h_\mu(f) \leq \delta(V_\mu)\int \log\|D_z f\|\, d\mu.$$

Applying the above procedure to f^n it follows that

$$h_\mu(f^n) \leq \delta(V_\mu)\int \log\|D_z f^n\|\, d\mu,$$

and by the Subadditive Ergodic Theorem [**4**] we obtain that

$$h_\mu(f) \leq \delta(V_\mu)\chi_\mu^+. \tag{2}$$

Note that we could have replaced R_q by any set of stable manifolds contained in $S_\chi^l(x)$ of positive measure.

In the rest of the proof of theorem 1 we shall follow very closely Mañé's proof of Pesin's formula [**5**].

Fix $\alpha > 0$ so small that $\mu(\Lambda_{\chi,l}) > 2\sqrt{\alpha}$. By Egorov's theorem there exists a compact set $\Lambda^1\subset\Lambda_\chi$, with $\mu(\Lambda^1)\geq 1-\alpha$, such that $T_xM = E_x^s\oplus E_x^u$ varies continuously on Λ^1 and for some $N>0$, if $g=f^N$, the inequalities

$$\|D_x g^n v\| \geq \exp nN(\chi_\mu^+ - \alpha)\|v\|,$$
$$\|D_x g^n v\| \leq 1,$$

hold for all $x\in\Lambda^1$, $n\geq 0$ and $v\in E_x^u$. Observe that the Ergodic Theorem implies that

$$\mu\left(\{x \mid \lim_{n\to\infty} l/n\,\#\{0\leq j<n \mid g^j(x)\notin\Lambda^1\}\leq\sqrt{\alpha}\}\right)\geq 1-\sqrt{\alpha}.$$

Then, applying Egorov's theorem once more, there exists a compact set $\Lambda^2\subset\Lambda^1$ with $\mu(\Lambda^2)\geq 1-2\sqrt{\alpha}$ and $N_0>0$ such that

$$\#\{0\leq j<n \mid g^j(x)\notin\Lambda^1\}\geq 2n\sqrt{\alpha},$$

for all $x\in\Lambda^2$. Now let $\Lambda^3=\Lambda^2\cap\Lambda_{\chi,l}$; clearly $\mu(\Lambda^3)\geq\mu(\Lambda_{\chi,l})-2\sqrt{\alpha}>0$.

Let us choose $x\in M$ such that x is a μ-density point for Λ^3. We define a measure $\tilde{\nu}_x$ on $W_{\mathrm{loc}}^u(x)$ as follows: if $A\subset W_{\mathrm{loc}}^u(x)\cap U(x,h/2)$, then

$$\tilde{\nu}_x(A)=\mu\left(\left(\bigcup_{z(y)\in A} W_{\mathrm{loc}}^s(y)\cap U(x,h/2)\right)\cap\Lambda^3\right)\Big/\mu(U(x,h/2)\cap\Lambda^3),$$

where $z(y)\in W_{\mathrm{loc}}^s(y)\cap W_{\mathrm{loc}}^u(x)$ and $y\in U(x,h/2)\cap\Lambda_{\chi,l}$.

Let us denote by ν the μ-conditional measure on $U(x, h/2)\cap\Lambda^3$ and let

$$B(y, \psi, n)=\{w\in M|d(g^i(w), g^i(y))\le\psi^i(g(y))\quad\text{for } 0\le i\le n\},$$

where $\psi: M\to(0, 1)$ is a function. Then since f is ergodic it follows that for almost every $y\in U(x, h/2)\cap\Lambda^3$,

$$\limsup_{n\to\infty} -\frac{1}{n}\log\nu(B(y, \psi, n))\le h_\mu(g),$$

provided that $\log\psi$ is μ-integrable. Now we are going to define the ψ to be used in this proof. For this we need some lemmas from [**5**].

If $E=E_1\oplus E_2$ is a normed space, we say that $G\subset E$ is an *(E_1, E_2)-graph* if there exists an open subset $U\subset E_2$ and a C^1 map $\phi: U\to E_1$ satisfying $G=\{(\phi(v), v): v\in U\}$. The number

$$\sup\{\|\phi(v_1)-\phi(v_2)\|/\|v_1-v_2\||v_1, v_2\in U\}$$

is called the *dispersion* of G.

LEMMA 3.1 [**5**]. *For all $c>0$ there exists $\xi>0$ such that if $y\in\Lambda^1$ and for $m>0$, $g^m(y)\in\Lambda^1$, then if W is a C^1 submanifold of M such that $\exp_y^{-1}W$ is an (E^s_y, E^u_y)-graph with dispersion $\le c$ and $W\subset B(x, \xi^m)$, then $\exp^{-1}_{g^m(y)}g^mW$ is a $(D_yg^mE^s_Y, D_yg^mE^u_y)$-graph with dispersion $\le c$.*

Remark 3.2. For each $y\in\Lambda_{\chi,l}$, $W^u_{\text{loc}}(y)$ can be lifted through the exponential map as an (E^s_y, E^u_y)-graph. Furthermore, as in remark 2.3.1 of [**9**], for any small $c>0$ we can find $\delta>0$ such that if $z\in B(y, \delta)\cap\Lambda_{\chi,l}$, then $W^u_{\text{loc}}(z)\cap B(y, \delta)$ can be lifted through the exponential map as an (E^s_y, E^u_y)-graph with dispersion $\le c$.

LEMMA 3.3 [**5**], [**8**]. *For $\alpha>0$ there exist $a>0$ and $c>0$ such that if $y\in\Lambda^1$, $z\in M$ and $d(z, y)<a$, then for any C^1 submanifold of M such that $z\in W$ and $\exp_y^{-1}W$ is an (E^s_y, E^u_y)-graph with dispersion $\le c$ we have*

$$|\log\|D_zg|T_zW\|-\log\|D_yg|E^u_y\||<\alpha.$$

Choose c, a, ξ and δ by the above lemmas and remark 3.2. Let

$$r(y)=\begin{cases}0 & \text{if } y\notin\Lambda^1\\ \text{the smallest integer } k>0 \text{ such that } g^k(y)\in\Lambda_1 & \text{if } y\in\Lambda_1,\end{cases}$$

define $\psi: M\to(0, 1)$ by

$$\psi(y)=\min\{a, \delta, \xi^{r(y)}, \exp(N(\chi^+_\mu+2\alpha))^{-r(y)}/\sqrt{2}\}.$$

Since $\int r(y)\,d\mu\le 1$, then $\log\psi$ is μ-integrable. For small $\rho>0$ define $n(\rho)$ to be the smallest positive integer satisfying

$$\rho\exp\{n(\rho)[N(\chi^+_\mu-\alpha)-\alpha-4NC\sqrt{\alpha}]\}\ge 1,$$

where $C=\max\{\sup_{y\in M}\log\|D_yf\|, \sup_{y\in M}\log\|D_yf^{-1}\|\}$. It is obvious that for small ρ, $2n(\rho)\sqrt{\alpha}>N_0$.

The proof of the following lemma is essentially borrowed from [5] and [12].

LEMMA 3.4. *For sufficiently small $\rho>0$ and any $y, w\in\Lambda^3\cap U(x,h)$ such that $w\in B(y,\psi,n(\rho))$, then*

$$W^u_{\mathrm{loc}}(w)\cap B(y,\psi,n(\rho))\subset B(z(w),\rho)\cap W^u_{\mathrm{loc}}(z(w)),$$

where $z(w)\in W^u_{\mathrm{loc}}(w)\cap W^s_{\mathrm{loc}}(y)$.

Proof. For any $y\in\Lambda^3\cap U(x,h)$ let $\{n_0,n_1,\ldots\}=\{n\ge 0: g^n(y)\in\Lambda^1\}$, assume that $n_0<n_1<\cdots<n_k<n(\rho)<n_{k+1}<\cdots$. Let us write $W^u(w,\psi,n(\rho))$ for $W^u_{\mathrm{loc}}(w)\cap B(y,\psi,n(\rho))$, for any $w\in B(y,\psi,n(\rho))\cap\Lambda^3\cap U(x,h/2)$. So for small $\rho>0$ then $n_k\ge N_0$, since $2n(\rho)\sqrt{\alpha}>N_0$ and $y\in\Lambda^2$. Denote $\{0\le n_i\le n\}$ by S_n. Thus if $n_i\in S_{n_k}$, by lemma 3.1, $g^{n_i}(W^u(w,\psi,n(\rho))$ can be lifted as an $(E^s_{g^{n_i}(y)},E^u_{g^{n_i}(y)})$-graph with dispersion $\le c$ (since from the definition of ψ, $g^{n_{i-j}}W^u,(w,\psi,n(\rho))\subset B(g^{n_{i-j}}(y),\xi^{n_i-n_{i-j}})$ for all $0\le j<i$). And by lemma 3.3 if $n_i\in S_{n_k}$

$$|\log\|D_{g^{n_i}(z')}g|T_{g^{n_i}(z')}g^{n_i}W^u(w,\psi,n(\rho))\|-\log\|D_{g^{n_i}(y)}g|E^u_{g^{n_i}(y)}\||<\alpha,$$

for any $z'\in W^u(w,\psi,n(\rho))$, also by the definition of ψ. Therefore, since $W^u(w,n(\rho))$ is one-dimensional

$$\begin{aligned}
&\log\|D_{z'}g^{n_k}|T_{z'}W^u(w,\psi,n(\rho))\|\\
&=\sum_{i=0}^{n_k-1}\log\|D_{g^i(z')}g|T_{g^i(z)}g^iW^u(w,\psi,n(\rho))\|\\
&\ge\sum_{i\in S_{n_k}}\log\|D_{g^i(z')}g|T_{g^i(z')}g^iW^u(w,\psi,n(\rho))\|-(n_k-\#S_{n_k})NC\\
&\ge\sum_{i\in S_{n_k}}\log\|D_{g^i(y)}g|E^u_{g^i(y)}\|-\alpha n_k-(n_k-\#S_{n_k})NC\\
&\ge\sum_{i=1}^{n_k-1}\log\|D_{g^i(y)}g|E^u_{g^i(y)}\|-\alpha n_k-2(n_k-\#S_{n_k})NC\\
&=\log\|D_yg^{n_k}|E^u_y\|-\alpha n_k-4n_kNC\sqrt{\alpha}\\
&\ge n_kN(\chi^+_\mu-\alpha)-\alpha n_k-4n_kNC\sqrt{\alpha}\\
&=n_k(N(\chi^+_\mu-\alpha)-\alpha-4NC\sqrt{\alpha}).
\end{aligned}$$

Thus

$$\|D_{z'}g^{n_k}|T_{z'}W^u(w,\psi,n(\rho))\|\ge\exp n_k(N(\chi^+_\mu-\alpha)-\alpha-4NC\sqrt{\alpha})\qquad(*)$$

for any $z'\in W^u(w,\psi,n(\rho))$.

Now let $d'_i(\cdot,\cdot)$ denote the restriction of the Riemannian metric to $g^iW^u(w,\psi,n(\rho))$. Since $g^{n_k}W^u(w,\psi,n(\rho))$ can be lifted as an $(E^s_{g^{n_k}(y)},E^u_{g^{n_k}(y)})$-graph with dispersion $\le c$, and c is small, it follows that $d'_{n_k}(\cdot,\cdot)\le\sqrt{2}\,d(\cdot,\cdot)$. Obviously $d(\cdot,\cdot)\le d'_0(\cdot,\cdot)$.

By the Mean Value Theorem if $w'\in W^u(w,\psi,n(\rho))$, then

$$d'_{n_k}(g^{n_k}(w'),g^{n_k}(z(w)))=\|D_{z'}g^{n_k}|T_{z'}W^u(w,\psi,n(\rho))\|d'_0(w',z(w)),$$

for some $z'\in W^u(w,\psi,n(\rho))$. Therefore, by $(*)$ and the definition of $n(\rho)$,

$$\begin{aligned}
d'_{n_k}(g^{n_k}(w'),g^{n_k}(z(w))&\ge\exp n_k(N(\chi^+_\mu-\alpha)-\alpha-4NC\sqrt{\alpha})d'_0(w',z(w))\\
&\ge\exp(-n(\rho)+n_k)(N(\chi^+_\mu-\alpha)-\alpha-4NC\sqrt{\alpha})\\
&\qquad\times(d'_0(w',z(w))/\rho).
\end{aligned}$$

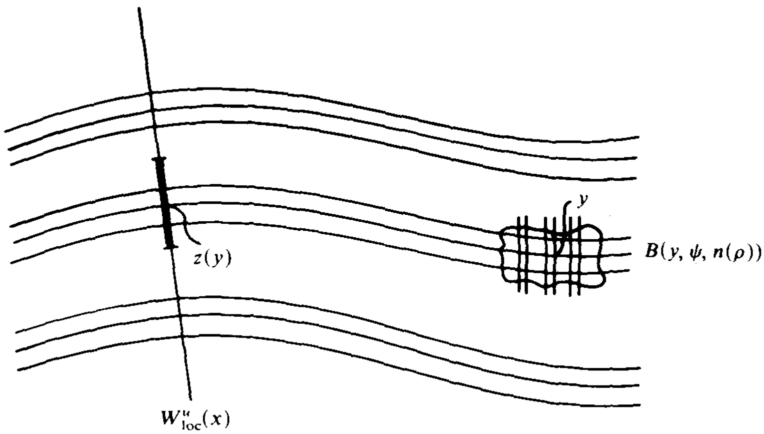
Now since $r(g^{n_k}(y)) = n_{k-1} - n_k$ and

$$d(g^{n_k}(w'), g^{n_k}(z(w))) \le \psi(g^{n_k}(y))$$
$$\le \exp(-n_{k+1} + n_k)(N(\chi_\mu^+ - \alpha) - \alpha - 4NC\sqrt{\alpha})/\sqrt{2},$$

it follows that $d(w', z(w)) \le \rho$ and the lemma is proved. □

Continuing with the main argument, let us note that the last lemma implies that

$$\nu(B(y, \psi, n(\rho)) \le \nu\Big(\bigcup_{w \in B(y,\psi,n(\rho)) \cap \Lambda^3} W^u(w, \psi, n(\rho))\Big)$$
$$\le \nu\Big(\bigcup_{w \in B(y,\psi,n(\rho)) \cap \Lambda^3} B(z(w), \rho) \cap W^u_{\text{loc}}(w)\Big).$$



The θ-Hölder property of $S_{\chi,l}(x)$ implies that the projection along $S_{\chi,l}(x)$ of $\bigcup_{w \in B(y,\psi,n(\rho)) \cap \Lambda^3} B(z(w), \rho) \cap W^u_{\text{loc}}(w)$ is contained in $B(z(y), K\rho^\theta) \cap W^u_{\text{loc}}(x)$, when $z(y) \in W^u_{\text{loc}}(x) \cap W^s_{\text{loc}}(y)$. Thus, by the definition of $\tilde{\nu}_x$

$$\nu(B(y, \psi, n(\rho)) \le \tilde{\nu}_x(B(z(y), K\rho^\theta).$$

Therefore

$$-\frac{1}{n(\rho)} \log \nu(B(y, \psi, n(\rho)) \ge (N(\chi_\mu^+ - \alpha) - \alpha - 4CN\sqrt{\alpha}) \frac{\log \tilde{\nu}_x(B(z(y)), K\rho^\theta))}{\log \rho}$$

and by propositions 1.3 and 1.4 it follows that

$$h_\mu(f) \ge \delta(\tilde{\nu}_x)(\chi_\mu^+ - \alpha - \alpha/N - 4C\sqrt{\alpha})\theta.$$

Now observe that $\Lambda^3 = \Lambda^3(\alpha)$ and $\tilde{\nu}_x = \tilde{\nu}_x(\alpha)$, and as $\alpha \to 0$ we have $\mu(\Lambda^3(\alpha)) \to \mu(\Lambda_{\chi,l})$ and $\delta(\tilde{\nu}_x(\alpha)) \to \delta(\tilde{\mu}_x)$, whence it follows, since $0 < \theta < 1$ is arbitrary, that

$$h_\mu(f) \ge \delta(\tilde{\mu}_x)\chi_\mu^+.$$ □

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