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POTENTIAL OPERATORS AND MULTIPLIERS ON LOCALLY COMPACT VILENKIN GROUPS

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Dedicated to Professor Satoru Igari on his 60th birthday

We study, under the setting of a locally compact Vilenkin group G, a weighted norm inequality for the potential operators of Riesz type and its applications to multipliers on G. We also consider the maximal operators of fractional type.

1. INTRODUCTION

In [**3**] we have given a characterisation of a two weights norm inequality for Riesz potential (fractional integral) operators defined on a locally compact Vilenkin group G, and, as a consequence, deduced a multiplier theorem of Hörmander type between power-weighted Hardy spaces on G. In this paper we shall continue to study the same subjects for weighted Lebesgue spaces on G. We shall consider a class of potential operators which includes the Riesz potential operator, Bessel potential operators and so on. Our main result for potential operators is Theorem 1. By combining this result with a multiplier theorem of the present author ([**1**, Theorem 1] or [**4**, Theorem 3.6]), we shall prove a multiplier theorem of Hörmander type between weighted Lebesgue spaces (see Theorem 2.) This result is considered to be an extension to G of multiplier theorems on weighted Lebesgue spaces on $\mathbf{R}^n$ due to Kurtz [**5**, Theorem 4.4] or Vinogradova [**10**, Theorem].

Throughout this paper G will denote a locally compact Vilenkin group, that is to say, G is a locally compact Abelian topological group containing a strictly decreasing sequence of compact open subgroups $(G_n)_{-\infty}^{\infty}$ such that

(i) $\bigcup_{-\infty}^{\infty} G_n = G$ and $\bigcap_{-\infty}^{\infty} G_n = \{0\}$,

(ii) $\sup\{\text{ order } (G_n/G_{n+1}) : n \in \mathbf{Z}\} := B < \infty$.

Examples of such groups are described in [**1**, Section 4.1.2]. Additional examples are given by the additive group of a local field (see [**9**]).

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Let Γ be the dual group of G and let Γ_n be the annihilator of G_n for each $n \in \mathbf{Z}$. Then $(\Gamma_n)_{-\infty}^{\infty}$ is a strictly increasing sequence of compact open subgroups of Γ such that

(i)' $\bigcup_{-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap_{-\infty}^{\infty} \Gamma_n = \{1\}$, and

(ii)' order $(\Gamma_{n+1}/\Gamma_n) =$ order (G_n/G_{n+1}).

We choose Haar measures dx on G and $d\gamma$ on Γ so that $|G_0| = |\Gamma_0| = 1$, where $|A|$ denotes the Haar measure of a measurable subset A of G or Γ. Then $|G_n|^{-1} = |\Gamma_n| := m_n$ for each $n \in \mathbf{Z}$. For $x \in G$, we set $|x| = (m_n)^{-1}$ if $x \in G_n \setminus G_{n+1}$ and $|x| = 0$ if $x = 0$. Similarly, we set $|\gamma| = m_{n+1}$ if $\gamma \in \Gamma_{n+1} \setminus \Gamma_n$ and $|\gamma| = 0$ if $\gamma = 1$. Since $2m_n \leqslant m_{n+1}$ for each $n \in \mathbf{Z}$, it follows that $\sum_{n=k}^{\infty} (m_n)^{-\alpha} \leqslant C(m_k)^{-\alpha}$ and $\sum_{n=-\infty}^{k} (m_n)^{\alpha} \leqslant C(m_k)^{\alpha}$ for any $\alpha > 0, k \in \mathbf{Z}$.

The symbols $\wedge$ and $\vee$ will denote the Fourier transform and inverse Fourier transform, respectively. We have $(\xi_{G_n})^{\wedge} = |\Gamma_n|^{-1} \xi_{\Gamma_n} := F_n$ and, hence, $(\xi_{\Gamma_n})^{\vee} = |G_n|^{-1} \xi_{G_n} := \Delta_n$ for each $n \in \mathbf{Z}$, where ξ_A denote the indicator function of a set A. A function f on G is said to be radial if f is constant on each set $G_n \setminus G_{n+1}$, $n \in \mathbf{Z}$, and said to be quasi-radial if f is constant on cosets of G_{n+k} in $G_n \setminus G_{n+1}$ for some integer k, $k \geqslant 1$, and all $n \in \mathbf{Z}$. Radial or quasi-radial functions on Γ are defined similarly.

We define $\mathcal{S}$ to be the set of all functions φ on G such that φ has compact support and is constant on the cosets of some G_n, $n \in \mathbf{Z}$. $\mathcal{S}$ is the space of testing functions for distributions on G (for details, see [9].) We set $\mathcal{S}_0 = \{f \in \mathcal{S} : \int_G f(x)dx = 0\}$. Cosets in G will be called intervals. Throughout this paper, I will be used to denote intervals in G.

Let $\omega(x)$ be a nonnegative locally integrable function on G. The Lebesgue space on G with respect to the weight measure $\omega(x)dx$ will be denoted by $L^p(\omega)$, $0 < p < \infty$. Weighted spaces $L^p(\omega)$ will be equipped with the norm $\|f\|_{p,\omega} = \left(\int_G |f(x)|^p \omega(x)dx\right)^{1/p}$. We denote $\omega(A) = \int_A \omega(x)dx$.

We say that ω satisfies the doubling condition if $\omega(I') \leqslant C\omega(I)$ for all $I = x + G_n$, $I' = x + G_{n-1}$, $x \in G$, $n \in \mathbf{Z}$.

We say that ω belongs to the class A_p $(\omega \in A_p)$, $1 \leqslant p < \infty$, if

$$\frac{1}{|I|}\int_I \omega(x)dx \left(\frac{1}{|I|}\int_I \omega(x)^{-1/(p-1)}dx\right)^{p-1} \leqslant C,$$

for all I. When $p = 1$, this should be interpreted as

$$\frac{1}{|I|}\int_I \omega(x)dx \leqslant C \text{ ess inf } \{\omega(x) : x \in I\}.$$

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We define $A_\infty = \bigcup_{p<\infty} A_p$. If $1 < p < \infty$ then $\omega \in A_p$ if and only if $\omega^{-1/(p-1)} \in A_{p'}$, $1/p + 1/p' = 1$. We denote by v_α the power weights $|x|^\alpha$, $\alpha \in \mathbf{R}$. Note that $v_\alpha \in A_p$, $1 < p < \infty$ if and only if $-1 < \alpha < p - 1$, also that $v_\alpha \in A_1$ if and only if $-1 < \alpha \leqslant 0$.

Let $M_{(\omega)}$ be the weighted Hardy-Littlewood maximal operator defined by

$$M_{(\omega)}f(x) = \sup_{I \ni x} \frac{1}{\omega(I)} \int_I |f(y)|\,\omega(y)dy.$$

When $\omega \equiv 1$, this is the usual Hardy-Littlewood maximal operator M.

If ω is doubling and $1 < p < \infty$, then $M_{(\omega)}$ is type (p,p) on $L^p(\omega)$. If $\omega \in A_p$, $1 < p < \infty$, then M is type (p,p) on $L^p(\omega)$.

We start with some simple lemmas.

LEMMA 1. *If $\omega \in A_p$, $1 \leqslant p < \infty$, then there is a constant C such that*

$$\frac{\omega(I)}{\omega(E)} \leqslant C\left(\frac{|I|}{|E|}\right)^p,$$

for all interval I and each measurable set $E \subset I$.

PROOF: Let E be any subset of I. When $1 < p < \infty$, the conclusion follows from an application of Hölder's inequality to the expression $\int_E \omega(x)^{1/p}\omega(x)^{-1/p}dx$. When $p = 1$, we have

$$\begin{aligned}|E| = \int_E \omega(x)\omega(x)^{-1}dx &\leqslant \int_E \omega(x)dx \sup\{\omega(x)^{-1} : x \in E\}\\ &\leqslant \omega(E)\left(\inf\{\omega(x) : x \in I\}\right)^{-1}\\ &\leqslant C\omega(E)\,|I|\,\omega(I)^{-1}.\end{aligned}$$

That is,

$$\frac{\omega(I)}{\omega(E)} \leqslant C\left(\frac{|I|}{|E|}\right). \qquad \square$$

From Lemma 1, we see that if $\omega \in A_\infty$ then ω is doubling, and there exist $\varepsilon, \delta > 0$ so that $\omega(I) \leqslant \delta\omega(E)$ whenever $|I| \leqslant \varepsilon\,|E|$.

LEMMA 2. [3, Lemma 4] *Let $\alpha > 0$, $0 < p,q < \infty$ and $\beta, \beta' > -1$. Then there is a constant C such that*

$$|I|^\alpha v_{\beta'}(I)^{1/q} \leqslant C v_\beta(I)^{1/p} \quad \textit{for all } I,$$

if and only if

$$\frac{\beta}{p} - \frac{\beta'}{q} = -\frac{1}{p} + \frac{1}{q} + \alpha \geqslant 0.$$

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2. Potential operators and Multipliers

Let Φ be a nonnegative locally integrable function on G. We define the potential operator $T = T_\Phi$ by

$$Tf(x) = T_\Phi f(x) = \int_G \Phi(x-y)f(y)dy.$$

Basic examples are provided by Riesz potential operators I_α with kernels $\Phi(x) = |x|^{\alpha-1}$, $0 < \alpha < 1$, and Bessel potential operators with kernels defined by means of its Fourier transform, $\widehat{\Phi}(\gamma) = (\max(1,|\gamma|))^{-\beta}$, $\beta > 0$. Both of these kernels are radial and decreasing. However, we here only assume the following growth condition on Φ:

There is a constant C so that

$$\text{(D)} \qquad \sup\{\Phi(x) : x \in G_n \setminus G_{n+1}\} \leqslant \frac{C}{|G_n|}\int_{G_n\setminus G_{n+1}} \Phi(x)dx \quad \text{for all } n \in \mathbf{Z}$$

(see [7]). Condition (D) is very general since radial functions are included.

For simplicity of notation, we set

$$\widetilde{\Phi}(t) = \int_{|x|\leqslant t} \Phi(x)dx, \quad t > 0.$$

Theorem 1. *Let $1 < p \leqslant q < \infty$, $w \in A_\infty$ and $v \in A_p$. Then the following statements are equivalent.*

(1) *$T : L^p(v) \longrightarrow L^q(w)$, bounded*

(2) *there is a constant C so that*

$$\text{(2.1)} \qquad \widetilde{\Phi}(|I|)\,|I|^{-1}\,w(I)^{1/q}\sigma(I)^{1/p'} \leqslant C \quad \textit{for all } I,$$

where $\sigma = v^{-1/(p-1)}$.

Remark. The assumption $v \in A_p$ is required for proving that condition (1) implies condition (2). For the proof of the converse implication, it is enough to assume that $\sigma \in A_\infty$.

Proof: (1) $\Rightarrow$ (2) : By testing condition (1) with $f = \xi_I$, we have

$$\widetilde{\Phi}(|I|)w(I)^{1/q} \leqslant Cv(I)^{1/p}.$$

Since $v \in A_p$, we have $|I|^{-p}\,w(I)\sigma(I)^{p-1} \leqslant C$. Hence

$$\widetilde{\Phi}(|I|)\,|I|^{-1}\,w(I)^{1/q}\sigma(I)^{1/p'} \leqslant C.$$

(2) $\Rightarrow$ (1) : Since T is a positive operator and the space of bounded functions with compact support, L_c^∞, is dense in $L^p(v)$, it is enough to prove that there is a constant C so that

$$\left(\int_G (Tf(x))^q\, w(x)dx\right)^{1/q} \leqslant C\left(\int_G (f(x))^p\, v(x)dx\right)^{1/p}$$

for any nonnegative $f \in L_c^\infty$. By duality, this inequality is equivalent to

$$\int_G Tf(x)g(x)w(x)dx \leqslant C\left(\int_G (f(x))^p\, v(x)dx\right)^{1/p}\left(\int_G (g(x))^{q'}\, w(x)dx\right)^{1/q'}$$

for any nonnegative $f, g \in L_c^\infty$.

Associated to any interval I ($I = x_0 + G_n$) we denote by Φ_I the value $\sup\{\Phi(y) : y \in G_n \setminus G_{n+1}\}$. Since $G = \bigcup_{-\infty}^{\infty} x + G_n \setminus x + G_{n+1}$ for any $x \in G$, we have

$$\begin{aligned} Tf(x) &= \sum_{n\in\mathbf{Z}} \int_{x+G_n\setminus x+G_{n+1}} \Phi(x-y)f(y)dy \\ &\leqslant \sum_{n\in\mathbf{Z}} \sup\{\Phi(y) : y \in G_n \setminus G_{n+1}\}\int_{x+G_n} f(y)dy \\ &= \sum_I \Phi_I \int_I f(y)dy\,\xi_I(x), \end{aligned}$$

where the sum $\sum_I$ is taken over all intervals I in G. Then

$$\begin{aligned} \int_G Tf(x)g(x)w(x)dx &\leqslant \int_G \sum_I \Phi_I \int_I f(y)dy\,\xi_I(x)g(x)w(x)dx \\ &= \sum_I \Phi_I \int_I f(y)dy \int_I g(x)w(x)dx. \end{aligned}$$

We shall replace the sum over all intervals by some "maximal" intervals. To do this, we let fix a constant $a > B$, and define

$$\Omega_k = \left\{I : \frac{1}{|I|}\int_I g(x)w(x)dx > a^k\right\}, \quad k \in \mathbf{Z}.$$

Since $g \in L_c^\infty$, $|I|^{-1}\int_I gw \to 0$ as $I \uparrow G$. This implies that if I is any element of Ω_k, then I is contained in an interval in Ω_k which is maximal with respect to inclusion. For each $k \in \mathbf{Z}$, let $\{I_{k,j}\}_j$ be a family of the maximal intervals in Ω_k. Then, the $I_{k,j}$ are disjoint in j for fixed k. Furthermore,

$$a^k < \frac{1}{|I_{k,j}|}\int_{I_{k,j}} g(x)w(x)dx \leqslant Ba^k,$$

where the second inequality can be seen as follows. If $I_{k,j} = x_0 + G_n$, we set $I'_{k,j} = x_0 + G_{n-1}$. Then by the maximality of $I_{k,j}$,

$$\frac{1}{|I_{k,j}|}\int_{I_{k,j}} g(x)w(x)dx \leqslant \frac{|I'_{k,j}|}{|I_{k,j}|}\left(\frac{1}{|I'_{k,j}|}\int_{I'_{k,j}} g(x)w(x)dx\right) \leqslant Ba^k.$$

We adapt now some ideas from [8] and [7]. We set $\mathcal{C}^k := \Omega_k \setminus \Omega_{k+1},\ k \in \mathbf{Z}$. It is easily seen that

(i) If $gw \not\equiv 0$ on I, there is a unique $k \in \mathbf{Z}$ such that $I \in \mathcal{C}^k$,
(ii) $I_{k,j} \in \mathcal{C}^k$ for all j,
(iii) If $I \in \mathcal{C}^k$, there exists j such that $I \subset I_{k,j}$,
(iv) If $I \in \mathcal{C}^k$ then

$$\frac{1}{|I|}\int_I g(x)w(x)dx \leqslant \frac{a}{|I_{k,j}|}\int_{I_{k,j}} g(x)w(x)dx \text{ for any } j.$$

Using these properties, we have

$$\begin{aligned}
(2.3)\qquad \int_G Tf(x)g(x)w(x)dx &\leqslant \sum_I \left(\Phi_I |I| \int_I f(x)dx\right)\left(\frac{1}{|I|}\int_I g(x)w(x)dx\right) \\
&= \sum_{k\in\mathbf{Z}}\sum_{I\in\mathcal{C}^k} \left(\Phi_I |I| \int_I f(x)dx\right)\left(\frac{1}{|I|}\int_I g(x)w(x)dx\right) \\
&= \sum_k\sum_j\sum_{I\subset I_{k,j}} \left(\Phi_I |I| \int_I f(x)dx\right)\left(\frac{1}{|I|}\int_I g(x)w(x)dx\right) \\
&\leqslant a\sum_k\sum_j\left(\sum_{I\subset I_{k,j}} \Phi_I |I| \int_I f(x)dx\right)\left(\frac{1}{|I_{k,j}|}\int_{I_{k,j}} g(x)w(x)dx\right).
\end{aligned}$$

We now claim that

$$\sum_{I\subset I_{k,j}} \Phi_I |I| \int_I f(x)dx \leqslant C\widetilde{\Phi}(|I_{k,j}|)\int_{I_{k,j}} f(x)dx.$$

In fact, if J is any interval then

$$\begin{aligned}
\sum_{I\subset J} \Phi_I |I| \int_I f(x)dx &= \sum_{k=n}^{\infty}\left(\sum_{I\subset J, |I|=m_k^{-1}} \Phi_I |I| \int_I f(x)dx\right) \\
&= \sum_{k=n}^{\infty} |G_k| \sup\{\Phi(y) : y \in G_k \setminus G_{k+1}\}\int_J f(x)dx \\
&\leqslant C\sum_{k=n}^{\infty}\int_{G_k\setminus G_{k+1}} \Phi(x)dx \int_J f(x)dx \\
&= C\int_{G_n} \Phi(x)dx \int_J f(x)dx = C\widetilde{\Phi}(|J|)\int_J f(x)dx,
\end{aligned}$$

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where the first inequality follows from condition (D) for Φ.

Hence, the right side of (2.3) is dominated by

$$
\begin{aligned}
&C\sum_k\sum_j\left(\widetilde{\Phi}(|I_{k,j}|)\int_{I_{k,j}}f(x)dx\right)\left(\frac{1}{|I_{k,j}|}\int_{I_{k,j}}g(x)w(x)dx\right)\\
&\leqslant C\sum_{k,j}\left[\sigma(I_{k,j})^{1/p}\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}}f(x)dx\right)\right]\left[w(I_{k,j})^{1/q'}\left(\frac{1}{w(I_{k,j})}\int_{I_{k,j}}g(x)w(x)dx\right)\right]\\
&\leqslant C\left[\sum_{k,j}\sigma(I_{k,j})\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}}f(x)dx\right)^{p}\right]^{1/p}\left[\sum_{k,j}w(I_{k,j})^{p'/q'}\left(\frac{1}{w(I_{k,j})}\int_{I_{k,j}}g(x)w(x)dx\right)^{p'}\right]^{1/p'}\\
&\leqslant C\left[\sum_{k,j}\sigma(I_{k,j})\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}}f(x)dx\right)^{p}\right]^{1/p}\left[\sum_{k,j}w(I_{k,j})\left(\frac{1}{w(I_{k,j})}\int_{I_{k,j}}g(x)w(x)dx\right)^{q'}\right]^{1/q'},
\end{aligned}
$$

where the first inequality follows from (2.1) and the last inequality follows from $p \leqslant q$.

We next set $D_k = \bigcup_j I_{k,j}$ and

$$E_{k,j} = I_{k,j} \setminus (I_{k,j} \cap D_{k+1}), \quad k \in \mathbf{Z}.$$

Then, $\{E_{k,j}\}_{k,j}$ is a disjoint family and

$$|I_{k,j} \cap D_{k+1}| < \frac{B}{a}|I_{k,j}| \tag{2.4}$$

$$|I_{k,j}| < \frac{a}{a-B}|E_{k,j}|. \tag{2.5}$$

We shall show (2.4), from which (2.5) is readily reduced.

$$
\begin{aligned}
|I_{k,j} \cap D_{k+1}| &= \sum_i |I_{k,j} \cap I_{k+1,i}| = \sum_{I_{k+1,i}\subset I_{k,j}} |I_{k+1,i}|\\
&\leqslant \sum_{I_{k+1,i}\subset I_{k,j}} \frac{1}{a^{k+1}}\int_{I_{k+1,i}} g(x)w(x)dx\\
&\leqslant \frac{1}{a^{k+1}}\int_{I_{k,j}} g(x)w(x)dx\\
&\leqslant \frac{1}{a^{k+1}}Ba^k|I_{k,j}| = \frac{B}{a}|I_{k,j}|.
\end{aligned}
$$

Now, since $\sigma \in A_\infty$, applying Lemma 1 to (2.5) yields

$$\sigma(I_{k,j}) \leqslant C\sigma(E_{k,j}).$$

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Hence, we have

$$\begin{aligned}\sum_{k,j}\sigma(I_{k,j})\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}}f(x)dx\right)^p &= \sum_{k,j}\sigma(I_{k,j})\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}}(f\sigma^{-1})(x)\sigma(x)dx\right)^p\\ &\leqslant C\sum_{k,j}\sigma(E_{k,j})\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}}(f\sigma^{-1})(x)\sigma(x)dx\right)^p\\ &\leqslant C\sum_{k,j}\int_{E_{k,j}}\left(M_{(\sigma)}(f\sigma^{-1})(x)\right)^p\sigma(x)dx\\ &\leqslant C\left\|M_{(\sigma)}(f\sigma^{-1})\right\|_{p,\sigma}^p\\ &\leqslant C\left\|f\sigma^{-1}\right\|_{p,\sigma}^p = C\|f\|_{p,v}^p.\end{aligned}$$

Similarly, we have, from $w \in A_\infty$

$$\begin{aligned}\sum_{k,j}w(I_{k,j})\left(\frac{1}{w(I_{k,j})}\int_{I_{k,j}}g(x)w(x)dx\right)^{q'} &\leqslant C\sum_{k,j}w(E_{k,j})\left(\frac{1}{w(I_{k,j})}\int_{I_{k,j}}g(x)w(x)dx\right)^{q'}\\ &\leqslant C\left\|M_{(w)}g\right\|_{q',w}^{q'} \leqslant C\|g\|_{q',w}^{q'}.\end{aligned}$$

Consequently, the right side of (2.3) is dominated by

$$C\|f\|_{p,v}\|g\|_{q',w} = C\left(\int_G (f(x))^p v(x)dx\right)^{1/p}\left(\int_G (g(x))^{q'} w(x)dx\right)^{1/q'}.$$

This completes the proof of (2.2). ∎

As a corollary of Theorem 1, we get a characterisation for I_α.

Corollary 1. *Let $0 < \alpha < 1$, $1 < p \leqslant q < \infty$ and $w, \sigma \in A_\infty$. Then the following statements are equivalent.*

(1) *$I_\alpha : L^p(v) \longrightarrow L^q(w)$, bounded*
(2) *there is a constant C so that*

$$|I|^{\alpha-1}w(I)^{1/q}\sigma(I)^{1/p'} \leqslant C \quad \textit{for all } I.$$

Proof: (1) $\Rightarrow$ (2) : If $f = \sigma\xi_I$,

$$I_\alpha f(x) = \int_I \frac{\sigma(y)}{|x-y|^{1-\alpha}}dy \geqslant |I|^{\alpha-1}\sigma(I) \quad \text{for } x \in I.$$

Hence,

$$|I|^{\alpha-1}\sigma(I)w(I)^{1/q} \leqslant \|I_\alpha f\|_{q,w} \leqslant C\,\|f\|_{p,v} = C\sigma(I)^{1/p},$$

which yields condition (2).

(2) $\Rightarrow$ (1) : Since the kernel of I_α satisfies condition (D) and $\widetilde{\Phi}(|I|) \sim |I|^\alpha$, condition (1) follows immediately from Theorem 1 (see Remark). ∎

In the last part of this section, we give a multiplier theorem of Hörmander type between two power-weighted Lebesgue spaces on G. In [**3**] we have given the same type of multiplier theorem for power-weighted Hardy spaces on G.

The generalised Hörmander classes of multipliers, $M(s,\lambda,\alpha)$ are defined as follows (see [**3**]). Let $\lambda > 0$, $1 \leqslant s \leqslant \infty$ and $\alpha \in \mathbf{R}$. For a function φ on Γ, we set $\varphi_j = \varphi\xi_{\Gamma_{j+1}\setminus\Gamma_j}$, $j \in \mathbf{Z}$. A function φ on Γ belongs to $M(s,\lambda,\alpha)$ if there is a constant C such that

$$|\varphi(\gamma)| \leqslant C\,|\gamma|^{-\alpha} \text{ and } \sup_{j\in\mathbf{Z}}\left\{(m_j)^{\lambda-1/s+\alpha}\left\|D^\lambda\varphi_j\right\|_s\right\} < \infty,$$

where D^λ is the fractional differential operator defined by $D^\lambda\varphi_j = \left(|x|^\lambda(\varphi_j)^\vee\right)^\wedge$.

$M(s,\lambda,0)$ coincides with the $M(s,\lambda)$ that was introduced in [**2**] and [**4**]. We note that if $\varphi \in L^\infty(\Gamma)$ is radial, or more generally quasi-radial, then $\varphi \in M(s,\lambda)$ for $1 \leqslant s \leqslant \infty$, $\lambda > 0$.

It is easily seen that $\varphi \in M(s,\lambda,\alpha)$ if and only if $\varphi(\gamma)\,|\gamma|^\alpha \in M(s,\lambda)$. Furthermore, $\varphi \in M(s,\lambda,\alpha)$ if and only if $\varphi(\gamma)/\widehat{k_\alpha}(\gamma) \in M(s,\lambda)$, where k_α is the kernel of I_α. This follows from the fact that the Fourier transform $\widehat{k_\alpha}$ of k_α is, in the distributional sense, a radial function on Γ and $\widehat{k_\alpha}(\gamma) \sim |\gamma|^{-\alpha}$ (see [**3**, Lemma 5]).

Theorem 2. *Let $0 < \alpha < 1$ and $1 < p \leqslant q < \infty$. Suppose that $\varphi \in M(s,\lambda,\alpha)$ for $1 \leqslant s \leqslant \infty$, $\lambda > \max(1/s, 1/2)$. Then*

$$\left\|\left(\varphi\hat{f}\right)^\vee\right\|_{q,v_{\beta'}} \leqslant C\,\|f\|_{p,v_\beta} \quad \text{for all } f \in \mathcal{S}_0,$$

if $-1 < \beta'$, $\max(-1,-p\lambda) < \beta < \min(p-1, p\lambda)$ and

$$\frac{\beta+1}{p} = \frac{\beta'+1}{q} + \alpha,\ 0 \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \alpha.$$

Proof: Let $\varphi_0(\gamma) = \varphi(\gamma)/\widehat{k_\alpha}(\gamma)$. Then $\varphi_0 \in M(s,\lambda)$ and $\varphi(\gamma) = \widehat{k_\alpha}(\gamma)\varphi_0(\gamma)$. By [**2**, Theorem 1] or [**4**, Theorem 3.6], we see $\varphi_0 \in \mathcal{M}(L^p(v_\beta))$. Since $v_\beta \in A_p$, we have, by Lemma 2,

$$|I|^{\alpha-1}v_{\beta'}(I)^{1/q}v_{-\beta/(p-1)}(I)^{1/p'} \leqslant C \quad \text{for all } I.$$

Hence, by Corollary 1,

$$\left\| \left(\varphi \hat{f}\right)^{\vee} \right\|_{q,v_{\beta'}} = \|I_\alpha g\|_{q,v_{\beta'}} \leqslant C\,\|g\|_{p,v_\beta} \leqslant C\,\|f\|_{p,v_\beta},$$

where $g = \left(\varphi_0 \hat{f}\right)^{\vee}$. Since $\mathcal{S}_0$ is dense in $L^p(v)$, (2.6) has a continuous extension to all of $L^p(v)$. This completes the proof of Theorem 2. ∎

3. Maximal operators

It is well known that the fractional integral operators I_α are closely related to the fractional maximal operators M_α defined by

$$M_\alpha f(x) = \sup_{x\in I} \frac{1}{|I|^{1-\alpha}} \int_I |f(y)|\,dy.$$

In this section we consider the corresponding maximal operators to the potential operators T_Φ discussed in the previous section. We define

$$M_\varphi f(x) = \sup_{x\in I} \frac{\varphi(|I|)}{|I|} \int_I |f(y)|\,dy,$$

where φ is a positive function defined on $(0,\infty)$. If $\varphi(t) = t^\alpha$, $\alpha > 0$ then M_φ is the fractional maximal operator M_α.

Here we assume that φ is essentially nondecreasing; that is, there is a constant ρ for which

$$\varphi(t) \leqslant \rho\varphi(s), \quad t \leqslant s \tag{3.1}$$

and

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0. \tag{3.2}$$

Notice that $\varphi(t) = t^\alpha$, $0 < \alpha < 1$, satisfies both of the above conditions.

The following theorem is our main result for M_φ.

Theorem 3. *Let $1 < p \leqslant q < \infty$ and $\sigma \in A_\infty$. Then the following statements are equivalent.*

(1) $M_\varphi : L^p(v) \longrightarrow L^q(w)$, *bounded,*
(2) *there is a constant C so that*

$$\varphi(|I|)\,|I|^{-1}\,w(I)^{1/q}\sigma(I)^{1/p'} \leqslant C \quad \textit{for all } I.$$

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PROOF: (1) $\Rightarrow$ (2) : As in the proof of Corollary 1, condition (2) follows from testing condition (1) with $f = v^{-1/(p-1)}\xi_I$.

(2) $\Rightarrow$ (1) : It is enough to show that there is a constant C such that

$$\left(\int_G (M_\varphi f(x))^q\, w(x)dx\right)^{1/q} \leqslant C\left(\int_G (f(x))^p\, v(x)dx\right)^{1/p} \tag{3.3}$$

for all nonnegative $f \in L_c^\infty$.

Fix a constant $a > B\rho$ and define

$$D_k = \{x \in G : M_\varphi f(x) > a^k\}, \quad k \in \mathbf{Z}.$$

Due to growth condition (3.2) for φ,

$$\frac{\varphi(|I|)}{|I|}\int_I f(y)dy \to 0 \quad \text{as } I \uparrow G.$$

Then, as in the proof of Theorem 1, we see that there is a family of maximal disjoint intervals $\{I_{k,j}\}_j$ such that $D_k = \bigcup_j I_{k,j}$ and furthermore,

$$a^k < \frac{\varphi(|I_{k,j}|)}{|I_{k,j}|}\int_{I_{k,j}} f(y)dy \leqslant B\rho a^k, \tag{3.4}$$

where the second inequality follows from (3.1) and the maximality of $I_{k,j}$. Hence,

$$\begin{aligned}\int_G (M_\varphi f(x))^q\, w(x)dx &= \sum_{k\in\mathbf{Z}}\int_{D_k\setminus D_{k+1}} (M_\varphi f(x))^q\, w(x)dx \\ &\leqslant \sum_k a^{(k+1)q} w(D_k \setminus D_{k+1}) \\ &\leqslant a^q \sum_{k,j}\left(\frac{\varphi(|I_{k,j}|)}{|I_{k,j}|}\int_{I_{k,j}} f(x)dx\right)^q w(I_{k,j}) \\ &\leqslant Ca^q \sum_{k,j}\left[\sigma(I_{k,j})^{1/p}\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}} f(x)dx\right)\right]^q.\end{aligned}$$

Since $q/p \geqslant 1$, the right side of the above inequality is dominated by

$$\begin{aligned}&Ca^q\left(\sum_{k,j}\left[\sigma(I_{k,j})^{1/p}\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}} f(x)dx\right)\right]^p\right)^{q/p} \\ &\qquad = Ca^q\left(\sum_{k,j}\sigma(I_{k,j})\left(\frac{1}{\sigma(I_{k,j})}\int_{I_{k,j}} (f\sigma^{-1})(x)\sigma(x)dx\right)^p\right)^{q/p}.\end{aligned}$$

Estimation of the above expression proceeds as in the proof of Theorem 1. We give an outline.

We set $E_{k,j} = I_{k,j} \setminus (I_{k,j} \cap D_{k+1})$. Then, $\{E_{k,j}\}_{k,j}$ is a disjoint family and by virtue of (3.4),

$$|I_{k,j}| < \frac{a}{a - B\rho} |E_{k,j}| \tag{3.5}$$

holds. Since $\sigma \in A_\infty$, we have $\sigma(I_{k,j}) \leqslant C\sigma(E_{k,j})$ by Lemma 1 and

$$\begin{aligned}
\sum_{k,j} \sigma(I_{k,j}) \left(\frac{1}{\sigma(I_{k,j})} \int_{I_{k,j}} (f\sigma^{-1})(x)\sigma(x)dx \right)^p
&\leqslant C \sum_{k,j} \sigma(E_{k,j}) \left(\frac{1}{\sigma(I_{k,j})} \int_{I_{k,j}} (f\sigma^{-1})(x)\sigma(x)dx \right)^p \\
&\leqslant C \int_G \left(M_{(\sigma)}(f\sigma^{-1})(x)\right)^p \sigma(x)dx \\
&\leqslant C \left\|f\sigma^{-1}\right\|_{p,\sigma}^p = C \left\|f\right\|_{p,v}^p ,
\end{aligned}$$

which concludes the proof of (3.3). □

Corollary 1 and Theorem 3 yield the following.

Corollary 2. *Let $0 < \alpha < 1$, $1 < p \leqslant q < \infty$ and $w, \sigma \in A_\infty$. Then the following statements are equivalent.*

(1) $I_\alpha : L^p(v) \longrightarrow L^q(w)$, *bounded,*
(2) $M_\alpha : L^p(v) \longrightarrow L^q(w)$, *bounded,*
(3) *there is a constant C so that*

$$|I|^{\alpha-1} w(I)^{1/q} \sigma(I)^{1/p'} \leqslant C \quad \text{for all } I.$$

We conclude this paper with a remark. As in the classical case, using good-λ inequality arguments and a pointwise estimate $M_\alpha f(x) \leqslant C I_\alpha f(x)$, we can prove that if $0 < p < \infty$, $0 < \alpha < 1$ and $w \in A_\infty$ then $\|I_\alpha f\|_{p,w} \sim \|M_\alpha f\|_{p,w}$. This equivalence together with Theorem 3 also implies Corollary 2 (see [**6**].)

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