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GROUPS WITH A QUOTIENT THAT CONTAINS THE ORIGINAL GROUP AS A DIRECT FACTOR

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We prove that given a finitely generated group G with a homomorphism of G onto $G \times H$, H non-trivial, or a finitely generated group G with a homomorphism of G onto $G \times G$, we can always find normal subgroups $N \neq G$ such that $G/N \cong G/N \times H$ or $G/N \cong G/N \times G/N$ respectively. We also show that given a finitely presented non-Hopfian group U and a homomorphism φ of U onto U, which is not an isomorphism, we can always find a finitely presented group $H \supseteq U$ and a finitely generated free group F such that φ induces a homomorphism of $U * F$ onto $(U * F) \times H$. Together with the results above this allows the construction of many examples of finitely generated groups G with $G \cong G \times H$ where H is finitely presented. A finitely presented group G with a homomorphism of G onto $G \times G$ was first constructed by Baumslag and Miller. We use a slight generalisation of their method to obtain more examples of such groups.

1. INTRODUCTION

Finitely generated groups G with $G \cong G \times H$, $H \neq 1$, or $G \cong G \times G$ were first constructed by Tyrer Jones [4], but it is still an open question whether there exist non-trivial finitely presented groups G with these properties. More insight into the structure and a new construction of finitely generated groups with the above properties are therefore still useful. The starting point of our investigations was the construction of a finitely presented group G with a homomorphism of G onto $G \times G$ by Baumslag and Miller [1].

Our paper consists of four sections. The main results are the following theorems which are proved in Sections 3 and 5 respectively. Theorem 2: A group G with a homomorphism of G onto $G \times H$ always contains a normal subgroup N such that $G/N \cong G/N \times H$. Theorem 4: A non-trivial group G with a homomorphism from G onto $G \times G$ always contains a normal subgroup $N \neq G$ such that $G/N \cong G/N \times G/N$.

Since in Sections 3 and 5 we need finitely presented groups G with homomorphisms of G onto $G \times H$ or of G onto $G \times G$, we study the construction of such groups in

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Sections 2 and 4. In Section 2 we use ideas contained in [**3**] to give a fairly general construction for groups G with a homomorphism of G onto $G \times H$. More precisely, we show that given a finitely presented non-Hopfian group U and a homomorphism φ of U onto U, which is not an isomorphism, we can always find a finitely presented group $H \supseteq U$ and a finitely generated free group F such that φ induces a homomorphism ϑ of $U * F$ onto $(U * F) \times H$. We use this to give an example of a one-relator group G with a homomorphism of G onto $G \times H$, $H \neq 1$. Together with Theorem 2 this allows the construction of many examples of finitely generated groups G with $G \cong G \times H$ where H is finitely presented. The construction of groups G with a homomorphism of G onto $G \times G$ given in Section 4 is just a slight generalisation, again based on ideas in [**3**], of the Baumslag-Miller group [**1**]. We start with a finitely generated abelian group A and a one-one homomorphism μ of A to A satisfying some additional conditions and construct a finitely presented group $G \supseteq A$ and a homomorphism ϑ from G onto $G \times G$ induced by μ.

2. Finitely presented groups G with a homomorphism from G onto $G \times H$

The following lemma is needed in the proof of Theorem 1.

Lemma 1. *Let U be finitely presented and $1 \neq x \in U$. Then there exists a finitely presented group $H \supseteq U$, such that x generates H as a normal subgroup.*

A proof can, for example, be modelled after the proof of Theorem 3.5 of [**2**], p.190.

Theorem 1. *Let U be finitely presented and non-Hopfian and let φ be a homomorphism of U onto U with non-trivial kernel. Then there exists a finitely presented group $H \supseteq U$ and a finitely presented group V such that φ induces a homomorphism ϑ of $V * U$ onto $(V * U) \times H$.*

Proof: Let $1 \neq x$ be an element in the kernel of φ. Form the group H as in Lemma 1 and let $\mu\colon U \to H$ be the embedding. Let V be any finitely presented group for which there exists a homomorphism ρ of V onto H. The map ϑ of $V * U$ onto $(V * U) \times H$ is given by

$$\vartheta\colon v \to (v,\, v\rho) \text{ for } v \in V \text{ of } V * U$$
$$\vartheta\colon u \to (u\varphi,\, u\mu) \text{ for } u \in U \text{ of } V * U.$$

In $(u\varphi, u\mu)$ the left-hand side, $u\varphi$, is in U of $V * U$. Because $V * U$ is a free product and φ and ρ are homomorphisms, ϑ extends to a homomorphism. Note, $x\vartheta = (1, x\mu)$. The map ρ is onto. $(V * U)\vartheta$ contains therefore an element (g, h) for all $h \in H$. Since $x\mu$ generates H as a normal subgroup, $(V * U)\vartheta$ contains $(1, h)$ for all $h \in H$ and

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therefore $(1, H)$. Therefore $(V * U)\vartheta$ contains $(v, 1)$ for all $v \in V$ and $(u\varphi, 1)$ for all $u \in U$. Because φ is onto this implies that also $(V * U, 1)$ of $(V * U) \times H$ is in $(V * U)\vartheta$. Together we get that ϑ is onto and the proof is complete. ∎

REMARK. There are two natural choices for V:

(i) $V = H$ and ρ the identity map.

(ii) V free. More exactly if $h_1, \ldots, h_n$ is a generating set of H then we may take $V = \langle f_1, \ldots, f_n\rangle$, the free group freely generated by $f_1, \ldots, f_n$ and ρ the homomorphism given by $f_i\rho = h_i$ for $i = 1, \ldots, n$. This gives the following corollary:

COROLLARY 1. *Let U be finitely presented and non-Hopfian. Then there exists a finitely generated free group F and a finitely presented group H containing U such that there is a homomorphism ϑ of $F * U$ onto $(F * U) \times H$.*

EXAMPLE 1: Let $U = \langle t, a; t^{-1}a^2t = a^3\rangle$ be the well known non-Hopfian group of Baumslag and Solitar. See [2], p.197, where it is also shown that $1 \neq x = [t^{-1}at, a]$ is in the kernel of φ given by $t\varphi = t$ and $a\varphi = a^2$. The group

$$H = \langle t, a, s, b; t^{-1}a^2t = a^3, s^{-1}b^2s = b^3, s = [t^{-1}at, a], t = [s^{-1}bs, b]\rangle,$$

which contains U and has the property that x generates H as a normal subgroup, was constructed in [3]. Let G_1 be the free product of H and U:

$$G_1 = \big\langle t, a, s, b, r, c; t^{-1}a^2t = a^3, s^{-1}b^2s = b^3, s = [t^{-1}at, a], \\ t = [s^{-1}bs, b], r^{-1}c^2r = c^3\big\rangle$$

By Theorem 1, the homomorphism ϑ of G_1 to $G_1 \times H$ given by

$$\vartheta: t \to (t, t),\ \vartheta: a \to (a, a),\ \vartheta: s \to (s, s),\ \vartheta: b \to (b, b), \\ \vartheta: r \to (r, t),\ \vartheta: c \to \big(c^2, a\big)$$

is onto.

EXAMPLE 2: Let U and H be the groups of Example 1. The relation $s = [t^{-1}at, a]$ of H can be used to eliminate the generator s, so that H has the following presentation on three generators and three relations:

$$H = \big\langle t, a, b; t^{-1}a^2t = a^3, [a, t^{-1}at]b^2[t^{-1}at, a] = b^3, t = [[a, t^{-1}at]b[t^{-1}at, a], b]\big\rangle.$$

Let G_2 be the one-relator group

$$G_2 = \langle r, c, f_t, f_a, f_b; r^{-1}c^2r = c^3\rangle.$$

Then by Corollary 1, the map ϑ of G_2 to $G_2 \times H$ given by

$$\vartheta: f_t \to (f_t, t),\ \vartheta: f_a \to (f_a, a),\ \vartheta: f_b \to (f_b, b), \\ \vartheta: r \to (r, t),\ \vartheta: c \to \big(c^2, a\big)$$

is onto.

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3. Finitely generated groups G with $G \cong G \times H$, H finitely presented

Theorem 2. *Let ϑ be a homomorphism of A onto $A \times B$. Then A contains a normal subgroup K such that $A/K \cong A/K \times B$.*

Proof: Let K be a maximal normal subgroup such that $K\vartheta$ is contained in $(K, 1)$. Then ϑ induces a map η of A/K onto $A/K \times B$. Let $L = \ker \eta$ and let N be its preimage under the quotient map $A \to A/K$. Then N contains K and has the property that $N\vartheta$ is contained in $(N, 1)$. By the maximality of K, we get that $K = N$ and therefore that η is an isomorphism and the proof is complete. ∎

We now show that a maximal normal subgroup K of A with the property that $K\vartheta$ is contained in $(K, 1)$ is unique.

The upper kernel of a homomorphism ϑ of A onto $A \times B$. Let $A_1 = (A, 1)\vartheta^{-1}$ and for $i > 0$, let $A_{i+1} = (A_i, 1)\vartheta^{-1}$. The A_i form a descending chain of normal subgroups. The intersection of the A_i contains the kernel of ϑ. We call it the upper kernel of ϑ and write $\operatorname{uker}(\vartheta)$ for it and claim that it has the property $A/\operatorname{uker}(\vartheta) \cong A/\operatorname{uker}(\vartheta) \times B$. Since ϑ maps all elements of $\operatorname{uker}(\vartheta)$ to $(\operatorname{uker}(\vartheta), 1)$, ϑ induces a map ϑ' from $A/\operatorname{uker}(\vartheta)$ to $A/\operatorname{uker}(\vartheta) \times B$, and since $a\vartheta \in (\operatorname{uker}(\vartheta), 1)$ implies $a \in \operatorname{uker}(\vartheta)$, ϑ' is an isomorphism.

Remark. If $a\vartheta \in (A, 1)$, we can consider $a\vartheta$ as an element of A and apply ϑ again to get $a\vartheta^2$. If $a\vartheta^2$ is again in $(A, 1)$ we can continue in the same way to get $a\vartheta^3$ and so on. The upper kernel $\operatorname{uker}(\vartheta)$ consists now precisely of those elements $a \in A$ with $a\vartheta^n$ in $(A, 1)$ for all n.

Proposition 1. *Let ϑ be a homomorphism of A onto $A \times B$. If K is a maximal normal subgroup of A such that $K\vartheta$ is contained in $(K, 1)$, then $K = \operatorname{uker}(\vartheta)$.*

Proof: Since $K\vartheta$ is contained in $(K, 1)$, $A_1 = (A, 1)\vartheta^{-1} \supseteq K$. Assume now that $A_i \supseteq K$. Then $K \supseteq K\vartheta$ implies $A_i \supseteq K\vartheta$ and $A_{i+1} = (A_i, 1)\vartheta^{-1} \supseteq K$. By induction we get $A_i \supseteq K$ for all i and therefore $\operatorname{uker}(\vartheta) \supseteq K$. By maximality of K this implies $K = \operatorname{uker}(\vartheta)$. ∎

The lower kernel of a homomorphism ϑ of A onto $A \times B$. The upper kernel is a method of finding the largest normal subgroup K with $K\vartheta$ in $(K, 1)$ and $A/K \cong A/K \times B$ as an intersection of a chain of subgroups. The smallest normal subgroup with $K\vartheta$ in $(K, 1)$ and $A/K \cong A/K \times B$ can be found as follows. Let $K_1 = \ker(\vartheta)$. Then ϑ induces a map ϑ_1 of A/K_1 onto $A/K_1 \times B$. Let K_2 be the preimage of $\ker(\vartheta_1)$ under the quotient map A to A/K_1. Inductively if ϑ_i maps A/K_i onto $A/K_i \times B$ we define K_{i+1} to be the preimage of $\ker(\vartheta_i)$ under the quotient map A to A/K_i. The K_i form an ascending chain of subgroups. Let $K = \cup K_i$. We call K the lower kernel

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of ϑ and write $\operatorname{lker}(\vartheta)$ for it. ϑ induces a map ϑ' from A/K onto $A/K \times B$ which is an isomorphism.

PROPOSITION 2. *Let ϑ be a homomorphism of A onto $A \times B$. If N is a normal subgroup of A such that $N\vartheta$ is contained in $(N, 1)$ and ϑ induces an isomorphism of A/N to $A/N \times B$, then $N \supseteq \operatorname{lker}(\vartheta)$.*

PROOF: Obviously, $N \supseteq K_1 = \ker(\vartheta)$. Assume now that $N \supseteq K_i$. Then $K_{i+1}N/N$ is in the kernel of the homomorphism A/N to $A/N \times B$ induced by ϑ. Since this map is an isomorphism, $N \supseteq K_{i+1}$. By induction we get $N \supseteq K_i$ for all i, therefore $N \supseteq \operatorname{lker}(\vartheta)$. □

REMARK. In general $\operatorname{lker}(\vartheta) \neq \operatorname{uker}(\vartheta)$. If $G \cong G \times B$ and φ is an isomorphism of G to $G \times B$ and H is any group then the map ϑ of $H \times G$ to $(H \times G) \times B$ given by $(h, g)\vartheta = (h, g\varphi)$ is an isomorphism. In this case we get $\operatorname{lker}(\vartheta) = 1$ and $\operatorname{uker}(\vartheta) \supseteq H$.

COROLLARY 2. *Let U be a finitely presented non-Hopfian group. Form the finitely presented group H as in Lemma 1. Then there exists a finitely generated group G such that $G \cong G \times H$.*

PROOF: By Theorem 1, there exists a homomorphism ϑ of $H * U$ onto $H * U \times H$. By Theorem 2, the group $G = H * U/\operatorname{uker}(\vartheta)$ has the property $G \cong G \times H$. □

EXAMPLE 3: Let G_2 and H be as in Example 2. From the above, there exists a quotient G_3 of G_2 such that $G_3 \cong G_3 \times H$.

4. FINITELY PRESENTED GROUPS G WITH A HOMOMORPHISM FROM G ONTO $G \times G$

Let $A = \langle a_1, \ldots, a_n \rangle$ be a finitely generated abelian group and let μ, ν be two one-one homorphisms from A to A which are not onto, such that $\mu\nu = \nu\mu$. It was shown in [**3**] that the HNN extension $U = \langle t, A; t^{-1}a_i\mu t = a_i\nu, i = 1, \ldots, n \rangle$ is non-Hopfian if $A = \langle A\mu, A\nu \rangle$. Theorem 3 generalises the construction of the group of Theorem C of Baumslag and Miller [**1**].

THEOREM 3. *Let A, μ, ν be as above. If the elements $a_i\mu(a_i\nu)^{-1}$, $i = 1, \ldots, n$ generate A, then there exists a finitely presented group G containing A with a homomorphism ϑ of G onto $G \times G$ induced by μ.*

PROOF: Let $B = \langle b_1, \ldots, b_n \rangle$ and $C = \langle c_1, \ldots, c_n \rangle$ be isomorphic to A such that $a_i \to b_i$ and $a_i \to c_i$, $i = 1, \ldots, n$ induce isomorphisms and form the HNN extensions

$$U = \langle t, A; t^{-1}a_i\mu t = a_i\nu, i = 1, \ldots, n \rangle,$$
$$V = \langle s, B \times C; s^{-1}b_i\mu s = b_i\nu, s^{-1}c_i\nu, i = 1, \ldots, n \rangle.$$

Since μ and ν are not onto, we can find $a_i \in A - A\mu$ and $a_j \in A - A\nu$. Then by Britton's Lemma (see [**2**], p.181), the element $[t^{-1}a_it, a_j]$ is not trivial and t and $[t^{-1}a_it, a_j]$ generate a free subgroup of U. Similarly $[s^{-1}b_is, b_j]$ is not trivial and s and $[s^{-1}b_is, b_j]$ generate a free subgroup of V.

We can therefore construct the amalgamation

$$G = \langle U, V; t = [s^{-1}b_is, b_j],\ s = [t^{-1}a_it, a_j]\rangle.$$

We claim that the map ϑ from G to $G \times G$ given on the generators by

$$\begin{aligned} &\vartheta\colon t \to (t, t),\ \vartheta\colon a_i \to (a_i, a_i), \quad i = 1, \dots, n,\\ &\vartheta\colon s \to (s, s),\ \vartheta\colon b_i \to (b_i, b_i), \quad i = 1, \dots, n,\\ &\vartheta\colon c_i \to (b_i(c_i\mu), c_i\mu), \qquad i = 1, \dots, n, \end{aligned}$$

is a homomorphism and onto.

To show that ϑ is a homomorphism, we have to prove that ϑ maps relations in G to relations of $G \times G$. Since all elements except c_i, $i = 1, \dots, n$, are mapped to the corresponding diagonal elements of $G \times G$, it suffices to show that $s^{-1}c_i\mu s = c_i\nu$, $i = 1, \dots, n$, and $[b_i, c_j] = 1$, for $i, j = 1, \dots, n$, are mapped to relations.

Note,

$$\begin{aligned} (s^{-1}c_i\mu s)\vartheta &= \big(s^{-1}b_i\mu(c_i\mu^2)s,\ s^{-1}c_i\mu^2 s\big)\\ &= \big(s^{-1}b_i\mu s\, s^{-1}(c_i\mu^2)s,\ s^{-1}c_i\mu^2 s\big). \end{aligned}$$

This can be simplified by the relations of V:

$$\big(s^{-1}b_i\mu s\, s^{-1}(c_i\mu^2)s,\ s^{-1}c_i\mu^2 s\big) = (b_i\nu c_i\mu\nu,\ c_i\mu\nu).$$

We now use the fact that $\mu\nu = \nu\mu$ to derive $(b_i\nu c_i\mu\nu, c_i\mu\nu) = (b_i\nu c_i\nu\mu, c_i\nu\mu)$. But $(b_i\nu c_i\nu\mu, c_i\nu\mu) = (c_i\nu)\vartheta$. $[b_i, c_j]\vartheta = (1, 1)$ is true because $B \times C$ is abelian.

It remains now to show that ϑ is onto. We observe, $[s^{-1}c_is, b_j]\vartheta = \big([s^{-1}b_ic_i\mu s, b_j], [s^{-1}c_i\mu s, b_j]\big) = \big([s^{-1}b_isc_i\nu, b_j], [c_i\nu, b_j]\big) = \big([s^{-1}b_is, b_j, 1]\big)$. From the relations of G we derive $(t, 1) \in G\vartheta$. Because $G\vartheta$ contains (a_i, a_i), $i = 1, \dots, n$, it contains $(a_i\mu, a_i\mu)$ for all i. Also $(a_i\mu, a_i\mu)(t, 1)^{-1}(a_i\mu, a_i\mu)^{-1}(t, 1) = \big(a_i\mu(a_i\nu)^{-1}, 1\big)$, $i = 1, \dots, n$. Since the elements $a_i\mu(a_i\nu)^{-1}$, $i = 1, \dots, n$ generate A, $(A, 1)$ is in $G\vartheta$. From $s = [t^{-1}a_it, a_j]$, it follows that also $(s, 1) \in G\vartheta$. The consideration above with s, B and C shows now that $(B, 1)$ and $(C, 1)$ are in $G\vartheta$. Together we get that $(G, 1) \in G\vartheta$. If we look at the values of ϑ on the generators we see that therefore $(1, t)$, $(1, s)$, $(1, a_i)$, $(1, b_i)$, $(1, c_i\mu)$, $i = 1, \dots, n$, are in $G\vartheta$. As before we conclude in a similar way that $(1, G) \in G\vartheta$. This completes the proof of Theorem 3. ∎

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The next example is the group constructed by Baumslag and Miller [1].

EXAMPLE 4: Let $A = \langle a \rangle$ be infinite cyclic, and μ, ν given by $a\mu = a^2$, $a\nu = a^3$. Then the conditions of the theorem are satisfied. Also $a \in A - A\mu$ and $a \in A - A\nu$. By Theorem 3 or rather the proof of Theorem 3, we construct the groups

$$U = \langle t, a; t^{-1}a^2t = a^3 \rangle,$$
$$V = \langle s, b, c; [b, c] = 1, s^{-1}b^2s = b^3, s^{-1}c^2s = c^3 \rangle$$

and conclude that for the group

$$G_4 = \langle t, a, s, b, c; t^{-1}a^2t = a^3, [b, c] = 1, s^{-1}b^2s = b^3, s^{-1}c^2s = c^3, s = [t^{-1}at, a], t = [s^{-1}bs, b] \rangle$$

the map ϑ from G_4 to $G_4 \times G_4$ given by

$$\vartheta\colon t \to (t, t),\ \vartheta\colon a \to (a, a),\ \vartheta\colon s \to (s, s),\ \vartheta\colon b \to (b, b),$$
$$\vartheta\colon c \to \left(bc^2, c^2\right)$$

is a homomorphism and onto.

EXAMPLE 5: Let $A = \langle a_1, a_2 \rangle$ be free abelian on two generators, and μ, ν given by $a_1\mu = a_1^2$, $a_2\mu = a_2$, $a_1\nu = a_1$, $a_2\nu = a_2^2$. Then the conditions of the theorem are satisfied. Also $a_1 \in A - A\mu$ and $a_2 \in A - A\nu$.

Again by the proof of Theorem 3, we construct the groups

$$U = \langle t, a_1, a_2; [a_1, a_2] = 1, t^{-1}a_1^2t = a_1, t^{-1}a_2t = a_2^2 \rangle,$$
$$V = \langle s, b_1, b_2, c_1, c_2; [b_1, b_2] = 1, s^{-1}b_1^2s = b_1, s^{-1}b_2s = b_2^2, [c_1, c_2] = 1, s^{-1}c_1^2s = c_1, s^{-1}c_2s = c_2^2, [b_i, c_j] = 1, i, j = 1, 2 \rangle$$

and conclude that for the group

$$G_5 = \langle t, a_1, a_2, s, b_1, b_2, c_1, c_2; [a_1, a_2] = 1, t^{-1}a_1^2t = a_1, t^{-1}a_2t = a^2, [b_1, b_2] = 1, s^{-1}b_1^2s = b_1, s^{-1}b_2s = b_2^2, [c_1, c_2] = 1, s^{-1}c_1^2s = c_1, s^{-1}c_2s = c_2^2, [b_i, c_j] = 1,\ i, j = 1, 2, s = [t^{-1}a_1t, a_2], t = [s^{-1}b_1s, b_2] \rangle$$

there exists a homomorphism ϑ from G_5 onto $G_5 \times G_5$.

5. Finitely generated groups $G \cong G \times G$

Theorem 4. *If a non-trivial finitely generated group G has a homomorphism ϑ from G onto $G \times G$, then there exists a non-trivial quotient E of G isomorphic to its own direct square.*

Proof: Let K be a maximal normal subgroup, $K \neq G$, such that, $K\vartheta$ is contained in $K \times K$. Such a normal subgroup exists for a finitely generated group G by Zorn's lemma. Set $E = G/K$. Then ϑ induces a map η of E onto $E \times E$. Let $L = \ker \eta$; then $L \neq E$ and its preimage N under the quotient map $G \to E$ contains K and has the property that $N\vartheta$ is contained in $N \times N$. By maximality of K, we get that $K = N$ and therefore that η is an isomorphism. This completes the proof of Theorem 4.

As in Section 3 we can find a normal subgroup K^* with $G/K^* \cong G/K^* \times G/K^*$ as follows. Let $K_1 = \ker(\vartheta)$; then ϑ induces a map ϑ_1 of G/K_1 onto $G/K_1 \times G/K_1$. Let K_2 be the preimage of $\ker(\vartheta_1)$ under the quotient map G to G/K_1. Inductively if ϑ_i maps G/K_i onto $G/K_i \times G/K_i$ we define K_{i+1} to be the preimage of $\ker(\vartheta_i)$ under the quotient map G to G/K_i. The K_i form an ascending chain of subgroups. Let $K^* = \cup K_i$. Then ϑ induces a map ϑ^* from G/K^* onto $G/K^* \times G/K^*$ which is an isomorphism. ∎

Proposition 3. *Let ϑ be a homomorphism of G onto $G \times G$. If N is a normal subgroup of G such that $N\vartheta$ is contained in $N \times N$ and ϑ induces an isomorphism of G/N to $G/N \times G/N$, then $N \supseteq K^*$ for $K^* = \cup K_i$ as above.*

Proof: Obviously, $N \supseteq K_1 = \ker(\vartheta)$. Assume now that $N \supseteq K_i$. Then $K_{i+1}N/N$ is in the kernel of the homomorphism G/N to $G/N \times G/N$ induced by ϑ. Since this map is an isomorphism, $N \supseteq K_{i+1}$. Therefore by induction, $N \supseteq K_i$ for all i and hence $N \supseteq K^*$. ∎

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